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Some asymptotic problems for Hamilton-Jacobi-Bellman equations and applications to global optimization

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Voyager, c'est bien utile, ça fait travailler l'imagination. Tout le reste n'est que déceptions et fatigues. Notre voyage à nous est entièrement imaginaire. Voilà sa force. *[Traveling is very useful, it makes the imagination work. Everything else is disappointment and fatigue. Our trip is entirely imaginary. This is its strength.]*

Céline (1952)
Voyage au bout de la nuit.

Le chemin est un hommage à l'espace. Chaque tronçon du chemin est en lui-même doté d'un sens et nous invite à la halte. *[The path is a tribute to space. Each section of the path is itself endowed with a meaning and invites us to stop.]*

Kundera (1990)
L'immortalité.

Abstract

The thesis deals with control theory and related topics both in deterministic and stochastic framework, with an emphasis on the analytical aspects and Hamilton-Jacobi equations. It is divided into four chapters.
The first chapter deals with periodic homogenization and singular perturbations in deterministic control problems. The main results concern the convergence and characterization of the limit value function and the underlying optimal trajectories, using relaxed control limits.
The second chapter is motivated by a recent algorithm in the context of Deep Learning, called *"Deep relaxation of Stochastic Gradient Descent"*, and concerns singular perturbations for stochastic control problems where the new difficulty with respect to the existing literature lies in the unboundeness of the data. The asymptotic behaviors in this context were obtained after developing new probabilistic methods, together with an adaptation of the viscosity instruments to problems with unbounded data. Then the results were applied to the previously mentioned algorithm and to its extension which also involves the optimal control of the so-called *learning-rate* parameter.
The third chapter is devoted to global optimization. It aims to construct a dynamic system that asymptotically reaches the global minimum of a given function. To do this, ideas from weak KAM theory and both deterministic and stochastic control problems are used. The main tools to prove convergence are occupational (random) measures and the asymptotic behavior of the solutions of Hamilton-Jacobi equations.
The last chapter provides a new method with new results for the solvability of the ergodic equations of Hamilton-Jacobi-Bellman in the viscous case with unbounded and merely measurable ingredients. The latter appears in various asymptotic problems present in the literature and among those addressed in the previous chapters. The results also extend to ergodic Mean-Field Games which are studied in the same context.

Sommario

La tesi tratta la teoria del controllo e argomenti correlati, sia in ambito deterministico che stocastico, con enfasi sugli aspetti analitici e sulle equazioni di Hamilton-Jacobi. È diviso in quattro capitoli.

Il primo capitolo tratta dell'omogeneizzazione periodica e delle perturbazioni singolari in problemi di controllo deterministici. I risultati principali riguardano la convergenza e la caratterizzazione della funzione valore limite e delle traiettorie ottimali sottostanti, utilizzando limiti di controlli rilassati.

Il secondo capitolo è motivato da un recente algoritmo nel contesto del Deep Learning, denominato *"Deep relaxation of Stochastic Gradient Descent"*, e riguarda perturbazioni singolari per problemi di controllo stocastico dove la nuova difficoltà rispetto alla letteratura esistente sta nell'illimitatezza dei dati. I comportamenti asintotici in questo contesto sono stati ottenuti dopo aver sviluppato nuovi metodi di tipo probabilistico, insieme ad un adattamento degli strumenti di viscosità a problemi con dati illimitati. Quindi i risultati sono stati applicati all'algoritmo precedentemente menzionato e ad una sua estensione che coinvolge anche il controllo ottimo del cosiddetto parametro *learning-rate*.

Il terzo capitolo è dedicato all'ottimizzazione globale. Mira a costruire un sistema dinamico che raggiunga asintoticamente il minimo globale di una data funzione. Per fare ciò vengono usate idee della teoria KAM debole e problemi di controllo sia deterministico che stocastico. I principali strumenti per dimostrare la convergenza sono misure occupazionali (aleatorie) e il comportamento asintotico delle soluzioni di equazioni di Hamilton-Jacobi.

L'ultimo capitolo fornisce un nuovo metodo con nuovi risultati per la risolubilità delle equazioni ergodiche di Hamilton-Jacobi-Bellman nel caso viscoso con ingredienti illimitati e meramente misurabili. Quest'ultimo compare in vari problemi asintotici presenti in letteratura e tra quelli affrontati nei capitoli precedenti. I risultati si estendono anche ai giochi a campo medio di tipo ergodico *(ergodic Mean-Field Games)* che sono studiati nello stesso contesto.

To my family

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Contents

Introduction

The thesis deals with control theory and related topics both in deterministic and stochastic framework, with an emphasis on the analytical aspects. The research was carried out mainly on Hamilton-Jacobi equations. Among the problems studied in this regard: homogenization, singular perturbations and asymptotic approximations. One of the main challenges that is needed in each of the latter topics is the solvability of the *ergodic problem*, which is addressed in a specific chapter. The scope of this research then extended to Mean-Field Games, with further applications to global optimization and stochastic gradient descent. The thesis is divided into four chapters as follows:

Chapter 1 deals with periodic homogenization and singular perturbations in deterministic control problems where the main results concern the convergence and characterization of the value function corresponding to an optimal control problem, in addition to the underlying optimal trajectories using limiting relaxed controls. The results are based on the manuscript

- *Homogenization of some optimal control problems and convergence of trajectories*, with M. Bardi and G. Terrone.

Chapter 2 is motivated by a recent algorithm, named "Deep relaxation of stochastic gradient descent" in the context of Deep Learning. It concerns singular perturbations for stochastic control problems with an additional difficulty of unboundedness (hence lack of compactness) of the data that is usually encountered in such applications. The asymptotic has been obtained with new methods based on probability together with an adaptation of viscosity tools to the unbounded setting. An application to the control of stochastic gradient descent is also shown. The results are based on the forthcoming manuscripts

- *Singular perturbations in stochastic optimal control with unbounded data*, with M. Bardi.
- *Deep relaxation of controlled Stochastic Gradient Descent via singular perturbations*, with M. Bardi.

Chapter 3 is devoted to global optimization. It aims at constructing strategies (dynamics) which asymptotically reach the global minimum of a given function. To do so, both deterministic and stochastic control problems have been put into action. The main tools to prove the convergence are (random) occupational measures and asymptotics of Hamilton-Jacobi equations. The results are based on the manuscripts

- *An Eikonal equation with vanishing Lagrangian arising in global optimization*, with M. Bardi. (Submitted)
- *Global optimization by a limiting discounted stochastic control approach*, with M. Bardi. (In progress)

Chapter 4 provides a new method with new results for the problem of *ergodic Hamilton-Jacobi-Bellman equation* in the viscous case with unbounded and measurable ingredients. The latter appears in various problems amongst those tackled in the previous chapters. The method is based on optimization over abstract Banach spaces combined with results from the theory of Dirichlet forms (and diffusion operators) and ultimately solves the ergodic viscous Hamilton-Jacobi-Bellman equation in addition to ergodic Mean-Field Games in the whole space. It is based on the manuscripts

- *A viscous ergodic problem with unbounded and measurable ingredients. Part 1: HJB equation.* (Submitted)
- *A viscous ergodic problem with unbounded and measurable ingredients. Part 2: Mean-Field Games.* (Submitted)

The problems tackled in the thesis required the use of a broad range of tools and methods, among them: control theory (deterministic and stochastic), PDEs (viscosity methods), set-valued analysis (real and stochastic), probability and stochastic analysis, abstract optimization and asymptotic approximations.

In what follows, we present an overview and the main contributions of each chapter.

Chapter 1. Periodic homogenization of deterministic control problems

We consider a dynamics in $\mathbb{R}^N$ ($N \in \mathbb{N}\backslash\{0\}$) of the following type:

$$\begin{cases} \dot{x}_1(t) = f_1\left(x(t), \frac{x_2(t)}{\varepsilon}, \alpha_1(t), \alpha_2(t)\right) \\ \dot{x}_2(t) = f_2\left(x(t), \frac{x_2(t)}{\varepsilon}, \alpha_2(t)\right) \\ x(0) = x. \end{cases}$$

The state variable is $x = (x_1, x_2)$, where $x_1 \in \mathbb{R}^{N_1}$ and $x_2 \in \mathbb{T}^{N_2}$, with $N_1 + N_2 = N$ ($N_1, N_2 \in \mathbb{N}\backslash\{0\}$). The controls (α_1, α_2) belong to the space of measurable functions defined on $[0, +\infty)$ and valued into compact metric spaces. Note that we assume only the state variables x_2 are oscillating, and that such components are driven by the α_2 component of the control variable only.

Together with the latter dynamics we consider the cost functional

$$J^{\varepsilon}(t, x, \alpha_1, \alpha_2) := \int_0^t \ell\left(s, x(s), \frac{x_2(s)}{\varepsilon}, \alpha_1(s), \alpha_2(s)\right) \mathrm{d}s + h(x(t)),$$

and the associated value function

$$v^{\varepsilon}(t, x) := \inf_{(\alpha_1, \alpha_2) \in \mathcal{A}} J^{\varepsilon}(t, x, \alpha_1, \alpha_2)$$

which solves, in viscosity sense, the Hamilton-Jacobi-Bellman equation

$$\begin{cases} \partial_t v^{\varepsilon} + \max\limits_{(\alpha_1, \alpha_2) \in A} \Big\{ -D_{x_1} v^{\varepsilon} \cdot f_1\left(x, \frac{x_2}{\varepsilon}, \alpha_1, \alpha_2\right) - D_{x_2} v^{\varepsilon} \cdot f_2\left(x, \frac{x_2}{\varepsilon}, \alpha_2\right) \\ \qquad -\ell\left(t, x, \frac{x_2}{\varepsilon}, \alpha_1, \alpha_2\right) \Big\} = 0 & \text{in } (0, +\infty) \times \mathbb{R}^N \\ v^{\varepsilon}(0, x) = h(x) & \text{in } \mathbb{R}^N. \end{cases}$$

where the Hamiltonian $H(x, y, p)$ writes as:

$$H(s, x, y, p) := \max_{(\alpha_1, \alpha_2) \in A} \Big\{ -p_1 \cdot f_1(x, y, \alpha_1, \alpha_2) - p_2 \cdot f_2(x, y, \alpha_2) - \ell(t, x, y, \alpha_1, \alpha_2) \Big\}.$$

The homogenization problem consists in studying the limit as $\varepsilon \to 0^+$. In the framework of viscosity solutions of Hamilton–Jacobi equations, this type of problems have been studied extensively in the last decades and are today very well understood. The general approach consists in proving the existence of an *effective Hamiltonian* $\bar{H}(t, x, p)$ such that the sequence $v^{\varepsilon}(t, x)$ converges locally uniformly in $[0, +\infty) \times \mathbb{R}^N$, as ε goes to 0, to $v(t, x)$, the unique viscosity solution of the limiting Hamilton-Jacobi equation

$$\begin{cases} \partial_t v + \bar{H}(t, x, Dv) = 0 & \text{in } (0, +\infty) \times \mathbb{R}^N \\ v(0, x) = h(x) & \text{in } \mathbb{R}^N. \end{cases}$$

The aim of this chapter is instead to find, for some classes of problems, a *limiting optimal control problem* associated to the limiting Hamilton-Jacobi equation given by the effective Hamiltonian $\bar{H}(t, x, p)$. This is a crucial issue in view of applications;

nonetheless, to the best of our knowledge, there are few results in this direction, mostly in the engineering literature. The first approach to homogenization of control problems originated in the light of the so-called Levinson–Tichonov theory for ODEs and then extended to deterministic control problems; see e.g. [33], [73], [112]. This approach requires the limit of the fast dynamics to be governed by an algebraic equation and the stationary points of the fast dynamics to be attractive. To treat more general situations, other averaging techniques, have been developed; see [9], [10], [11], [12], [90]. These results are based on the analysis of trajectories and make use of invariant measures and occupational measure of the fast dynamics. In [155], [156], by using special measures called *limiting relaxed controls*, some of these results have been interpreted in the light of singular perturbation theory and connected to Hamilton-Jacobi-Bellman equations.

The approach in this chapter is based on the results obtained in the theory of homogenization of Hamilton–Jacobi equations. Our strategy consists in studying whether $\bar{H}$ can be written as a Bellman Hamiltonian, and in describing the unique solution of the limiting Hamilton-Jacobi equation as a value function of some optimal control problem. This is indeed one of the main challenges of this chapter, since except in few cases it is hard to find explicit formulas for the $\bar{H}$ associated to an ergodic Hamiltonian. To this scope, we require the dynamics and the costs to satisfy from time to time different structure hypotheses, as well as some controllability assumptions. We will then construct the corresponding limiting optimal control problem as a differential inclusion obtained by averaging the vector field with respect to limiting relaxed controls. And therefore we shall be interested in the convergence of the trajectories, in their singular perturbation form, as $\varepsilon \to 0$. Indeed, after the convergence of the value function is proven, it is natural to wonder whether the singularly perturbed trajectories do converge to some trajectories controlled by limiting relaxed controls. It is also of much interest to consider the converse of such result, that is, whether each trajectory driven by a limiting relaxed control can be approximated by a sequence of singularly perturbed trajectories. The interest of the latter lies in application in engineering or computer science, where it can be more profitable to consider approximating dynamics instead of the real one, provided the convergence holds true.

Main contributions of chapter 1

Let us denote by $\mathcal{P}(\mathbb{T}^{N_2} \times A_2)$ the set of Radon probability measures on $\mathbb{T}^{N_2} \times A_2$. For any $t > 0$, $y \in \mathbb{T}^{N_2}$ and $\alpha_2 \in \mathcal{A}_2$ consider the solution $y(s)$ of

$$\dot{y}(s) = f_2(x, y(s), \alpha_2(s)),\ y(0) = y \in \mathbb{T}^{N_2},$$

and define a measure in $\mathcal{P}(\mathbb{T}^{N_2} \times A_2)$, called occupational measure, as

$$\mu_{t,x,y,\alpha_2} := \frac{1}{t}\int_0^t \delta_{(y(s),\alpha_2(s))}\mathrm{d}s,$$

where δ is the Dirac's delta.

We denote by $Z(x)$ the set of weak star limits of these measures, i.e. the set of measures $\mu \in \mathcal{P}(\mathbb{T}^{N_2} \times A_2)$ such that

$$\mu = \lim_{n\to\infty} \mu_{t_n,x,y,\alpha_2} \quad \text{weak-star}$$

for some $t_n \to +\infty$, $\alpha_2 \in \mathcal{A}_2$ and $y \in \mathbb{R}^{N_2}$. A measure in $Z(x)$ is called limiting occupational measure, or limiting relaxed control.

Our **first main result** can be stated as follows: assuming the controlled dynamics $y(s)$ satisfies a strong controllability assumption, then as $\varepsilon \to 0$, the sequence $v^\varepsilon(t,x)$ converges locally uniformly on $(0,+\infty) \times \mathbb{R}^N$ to

$$\bar{v}(t,x) := \inf\left\{\int_0^t \bar{\ell}(s, x(s), \alpha_1(s), \mu(s))\mathrm{d}s + h(x(t))\right\},$$

where the infimum is taken over trajectories $x(\cdot)$ satisfying

$$\begin{cases} \dot{x}_1(t) = \bar{f}_1\left(x(t), \alpha_1(t), \mu(t)\right) \\ \dot{x}_2(t) = 0 \\ \alpha_1 \in \mathcal{A}_1, \quad \mu(t) \in Z(x(t)) \\ x(0) = x. \end{cases}$$

where we define the averaged vector field and running cost as follows:

$$\bar{f}_1(x, \alpha_1, \mu) := \int_{\mathbb{R}^{N_2}\times A_2} f_1\big(x, y, \alpha_1, \alpha_2\big)\mathrm{d}\mu(y, \alpha_2);$$
$$\bar{\ell}(s, x, \alpha_1, \mu) := \int_{\mathbb{R}^{N_2}\times A_2} \ell\big(s, x, y, \alpha_1, \alpha_2\big)\mathrm{d}\mu(y, \alpha_2).$$

Our **second main result** concerns the convergence of trajectories. In particular, we are interested in *the perturbed dynamics*

$$\begin{cases} \dot{x}_1(t) = f_1\left(x(t), y(t), \alpha_1(t), \alpha_2(t)\right) \\ \dot{x}_2(t) = f_2\left(x(t), y(t), \alpha_2(t)\right) \\ \dot{y}(t) = \frac{1}{\varepsilon} f_2\left(x(t), y(t), \alpha_2(t)\right) \\ x(0) = x, \quad y(0) = y \end{cases}$$

where y plays the role of x_2/ε in the original system, and in *the relaxed dynamics*

$$\begin{cases} \dot{x}_1(t) = \bar{f}_1\left(x(t), \alpha_1(t), \mu(t)\right) \\ \dot{x}_2(t) = 0 \\ \alpha_1 \in \mathcal{A}_1, \quad \mu(t) \in Z(x(t)) \\ x(0) = x. \end{cases} \qquad \Leftrightarrow \qquad \begin{cases} \dot{x}_1(t) \in \bar{f}_1\left(x(t), A_1, Z(x(t)\right) \\ \dot{x}_2(t) = 0 \\ x(0) = x. \end{cases}$$

We show (again under the controllability assumption) that every solution to *the relaxed dynamics* is an accumulation point to a sequence of *the perturbed dynamics*, as $\varepsilon \to 0$, with respect to the uniform convergence topology. And reciprocally, every accumulation point with respect to the uniform convergence topology of a sequence of *the perturbed dynamics* as $\varepsilon \to 0$ is a solution to the *convexified relaxed dynamics*

$$\begin{cases} \dot{x}_1(t) \in \overline{\text{co}} \bar{f}_1\left(x(t), A_1, Z(x(t)\right) \\ \dot{x}_2(t) = 0 \\ x(0) = x. \end{cases}$$

Chapter 2. Deep relaxation via singular perturbations of stochastic control problems

Our motivation in this chapter is the Stochastic Gradient Descent algorithm in the context of Deep Learning and Big Data analysis, where one needs to take into account the possible unboundedness of the data and the state space. More precisely, a recent algorithm for a Stochastic Gradient Descent, named *Deep Relaxation*, has been introduced in [64] and is based on the following singularly perturbed system of SDEs

$$\begin{aligned} \mathrm{d}X_s^\varepsilon &= -\nabla_x V(Y_s^\varepsilon, X_s^\varepsilon)\,\mathrm{d}s, \quad X_0^\varepsilon = x \in \mathbb{R}^n \\ \mathrm{d}Y_s^\varepsilon &= -\frac{1}{\varepsilon}\nabla_y V(Y_s^\varepsilon, X_s^\varepsilon)\,\mathrm{d}s + \sqrt{\frac{2}{\varepsilon}}\beta^{-1/2}\,\mathrm{d}W_s, \quad Y_0^\varepsilon = y \in \mathbb{R}^n. \end{aligned}$$

where

$$V(y,x) := f(y) + \frac{1}{2\gamma}|x-y|^2,$$

and f is the function to be minimized. Their main idea is that when $\varepsilon \to 0$, the limit in the latter system of singularly perturbed SDEs is expected to be

$$\mathrm{d}\hat{X}_s = \int_{\mathbb{R}^n} -\frac{1}{\gamma}(X_s - y)\rho_\beta^\infty(\mathrm{d}y; X_s)\,\mathrm{d}s, \quad \hat{X}_0 = x \in \mathbb{R}^n$$

where ρ_β^∞ is Gibbs measure corresponding to the potential V. The latter writes as

$$\mathrm{d}\hat{X}_s = -\nabla f_\gamma(\hat{X}_s)\mathrm{d}s, \quad \hat{X}_0 = x \in \mathbb{R}^n$$

that is the gradient descent (not stochastic) of the regularized loss function defined by

$$f_\gamma := -\frac{1}{\beta}\log\left(G_{\beta^{-1}\gamma} * \exp(-\beta f(x))\right)$$

where

$$G_{\beta^{-1}\gamma}(x) := (2\pi\gamma)^{-n/2}\exp\left(-\frac{\beta}{2\gamma}|x|^2\right)$$

is the heat kernel, and $\beta, \gamma > 0$ are fixed parameter. The function f_γ plays the role of a local entropy and, as $\gamma \to 0$, it is a smooth approximation of f. The parameter β corresponds in physics to the inverse of the temperature (see [144, Chapter 6]) and as $\beta \to \infty$, the heat kernel tends to Dirac measure supported on 0 (see [144, Chapter 7, p.236]).

In the present chapter, we shall consider, in addition to the above system of singularly perturbed SDEs, a more general class of SDEs with unbounded data with a control parameter which plays the role of the so-called *learning rate*. In the previous system of singularly perturbed SDEs, this consists of introducing a control u_s in the slow dynamics

$$\mathrm{d}X_s = -u_s\,\nabla_x V(Y_s, X_s)\,\mathrm{d}s, \quad X_0 = x \in \mathbb{R}^n.$$

Controlling the latter parameter has been considered in [123] where it was shown that the performance of the *controlled stochastic gradient descent* improves with respect to the classical stochastic gradient descent. Our model combines the two algorithms in [64] and [123], that is, a *Deep relaxation of controlled stochastic gradient descent.*

More generally, we are interested in studying the asymptotic behavior as $\varepsilon \to 0$ of a system of controlled and singularly perturbed stochastic differential equations

$$\begin{aligned} \mathrm{d}X_t^\varepsilon &= f(X_t^\varepsilon, Y_t^\varepsilon, u_t)\,\mathrm{d}t + \sqrt{2}\,\sigma^\varepsilon(X_t^\varepsilon, Y_t^\varepsilon, u_t)\,\mathrm{d}W_t,\\ \mathrm{d}Y_t^\varepsilon &= \frac{1}{\varepsilon}\,b(X_t^\varepsilon, Y_t^\varepsilon)\,\mathrm{d}t + \sqrt{\frac{2}{\varepsilon}}\,\varrho(X_t^\varepsilon, Y_t^\varepsilon)\,\mathrm{d}W_t, \end{aligned}$$

where $X_t^\varepsilon \in \mathbb{R}^n$ is the *slow* dynamics, $Y_t^\varepsilon \in \mathbb{R}^m$ is the *fast* dynamics, u_t is the control taking values in a given compact set U and W_t is a multidimensional Brownian motion. Note that f here is a drift vector field and not the loss function to be minimized (as previously denoted). We will allow the components of the drift and the diffusion of the slow dynamics to grow at most linearly with respect to the fast process Y, as in the model of [64]. And while the diffusion coefficient of the process X can be degenerate (i.e. $\sigma^\varepsilon = 0$ is possible), the diffusion coefficient of the process Y is required to be nondegenerate. The precise assumptions are give in Section 2.2. We carry out our analysis in the context of stochastic optimal control problems of the form

$$\sup_u \; J(t,x,y,u) := \mathbb{E}\left[e^{\lambda(t-T)} g(X_T^\varepsilon, Y_T^\varepsilon) + \int_t^T \ell(s, X_s^\varepsilon, Y_s^\varepsilon, u_s) e^{\lambda(s-T)} \mathrm{d}s \,\middle|\, X_t^\varepsilon = x, Y_t^\varepsilon = y \right].$$

Such a quantity is denoted by $V^\varepsilon(t,x,y)$ and refers to the value function which solves in the viscosity sense a fully nonlinear parabolic degenerate PDE in $(0,T) \times \mathbb{R}^n \times \mathbb{R}^m$

$$-V_t^\varepsilon + F^\varepsilon\left(t, x, y, V^\varepsilon, D_x V^\varepsilon, \frac{D_y V^\varepsilon}{\varepsilon}, D_{xx}^2 V^\varepsilon, \frac{D_{yy}^2 V^\varepsilon}{\varepsilon}, \frac{D_{x,y}^2 V^\varepsilon}{\sqrt{\varepsilon}}\right) = 0,$$

complemented with the terminal condition $V^\varepsilon(T,x,y) = g(x,y)$, for a suitable Hamilton-Jacobi-Bellman operator F^ε.

In chapter 2, we show how our results allow us to prove such a convergence with or without a control. This also captures the previous results in [21, 23], where the coefficients in the slow variable are assumed to be bounded with respect to the fast variables. Moreover, we rely in our analysis on arguments and methods sometimes different from those in [21, 23], and borrowed from probability theory, which were key ingredients for handling unboundedness of the data and the state domain.

There is a wide literature on singular perturbations for control systems that goes back to [112] in the late 60's, and also for diffusion processes, with and without control, and different models with fast variables have been studied since then, both in deterministic and stochastic settings, and using methods of probability, analysis, measure theory, or control. We refer the reader to the introduction in [21, 23], where a large but non-exhaustive list of references on these topics are provided. Let us also mention that our results recover in addition a large range of applications in finance, e.g. models of pricing and trading derivative securities in financial markets with stochastic volatility, or applications in economics and advertising as it has been done in [21, 23].

In chapter 2, we analyse the convergence of singular perturbations in the framework of stochastic control and we rely on the associated Hamilton-Jacobi-Bellman equation. We also insist on making assumptions that can be easily checked and which do comply with the applications we are interested in.

Following the diagram below, we are interested in the two arrows with question marks. We start first by embedding this system of SDEs which depends on $\frac{1}{\varepsilon}$ in a family of control problems that we identify through their value function $V^\varepsilon(t,x,y)$. The latter is characterized as the unique viscosity solution to a HJB equation which depends on ε. Then we rely on ergodicity of the fast process to construct the effective Hamiltonian and initial data that allow us to set the limit Cauchy problem. Using methods from homogenization and viscosity theory, we prove the convergence of $V^\varepsilon(t,x,y)$ to the unique viscosity solution of the limit Hamilton-Jacobi equation. And only after proving the effective Hamiltonian is of Bellman type, by means of a selection argument, we can consider the limit PDE as a Hamilton-Jacobi-Bellman equation and we can identify its unique viscosity solution $V(t,x)$ with the value function of an optimal control problem with a stochastic differential inclusion. Therefore we have the convergence of the value function of a family of optimal control problems to a value function of another family of control problems subject to stochastic differential inclusions (SDI).

$$\begin{tikzcd}
(HJB)_\varepsilon \arrow[d, dash, "Ergodicity"] & V^\varepsilon(t,x,y) \arrow[l, "\textbf{Viscosity}"'] \arrow[r, leftrightarrow, "\textbf{control prob}"'] \arrow[dd, "?"] & SDE(\frac{1}{\varepsilon}) \arrow[dd, leftrightarrow, "?"] \\
\text{Effective data} \arrow[d, "Homogenization"] & & \\
HJ(B) \arrow[r, bend left, "\textbf{Bellman Ham.}", "(Selection\, arg.)"'] & V(t,x) \arrow[l, bend left, "\textbf{Viscosity}"] \arrow[r, leftrightarrow, "\textbf{control prob}"] & (SDI)
\end{tikzcd}$$

To do so, we shall make use of new techniques and strategy, amongst:

- The study of the behavior of the first exit time of a stochastic process in bounded domains as their diameter gets larger.
- A sequential construction of an effective Hamiltonian (ergodic constant in the *ergodic problem*, also called *cell problem*) and of the initial data (long-time behavior of the solution to an initial Cauchy problem with unbounded data in the full space).
- An adaptation of the perturbed test function method to unbounded setting in order to prove the convergence of the value function V^ε as $\varepsilon \to 0$.
- A control representation of the limit PDE based on a selection argument.

Main contributions of chapter 2

Our **first main result** is the convergence, as $\varepsilon \to 0$, of $V^\varepsilon(t,x,y)$ to $\overline{V}(t,x)$ solution to an *effective* Hamilton-Jacobi equation, using a combination of viscosity methods together with a sequential construction of the *effective* data by probabilistic arguments. Although this result is somehow expected in regards to the classical theory of homogenization and singular perturbations for HJB equations, our setting does not comply with the classical assumptions (mainly because of the unboundedness of the data, and hence lack of compactness) which prevents using directly the classical methods.

Our **second main result** consists in a control representation of the limit $\overline{V}$. We show that it is in fact the value function of a control problem subject to a stochastic differential inclusion. This permits in particular to prove that, for a given minimization problem, one reaches a lower value using the previously mentioned *deep relaxation of controlled stochastic gradient descent*, when using the classical stochastic gradient descent.

Our **third main result** concerns the convergence of the controlled and singularly perturbed trajectories in the case

$$\lim_{\varepsilon\to 0} \sigma^\varepsilon(x,y,u) = 0, \quad \text{loc. uniformly},$$

which is the situation in the *Deep Relaxation* algorithm that is the main motivation of this chapter. More precisely, given $\nu(\cdot) \in L^\infty(\mathbb{R}^m)$ with values in U and $\mu_{\hat{x}_t}$ the invariant measure of the fast process, we show that every solution $\hat{x}_\cdot$ to the *effective dynamics*

$$\frac{\mathrm{d}\hat{x}_t}{\mathrm{d}t} = \int_{\mathbb{R}^m} f(\hat{x}_t, y, \nu(y))\mathrm{d}\mu_{\hat{x}_t}(y), \quad \hat{x}_0 = x \in \mathbb{R}^n$$

is an accumulation point to a sequence $\{X^\varepsilon_\cdot\}_{\varepsilon>0}$ of trajectories

$$\begin{aligned} \mathrm{d}X^\varepsilon_t &= f(X^\varepsilon_t, Y^\varepsilon_t, u_t)\,\mathrm{d}t + \sqrt{2}\,\sigma^\varepsilon(X^\varepsilon_t, Y^\varepsilon_t, u_t)\,\mathrm{d}W_t, \\ \mathrm{d}Y^\varepsilon_t &= \frac{1}{\varepsilon}\,b(X^\varepsilon_t, Y^\varepsilon_t)\,\mathrm{d}t + \sqrt{\frac{2}{\varepsilon}}\,\varrho(X^\varepsilon_t, Y^\varepsilon_t)\,\mathrm{d}W_t, \end{aligned}$$

in the sense

$$\lim_{\varepsilon\to 0} \mathbb{E}\left[|X^\varepsilon_s - \hat{x}_s|^p\right] = 0, \quad \text{a.e. } s \in [t,T]$$

for some $p \in (0,2]$. Conversely, we show that if a given sequence of controlled processes $X^\varepsilon_\cdot$ converges to some (deterministic) process $\overline{x}_\cdot$ in the latter sense, for some $p \in [1,2]$, then $\overline{x}_\cdot$ is a solution to the *closed and convexified effective* dynamics

$$\frac{\mathrm{d}\overline{x}_t}{\mathrm{d}t} = \overline{\mathrm{co}} \int_{\mathbb{R}^m} f(\overline{x}_t, y, \nu(y))\mathrm{d}\mu_{\overline{x}_t}(y), \quad \overline{x}_0 = x \in \mathbb{R}^n$$

where $\overline{\mathrm{co}}$ denotes the closed convex hull. The two effective dynamics for $\hat{x}$ and $\overline{x}$ shall be expressed in terms of differential inclusions.

Chapter 3. Global optimization: an optimal control approach

This chapter is divided into three parts. The **first part** provides preliminary results on a control problem. In particular we show some estimates on the value function depending on the viscosity coefficient, the discount factor and the time horizon. These estimates are then used in the rest of this chapter where we show a connection between global unconstrained optimization of a continuous function f and weak KAM theory for an eikonal-type equation arising also in ergodic control.

Let $f \in C(\mathbb{R}^n)$ be a bounded function attaining the global minimum. Global optimization is concerned with the search of the minimum points, i.e., finding the set $\mathfrak{M} = \operatorname{argmin} f$. For convex smooth functions this is achieved by the gradient flow, i.e., by following the trajectories of $\dot{y}(s) = -\nabla f(y(s))$ from any initial point $x = y(0)$. However, if the function f is not convex the trajectory $y(\cdot)$ may converge to a local minimum or a saddle point. Several alternative algorithms have been derived to handle non-convex optimization, such as the stochastic gradient descent, simulated annealing, or consensus-based methods. In particular the case of non-smooth f in high dimensions is important for the applications to machine learning, see, e.g., the recent paper [59] and the references therein.

In the **second part** of this chapter, we construct and study a Lipschitz function $v : \mathbb{R}^n \to \mathbb{R}$ such that the following normalized **non-smooth gradient descent differential inclusion**

$$\dot{y}(s) \in \left\{ -\frac{p}{|p|}\ ,\ p \in D^- v(y(s)) \right\},\quad \text{for a.e. } \ s > 0,$$

has a solution for any initial condition $x = y(0)$ and all solutions converge to $\mathfrak{M}$ as $t \to +\infty$. Here $D^- v$ is the sub-differential of the theory of viscosity solutions (see, e.g., [19]). The construction of such a generating function v is based on a classical problem for Hamilton-Jacobi equations: find a constant c such that the stationary equation

$$H(x, Dv) = c \quad \text{in } \mathbb{R}^n$$

has a solution v. The minimal c with this property is the critical value of the Hamiltonian H and, if $H(x, \cdot)$ is convex, it is also the value of an optimal control problem with ergodic cost having H as its Bellman Hamiltonian. If the critical solution v is interpreted in the viscosity sense, the problem fits in the weak KAM theory, and it is well-known that, for $H = |p|^2 - f$ with f periodic, $c = -\min f$ [81, 127]; moreover the same holds for any bounded $f \in C^2(\mathbb{R}^n)$ by a result of Fathi and Maderna [83]. In this chapter we extend such result to non-smooth f, provided it is Lipschitz and semiconcave. We also prove that $\min f$ and v solving the critical equation

$$\min f + \frac{1}{2}|\nabla v(x)|^2 = f(x) \quad \text{ in } \mathbb{R}^n$$

can be approximated in two ways: by the solution of the stationary equation

$$\lambda u_\lambda + \frac{1}{2}|Du_\lambda|^2 = f(x), \quad x \in \mathbb{R}^n,$$

as $\lambda \to 0+$, the so-called small discount limit, as well as by the long-time limit of the solution of the evolutive equation

$$\partial_t u + \frac{1}{2}|Du|^2 = \ f(x), \ \text{ in } \ \mathbb{R}^n \times (0, +\infty), \quad u(x, 0) = 0.$$

More precisely, for the evolutive equation we prove

$$\lim_{t \to +\infty} (u(x, t) - t \min f) = v(x) \quad \text{locally uniformly in } \mathbb{R}^n.$$

Note that the two PDE problems (stationary and evolutive) do not require the a-priori knowledge of $\min f$ and $\operatorname{argmin} f$. Moreover we show that Du_λ and $D_x u(\cdot, t)$ both converge (a.e.) to Dv, therefore giving an approximation of the abovementioned non-smooth gradient descent differential inclusion.

In the **third part** of this chapter we also study the approximation of v and $\mathfrak{M}$ by vanishing viscosity. We add to the stationary equation a term $-\varepsilon\Delta u_\lambda$ and let $\lambda \to 0+$ to get the viscous critical equation

$$U^\varepsilon - \varepsilon\Delta v^\varepsilon(x) + \frac{1}{2}|\nabla v^\varepsilon(x)|^2 = f(x) \quad \text{ in } \mathbb{R}^n,$$

where U^ε is a constant. We prove that $0 \le U^\varepsilon - \min f \le C\varepsilon^\beta$ for some $\beta > 0$. Then we define the approximate stochastic gradient descent

$$\mathrm{d}X_s = -\nabla u_\lambda(X_s)\,\mathrm{d}s + \sqrt{2\varepsilon}\,\mathrm{d}W_s,$$

and show that the trajectories converge to $\mathfrak{M}$ in a suitable sense, for small λ and ε.

<u>Main contributions of chapter 3</u>

Our **first main result** is the convergence of the gradient descent trajectories to the set $\mathfrak{M}$ of minima of f. This is done after observing that v solves also the Dirichlet problem for the eikonal equation

$$\begin{cases} |\nabla v(x)| = \ell(x), & x \in \mathbb{R}^n \setminus \mathfrak{M} \\ v(x) = 0, & x \in \mathfrak{M} \end{cases}$$

with $\ell(x) := \sqrt{2(f(x) - \min f)}$. (In fact, our analysis of this problem requires only that $\ell \in C(\mathbb{R}^n)$ is bounded, non-negative, and $\mathfrak{M} = \{x : \ell(x) = 0\}$). We exploit that the unique solution of the latter Dirichlet problem is the value function

$$v(x) = \inf_{\alpha(\cdot)} \int_0^{t_x(\alpha)} \ell(y_x^\alpha(s))\,\mathrm{d}s, \quad \dot{y}_x^\alpha(s) = \alpha(s), \text{ for } s > 0, \quad y_x^\alpha(0) = x,$$

where α is measurable, $|\alpha(s)| \le 1$, and $t_x(\alpha)$ is the first time the trajectory y_x^α hits $\mathfrak{M}$. We show that optimal trajectories exist, satisfy the gradient descent inclusion

$$\dot{y}(s) \in \left\{ -\frac{p}{|p|}\,,\ p \in D^- v(y(s)) \right\}, \quad \text{for a.e. } s > 0,$$

and tend to $\mathfrak{M}$ as $t \to +\infty$ under a slightly strengthened positivity condition on ℓ. A crucial new tool for the proof are the occupational measures associated to these functions. Finally, we give a sufficient condition for such trajectories to reach $\mathfrak{M}$ in finite time.

Our **second main result** is a stochastic approximation of the previous result. First, we define random occupational measures by

$$\mu_{\lambda,x,\alpha}(Q) := \lambda \int_0^\infty \mathbb{1}_Q(X_s)e^{-\lambda s}\,\mathrm{d}s$$

where Q is any Borel set of $\mathbb{R}^n$, $\mathbb{1}_Q(x)$ is the indicator function ($= 1$ if $x \in Q$ and $= 0$ otherwise), $X.$ is a given stochastic differential equation. Then, recalling $\underline{f} := \min_{z\in\mathbb{R}^n} f(z)$, we define for $\delta \geq 0$ fixed, the set of quasi-minimizers (or quasi-optimal sublevel set) as follows

$$K_\delta := \{\, y \in \mathbb{R}^n \mid f(y) \leq \underline{f} + \delta \,\}$$

Our main result consists on constructing a stochastic dynamics X^* such that

$$\rho_\lambda^{\delta,\varepsilon} := \mu_{\lambda,x,\alpha^*}(K_\delta^c) = \lambda \int_0^\infty \mathbb{1}_{K_\delta^c}(X_s^*)e^{-\lambda s}\,\mathrm{d}s$$

where $K_\delta^c = \mathbb{R}^n \setminus K_\delta$, satisfies

$$\lim_{\varepsilon\to 0}\lim_{\lambda\to 0} \mathbb{P}\left(\rho_\lambda^{\delta,\varepsilon} > a\right) = 0, \quad \forall\, a > 0, \delta > 0,$$

where ε is the diffusion coefficient and λ is the discount factor. This means that X^* tends to concentrate on the global minimum of a given function f in the small-noise limit and when the discount factor goes to zero. This is based on a discounted stochastic control problem, together with our result on the small-noise limit of the ergodic constant

$$\lim_{\varepsilon\to 0} U^\varepsilon = \min_{z\in\mathbb{R}^n} f(z).$$

Chapter 4. The viscous ergodic problem

This chapter is devoted to one of the key problems that is in common with the previous three chapters and extends the existing results about it. It concerns the problem of existence of solutions to some ergodic partial differential equations in the whole space domain $\mathbb{R}^m$ with unbounded data satisfying a subexponential growth. Such problems take the form of

$$\textit{Find } (c, u(\cdot)) \in \mathbb{R} \times \mathcal{X}(\mathbb{R}^m) \textit{ s.t.: } F(x, \nabla u(x), D^2 u(x)) = c, \quad \text{in } \mathbb{R}^m$$

where $\mathcal{X}$ is a functional space (part of the unknowns) and F is either

- a linear operator of the form $F := -\mathcal{L}u(x) + f(x)$, or
- a Bellman Hamiltonian of one of the two forms

$$F \coloneqq \min_{\alpha \in A} \{ -\mathcal{L}_\alpha u(x) + f(x, \alpha) \} \quad \text{or} \quad F \coloneqq \max_{\alpha \in A} \{ -\mathcal{L}_\alpha u(x) + f(x, \alpha) \}$$

and where $\mathcal{L}$ is a diffusion operator

$$\mathcal{L}\varphi(x) := \text{trace}(a(x) D^2 \varphi(x)) + b(x) \cdot \nabla \varphi(x)$$

and $\mathcal{L}_\alpha$ is analogously defined with $b := b(x, \alpha)$ and $a := a(x, \alpha)$, and $\alpha \in A$ a compact subset of $\mathbb{R}^k$ for some $k > 0$. Such problems arise in ergodic stochastic control, weak KAM theory, homogenization, singular perturbations and asymptotic approximations in partial differential equations (long-time behavior, vanishing discount factor).

The main difficulty and novelty in this setting is that we are looking for solutions in the whole space $\mathbb{R}^m$ while both b and f are unbounded. An assumption which will play an important role in the sequel is the *Recurrence condition* (see [141]) which writes as

$$\lim_{|x| \to \infty} \sup_{\alpha \in A} b(x, \alpha) \cdot x = -\infty.$$

Usually, we refer to c as the *ergodic constant* and $u(\cdot)$ as the *corrector*. The differential operator $\mathcal{L}_\alpha$ can be interpreted as the infinitesimal generator of the controlled stochastic process

$$dX_t = b(X_t, \alpha_t) dt + \sqrt{2} \varrho(X_t, \alpha_t) dB_t$$

where ϱ is such that $a = \varrho \varrho^\top$ and B_t is a Wiener process. Similarly, the operator $\mathcal{L}$ would correspond to the same stochastic process where we drop the dependency on the parameter α. Note that the latter SDE should be understood in its weak sense (see e.g. [114, 115]).

We will also provide an analogous result in the manifold setting, and derive an estimate on the difference of two ergodic constants corresponding to two ergodic HJB equations.

The usual method. As we did in the previous chapters, the ergodic problem is studied as being a limiting problem of either the long-time behavior ($t \to +\infty$) of parabolic equations

$$\partial_t \omega + H(x, D\omega, D^2\omega) = 0, \quad \text{in } (0, +\infty) \times \mathbb{R}^m$$

or to vanishing-discount coefficient ($\delta \to 0$) in elliptic equations

$$\delta\omega + H(x, D\omega, D^2\omega) = 0, \quad \text{in } \mathbb{R}^m$$

The main questions then are the study of the limits $\lim\limits_{t\to+\infty} \frac{1}{t}\omega(x,t)$ (or $\lim\limits_{\delta\to 0} \delta\omega(x)$) and $\lim\limits_{t\to+\infty} \omega(x,t) - \omega(x_\circ, t)$ (or $\lim\limits_{\delta\to 0} \omega(x) - \omega(x_\circ)$) for some fixed $x_\circ$. In our setting these limits (in time or in the discount factor) are hard to obtain and remain, to our knowledge, an open question. However these methods are extremely powerful and provide a better insight on the problem (and motivates where the ergodic problem comes from). We refer to [4] (and references therein) for many more details on the latter.

Our method in chapter 4 relies on duality tools together with the extension of the diffusion operator $\mathcal{L}$. The idea is to isolate the two terms c and f which make the PDE in the problem in question difficult to solve and consider them as (part of) objective functions in suitable optimization problems which are dual to each other. Then we interpret a solution $(c, u(\cdot))$ as a Lagrange multiplier of an optimization problem over the space of measures μ with admissible set the measures solving $\mathcal{L}^*\mu = 0$. Provided we can solve the latter equation, which is in fact a stationary *Fokker-Planck-Kolmogorov* equation, we can describe the admissible set of the optimization problem and hence recover existence and uniqueness of its corresponding dual variables i.e. the Lagrange multipliers, which turn out to be the solution of the ergodic equation.
In fact, this method allows us to transpose to such a problem the information one can get from the study of the operator $\mathcal{L}$ and its adjoint $\mathcal{L}^*$ through a duality scheme for suitably chosen optimization problems.

This optimization view point is not totally new since it is briefly mentioned in [7, §6.6] and is also reminiscent of [77]. However, to our knowledge, this analysis has never been used to address a PDE problem such as the solvability of an ergodic HJB equation in our setting. Another interesting direction is the one considered in [5] where the problem of uniqueness of solutions to viscous HJB is addressed via similar duality methods, unlike in our manuscript where we use duality to prove existence only and rely rather on Liouville type results in [22] to prove uniqueness. We would like also to mention that our method allows to deal with the ergodic HJB equation under weak regularity assumptions, in particular the dependency on the space variable is assumed to be measurable only and with a subexponential growth. Moreover our assumptions concern the coefficients of the diffusion operator (or the underlying stochastic differential equation) which is a way of presentation that is different from the classical references (amongst the abovementioned) that rather rely on structural assumptions on the Hamiltonian. Finally, we show how the method also applies for HJB equations on non-compact Riemannian manifolds, and we extend the results to deal with ergodic Mean-Field Games in the same setting.

Main contributions of chapter 4

Our **first main result** concerns ergodic HJB equation when F is defined as before, and can be informally stated as follows:

(i) (Existence) There exists a constant $c_\circ \in \mathbb{R}$ such that the PDE

$$F(x, \nabla u(x), D^2u(x)) = c_\circ$$

admits an almost everywhere solution $u(\cdot) \in W^{r,2}_{loc}(\mathbb{R}^m)$ with $r \in [1, +\infty)$ and satisfying $|u(x)| \leq K(1 + |x|^\kappa)$ where $K > 0$ and $\kappa \geq 1$ that depends on the data.

(ii) (Uniqueness) If we assume moreover that b is locally Lipschitz continuous with at most a linear growth, then $u(\cdot)$ is unique in $W^{r,2}_{loc}(\mathbb{R}^m)$ with $r > \frac{m}{2}$, up to an additive constant. That is, if $(c_\circ, u(\cdot))$ and $(c_\circ, v(\cdot))$ are two solutions in the sense of (i), then $u(\cdot) - v(\cdot)$ is a constant.

Moreover, the result of (i) is valid when we replace $\mathbb{R}^m$ with a non-compact complete and connected smooth Riemannian manifold of dimension $m \geq 2$.

Our **second main result** concerns Mean-Field Games and can be informally stated as follows:
There exists $(c_\circ, u_\circ, q_\circ) \in \mathbb{R} \times W^{r,2}_{loc}(\mathbb{R}^m) \times W^{s,1}_{loc}(\mathbb{R}^m)$ for any $r \in [1, +\infty)$, $s > m$, and there exists a measurable function $\alpha_\circ(\cdot) : \mathbb{R}^m \to A$ that solve the coupled system

$$\begin{cases} -\,trace\big(a(x, \alpha_\circ(x))D^2u_\circ(x)\big) - b(x, \alpha_\circ(x)) \cdot \nabla u_\circ(x) + f(x, \alpha_\circ(x), q_\circ) = c_\circ \\ -\,trace\big(D^2(a(x, \alpha_\circ(x))q_\circ(x))\big) + div\big(b(x, \alpha_\circ)q_\circ(x)\big) = 0, \qquad a.e.\ in\ \mathbb{R}^m \end{cases}$$

and such that

(i) the constant $c_\circ$ *is defined by* $c_\circ = \int_{\mathbb{R}^m} f(x, \alpha_\circ(x), q_\circ)\, dq_\circ(x)$,

(ii) the function $\alpha_\circ(\cdot)$ *satisfies*

$$\alpha_\circ(x) \in \operatorname*{argmin}_{\alpha \in A} \{-\mathcal{L}_\alpha u_\circ(x) + f(x, \alpha, q_\circ)\}, \quad \textit{a.e.} \;\; x \in \mathbb{R}^m.$$

We will also discuss uniqueness of the solution, show that $c_\circ$ is the critical value (at least in the case of ergodic HJB equation) and conclude with further possible extensions and some open problems.

Chapter 1

Periodic homogenization of deterministic control problems

1.1 Introduction and problem setting

Consider a dynamics in $\mathbb{R}^N$ ($N \in \mathbb{N}\backslash\{0\}$) of the following type:

$$\begin{cases} \dot{x}_1(t) = f_1\left(x(t), \frac{x_2(t)}{\varepsilon}, \alpha_1(t), \alpha_2(t)\right) \\ \dot{x}_2(t) = f_2\left(x(t), \frac{x_2(t)}{\varepsilon}, \alpha_2(t)\right) \\ x(0) = x. \end{cases} \tag{1.1.1}$$

The state variable is $x = (x_1, x_2)$, where $x_1 \in \mathbb{R}^{N_1}$ and $x_2 \in \mathbb{R}^{N_2}$, with $N_1 + N_2 = N$ ($N_1, N_2 \in \mathbb{N}\backslash\{0\}$). The controls (α_1, α_2) belong to the space $\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2$ of measurable functions defined on $[0, +\infty)$ and valued into the product space $A = A_1 \times A_2$, with both A_1 and A_2 compact metric spaces. We stress that we assume that oscillations only affect certain components of the state variable, labeled x_2, and that such components are driven by the α_2 component of the control variable only.

Together with the dynamics (1.1.1) we consider the cost functional

$$J^\varepsilon(t, x, \alpha_1, \alpha_2) := \int_0^t \ell\left(s, x(s), \frac{x_2(s)}{\varepsilon}, \alpha_1(s), \alpha_2(s)\right) \mathrm{d}s + h(x(t)). \tag{1.1.2}$$

We will assume through this chapter that:

– The functions $f_1(x, y, \alpha_1, \alpha_2) : \mathbb{R}^N \times \mathbb{R}^{N_2} \times A_1 \times A_2 \to \mathbb{R}^{N_1}$ and $f_2(x, y, \alpha_2) : \mathbb{R}^N \times \mathbb{R}^{N_2} \times A_2 \to \mathbb{R}^{N_2}$ are bounded and continuous in all their arguments and Lipschitz–continuous in (x, y) uniformly with respect to α;

– The functions $\ell(s,x,y,\alpha_1,\alpha_2)$ and $h(x)$ are bounded uniformly continuous functions respectively from $[0,+\infty)\times\mathbb{R}^N\times\mathbb{R}^{N_2}\times A_1\times A_2$ and $\mathbb{R}^N$ to $\mathbb{R}$;

– The functions f_1, f_2 and ℓ are $\mathbb{Z}^{N_2}$–periodic with respect to y, that is, for any $s\in[0,+\infty)$, any $x\in\mathbb{R}^N$ and any $(\alpha_1,\alpha_2)\in A_1\times A_2$, $\phi(s,x,\cdot+k,\alpha_1,\alpha_2)=\phi(s,x,\cdot,\alpha_1,\alpha_2)$ for any $k\in\mathbb{Z}^{N_2}$ (for $\phi=f_1,f_2,\ell$).

– The state variable labeled x_2 is defined on $\mathbb{T}^{N_2}$ the N_2-dimensional torus $\mathbb{R}^{N_2}/\mathbb{Z}^{N_2}$ via the identification:

$$\mathbb{R}^{N_2}\ni x_2=(x_2^1,\dots,x_2^{N_2})=(x_2^1+m^1,\dots,x_2^{N_2}+m^{N_2}),\quad \forall\, m=(m^1,\dots,m^{N_2})\in\mathbb{Z}^{N_2}$$

and the distance between two elements x_2 and x_2' is the one given by

$$\operatorname{dist}(x_2,x_2'):=\inf_{m\in\mathbb{Z}^{N_2}}|x_2-x_2'+m|$$

where $|\cdot|$ is the Euclidean distance. The latter induces a norm on $\mathbb{T}^{N_2}$ that we denote $|\cdot|_{_T}$ and define as $|\cdot|_{_T}:\mathbb{T}^{N_2}\to[0,\sqrt{N_2})$ such that

$$|x_2|_{_T}=\operatorname{dist}(x_2,0) \tag{1.1.3}$$

The solution $t\mapsto x_2(t)$ is then obtained by taking the one in $\mathbb{R}^{N_2}$ and projecting it down to $\mathbb{T}^{N_2}$.

The value function associated to (1.1.1), (1.1.2) is

$$v^\varepsilon(t,x):=\inf_{(\alpha_1,\alpha_2)\in\mathcal{A}} J^\varepsilon(t,x,\alpha_1,\alpha_2)\,. \tag{1.1.4}$$

It is well known (see, e.g. [19]) that under the previous assumptions v^ε solves, in viscosity sense, the Hamilton-Jacobi-Bellman equation

$$\begin{cases}\partial_t v^\varepsilon+\max\limits_{(\alpha_1,\alpha_2)\in A}\Big\{-D_{x_1}v^\varepsilon\cdot f_1\left(x,\dfrac{x_2}{\varepsilon},\alpha_1,\alpha_2\right)-D_{x_2}v^\varepsilon\cdot f_2\left(x,\dfrac{x_2}{\varepsilon},\alpha_2\right)\\ \qquad\qquad -\ell\left(t,x,\dfrac{x_2}{\varepsilon},\alpha_1,\alpha_2\right)\Big\}=0 \qquad\qquad \text{in } (0,+\infty)\times\mathbb{R}^N\\ v^\varepsilon(0,x)=h(x) \qquad\qquad\qquad\qquad\qquad\qquad \text{in } \mathbb{R}^N.\end{cases} \tag{1.1.5}$$

For further references, we denote by $H(x, y, p)$ the Hamiltonian in (1.1.5):

$$H(s, x, y, p) := \max_{(\alpha_1,\alpha_2)\in A} \Big\{ -p_1 \cdot f_1(x, y, \alpha_1, \alpha_2) - p_2 \cdot f_2(x, y, \alpha_2) - \ell(t, x, y, \alpha_1, \alpha_2) \Big\}. \tag{1.1.6}$$

The homogenization problem consists in passing to the limit as $\varepsilon \to 0^+$. In the framework of viscosity solutions of Hamilton–Jacobi equations, this type of problems have been studied extensively in the last decades and are today very well understood. The general approach consists in finding an *effective Hamiltonian* $\bar{H}(t, x, p)$ such that the sequence $v^\varepsilon(t, x)$ of solutions of (1.1.5) converges locally uniformly in $[0, +\infty) \times \mathbb{R}^N$, as ε goes to 0, to $v(t, x)$, the unique viscosity solution of

$$\begin{cases} \partial_t v + \bar{H}(t, x, Dv) = 0 & \text{in } (0, +\infty) \times \mathbb{R}^N \\ v(0, x) = h(x) & \text{in } \mathbb{R}^N. \end{cases} \tag{1.1.7}$$

The aim of this chapter is instead to find, for some classes of problems, a *limiting optimal control problem* associated to (1.1.7). This is a crucial issue in view of applications; nonetheless, to the best of our knowledge, there are few results in this direction, mostly in the engineering literature. The first approach to homogenization of control problems originated in the light of the so-called Levinson–Tichonov theory for ODEs and then extended to deterministic control problems; see e.g. [33], [73], [112]. This approach requires the limit of the fast dynamics to be governed by an algebraic equation and the stationary points of the fast dynamics to be attractive. To treat more general situations, other averaging techniques, have been developed; see [9], [10], [11], [12], [90]. These results are based on the analysis of trajectories and make use of invariant measures and occupational measure of the fast dynamics. In [155], [156], by using special measures called *limiting relaxed controls*, some of these results have been interpreted in the light of singular perturbation theory.

The approach in this chapter is based on the results obtained in the theory of homogenization of Hamilton–Jacobi equations. Our strategy consists in studying whether $\bar{H}$ can be written as a Bellman Hamiltonian, and consequently in describing the unique solution of (1.1.7) as a value function of some optimal control problem. This is indeed one of the main challenges of this chapter, since except in few cases it is hard to find explicit formulas for the $\bar{H}$ associated to an ergodic Hamiltonian. To this scope, we require the dynamics and the costs to satisfy from time to time different structure hypotheses, as well as some controllability assumptions.

We describe next the structure and main results in the chapter. In **Section 1.2** we recall how the theory of singular perturbation can be applied to periodic homogenization of Hamilton-Jacobi-Bellman equation, and we establish our first homogenization results. We prove in Theorem 1.2.1 that if the dynamics for the oscillating variables, i.e.

$$\dot{y}(t) = f_2(x, y(t), \alpha(t)), \quad y(0) = y, \quad x \text{ frozen} \tag{1.1.8}$$

is just controllable in bounded time then a general weak convergence result holds; namely, the upper and lower semilimits of v^ε are respectively a viscosity subsolution and supersolution of the limiting problem (1.1.7). If instead (1.1.8) is controllable in a stronger sense (see formula (1.3.1)) then, in **Section 1.3**, and by adapting a result in [156] we are able to represent the limiting optimal control problem as a differential inclusion obtained by averaging the vector field with limiting relaxed controls (see Theorem 1.3.1). The particular case in which f_2 is independent of the slow variable is considered in **Section 1.4.1** (see Theorem 1.4.1). A different situation is studied in **Section 1.4.2** where we assume the two components of the vector field f_1 and f_2 are driven only by α_1 and α_2 respectively. This partially decoupled structure combined with some controllability assumption on the vector field f_2 (see formula (1.4.12)) allows to represent the unique solution of the effective Cauchy problem as a value function of a dynamics for x_1 controlled by $(\alpha_1, y) \in A_1 \times \mathbb{T}^{N_2}$ (see Theorem 1.4.2). In **Section 1.3.2** we are interested in the convergence of the trajectories (1.1.1), in their singular perturbation form, as $\varepsilon \to 0$. After the convergence of the value function is proven, it is a natural question to wonder whether the singularly perturbed trajectories (1.2.1) do converge to trajectories of the form (1.3.3) with limiting relaxed controls. It is also of much interest to consider the converse of such result, that is, whether each trajectory of the form (1.3.3) with limiting relaxed controls can be indeed approximated by a sequence of singularly perturbed trajectories (1.1.1) (see Theorem 1.3.2 and Theorem 1.3.3). The interest of the latter lies in application in engineering or computer science, where it can be more profitable to consider approximating dynamics instead of the real one, provided the convergence holds true. In **Section 1.5** we consider the case in which f_2 does not depend on the control and, for any x, the dynamics $\dot{y}(t) = f_2(x, y(t))$ has a unique invariant measure μ_x. Under these assumptions we discuss a relaxation procedure to average the dynamics and the cost functional, which is rather different from that considered in Section 1.3. We prove first a weak convergence result (see Theorem 1.5.1). Then, in the case in which the unique invariant measure is independent on the slow state, we establish in **Section 1.5.2** local uniform convergence of the value function as well as a representation for the limiting control problem (see Corollary 1.5.1). And in the same vein as in Section 1.3.2, the convergence of trajectories holds true again in this context. A simplified expression for the limiting dynamics is provided in **Section 1.5.3** where a decoupling assumption is made on f_1 and the running cost. In agreement with our results in Section 1.3, we represent the unique solution of the effective Cauchy problem as a value function of a dynamics obtained by averaging the original vector field f and the running cost ℓ with the unique invariant measure (see Corollary 1.5.4). We conclude this chapter by an appendix in **Section 1.6** where we provide the proof of upper-semicontinuity of the set-valued map of occupational measures (limiting relaxed controls), and where we also study an example in which the effective Hamiltonian is not regular enough to guarantee the uniform convergence. The example consists of an homogenization problem which is ergodic, but whose value function does not converge uniformly on compact sets; it is obtained by slightly modifying a counterexample to homogenization presented in [4] and is useful to check some of the assumptions made to prove our results. Let us finally mention that similar results can be extended to the stochastic case and will be tackled in **Chapter 2**.

1.2 Mathematical background

1.2.1 Homogenization as a singular perturbation

The approach in this chapter is based on the results obtained in the theory of homogenization and singular perturbation of Hamilton–Jacobi equations. In fact, our homogenization problem can be seen as a singular perturbation problem in $\mathbb{R}^N \times \mathbb{R}^{N_2}$ by introducing the fast variable $y = x_2/\varepsilon$. Then, the dynamics (1.1.1) becomes

$$\begin{cases} \dot{x}_1(t) = f_1\left(x(t), y(t), \alpha_1(t), \alpha_2(t)\right) \\ \dot{x}_2(t) = f_2\left(x(t), y(t), \alpha_2(t)\right) \\ \dot{y}(t) = \frac{1}{\varepsilon} f_2\left(x(t), y(t), \alpha_2(t)\right) \\ x(0) = x, \quad y(0) = y \end{cases} \tag{1.2.1}$$

and the cost functional to be considered is

$$J\left(t, x, y, \alpha_1, \alpha_2\right) := \int_0^t \ell\left(s, x(s), y(s), \alpha_1(s), \alpha_2(s)\right) \mathrm{d}s + h(x(t)).$$

If we look for a solution of (1.1.5) of the form $v^\varepsilon(t,x) = u^\varepsilon(t,x,x_2/\varepsilon)$, then $u^\varepsilon(t,x,y)$ solves

$$\begin{cases} \partial_t u^\varepsilon + H\left(t,x,y,D_{x_1}u^\varepsilon,\left(D_{x_2}u^\varepsilon + \frac{1}{\varepsilon}D_y u^\varepsilon\right)\right) = 0 & \text{in } (0,+\infty)\times\mathbb{R}^N\times\mathbb{R}^{N_2} \\ u^\varepsilon(0,x,y) = h(x) & \text{in } \mathbb{R}^N\times\mathbb{R}^{N_2}, \end{cases} \tag{1.2.2}$$

where H is defined in (1.1.6). The value function

$$u^\varepsilon(t,x,y) = \inf_{(\alpha_1,\alpha_2)\in\mathcal{A}} J(t,x,y,\alpha_1,\alpha_2)$$

for the control system (1.2.1) is the unique viscosity solution of (1.2.2).

The definition of the effective Hamiltonian $\bar{H}$ is related with a property of H, called *ergodicity*, extensively studied in [3], [4]. For completeness and further references, we recall briefly its definition. Consider, for fixed $\bar{t}\in(0,+\infty)$ and $\bar{x},\bar{p}\in\mathbb{R}^N$, the *cell problem*:

$$\lambda + H\left(\bar{t},\bar{x},y,\bar{p}_1,\bar{p}_2 + D\chi\right) = 0, \qquad \chi \text{ periodic.} \tag{1.2.3}$$

It is well known that there exists at most one value $\lambda\in\mathbb{R}$ such that (1.2.3) has a continuous viscosity solution χ; see, e.g. [127], [78], [79]. In general, there may be no pairs (λ,χ) with χ continuous solving (1.2.3). We say that H is *ergodic* at $(\bar{t},\bar{x},\bar{p})$ if

$$\sup\left\{\lambda \,\middle|\, (1.2.3) \text{ has a subsolution}\right\} = \inf\left\{\lambda \,\middle|\, (1.2.3) \text{ has a supersolution}\right\} =: \bar{\lambda}.$$

In this case, we set $\bar{H}(\bar{t},\bar{x},\bar{p}) = -\bar{\lambda}$. Equivalent definitions of ergodicity, based on alternative formulations of the cell problem will be considered in the sequel.

Ergodicity is, in general, not sufficient for homogenization, in fact the effective Hamiltonian $\bar{H}$ will be only continuous; consequently the comparison principle for the effective Cauchy problem (1.1.7) and then the uniform convergence of u^ε may not hold. In general, only a *weak convergence result* can be shown; see [4, Proposition 2.6] and [3, Theorem 1]:

Proposition 1.2.1. *Let u^ε be the solution of* (1.2.2)*. Assume that the Hamiltonian H defined in* (1.1.6) *is ergodic. Then the upper semilimit of u^ε, defined as*

$$\bar{u}^*(t,x) := \limsup_{\substack{(t',x')\to(t,x)\\ \varepsilon\to 0}} \sup_y u^\varepsilon(t',x',y), \quad \text{if } t>0$$

$$\bar{u}^*(0,x) := \limsup_{\substack{(t',x')\to(0,x)\\ t'>0}} \bar{u}^*(t,x), \quad \text{if } t=0,$$

(resp. the lower semilimit $\underline{u}_*$*, defined replacing* lim sup *with* lim inf *and* sup *with* inf*) is a viscosity subsolution (resp. supersolution) of the effective problem* (1.1.7)*.*

Under extra assumptions, more regularity with respect to x can be established for $\bar{H}$ and then uniform convergence of u^ε to u, unique solution of (1.1.7) can be proved.

Remark 1.2.1. *The semilimits of* $v^\varepsilon(t,x)$ *are defined in a slightly different way: the upper semilimit of* v^ε *is*

$$\begin{aligned} \bar{v}^*(t,x) &:= \limsup_{\substack{(t',x')\to(t,x)\\ \varepsilon\to 0}} v^\varepsilon(t',x'), \quad \text{if } t>0 \\ \bar{v}^*(0,x) &:= \limsup_{\substack{(t',x')\to(0,x)\\ t'>0}} \bar{v}^*(t,x), \quad \text{if } t=0; \end{aligned}$$

the lower semilimit $\underline{v}_*$ *is defined analogously, with* lim inf *in place of* lim sup*. If we look for a solution of* (1.1.5) *of the form* $v^\varepsilon(t,x) = u^\varepsilon(t,x,x_2/\varepsilon)$*, then*

$$\begin{aligned} \underline{v}_*(t,x) &= \liminf_{\substack{(t',x')\to(t,x)\\ \varepsilon\to 0}} v^\varepsilon(t',x') \\ &= \liminf_{\substack{(t',x')\to(t,x)\\ \varepsilon\to 0}} u^\varepsilon\left(t',x',\frac{x_2'}{\varepsilon}\right) \geq \liminf_{\substack{(t',x')\to(t,x)\\ \varepsilon\to 0}} \inf_y u^\varepsilon(t',x',y) = \underline{u}_*(t,x) \end{aligned}$$

and similarly $\bar{v}^* \leq \bar{u}^*$*. Moreover, it has been proved (see [4, Theorem 2.7]) that there exist a minimal (l.s.c.) subsolution* $u_\sharp$ *and a maximal (u.s.c.) supersolution* $u^\sharp$ *of* (1.1.7)*. We conclude:*

$$u_\sharp \leq \underline{u}_* \leq \underline{v}_* \leq \bar{v}^* \leq \bar{u}^* \leq u^\sharp \quad \text{in } [0,+\infty)\times\mathbb{R}^N. \tag{1.2.4}$$

In particular, if $u^\varepsilon(t,x,y)$ *converges locally uniformly in* $(0,+\infty)\times\mathbb{R}^N\times\mathbb{R}^{N_2}$ *to a function* $u(t,x)$ *then* $\underline{u}_* = \bar{u}^* = u$ *in* $[0,+\infty)\times\mathbb{R}^N$*. Since* u^ε *is periodic in* y*, then also* $v^\varepsilon(t,x)$ *converges uniformly to the same function* u*.*

For any fixed $x \in \mathbb{R}^N$ the *fast subsystem* of (1.2.1) is

$$\dot{y}(t) = f_2\left(x, y(t), \alpha_2(t)\right), \qquad y(0) = y. \tag{1.2.5}$$

Definition 1.2.1. *The system* (1.2.5) *is said bounded time controllable if there exists* $T>0$ *such that, for any* $y, y' \in \mathbb{R}^{N_2}$ *there exists a control* $a_2 \in \mathcal{A}_2$ *such that the corresponding solution of* (1.2.5) *satisfies* $y(t') = y'$ *for some* $t' \leq T$*.*

In the following Lemma we recall for later use two important consequences of bounded time controllability. The first is a sufficient condition for ergodicity; the second is that the effective Hamiltonian can be represented as an optimal average cost of an ergodic control problem for the fast subsystem.

Lemma 1.2.1. *Let* $\bar{t} \in (0, +\infty)$ *and* $\bar{x}, \bar{p} = (\bar{p}_1, \bar{p}_2) \in \mathbb{R}^N$. *Assume that* (1.2.5) *is bounded time controllable for* $x = \bar{x}$. *Then the Hamiltonian* H *defined in* (1.1.6) *is ergodic at* $(\bar{t}, \bar{x}, \bar{p})$. *Moreover*

$$\bar{H}(\bar{t}, \bar{x}, \bar{p}) = \sup_{\substack{\alpha_1 \in A_1 \\ \alpha_2 \in \mathcal{A}_2}} \limsup_{t \to +\infty} \left\{ \frac{1}{t} \int_0^t -\bar{p}_1 \cdot f_1\big(\bar{x}, y(s), \alpha_1, \alpha_2(s)\big) \right.$$
$$\left. - \bar{p}_2 \cdot f_2(\bar{x}, y(s), \alpha_2(s)) - \ell\big(\bar{t}, \bar{x}, y(s), \alpha_1, \alpha_2(s)\big) \mathrm{d}s \,\middle|\, y(\cdot) \text{ solves } (1.2.5) \right\}. \quad (1.2.6)$$

Proof. The statement of this Lemma is a particular case of [4, Theorem 6.1]. Consider the δ-cell problem

$$\delta w_\delta + H\big(\bar{t}, \bar{x}, y, \bar{p}_1, \bar{p}_2 + D_y w_\delta\big) = 0 \quad \text{in } \mathbb{R}^{N_2}, \qquad w_\delta \text{ periodic.} \quad (1.2.7)$$

For fixed $y, y' \in \mathbb{R}^{N_2}$, the assumed bounded time controllability and the Dynamic Programming Principle for w_δ imply the estimate $\delta w_\delta(y) - \delta w_\delta(y') \le C(1 - e^{-\delta T})$ for some $C > 0$ independent of δ and T as in Definition 1.2.1. Then,

$$\lim_{\delta \to 0^+} |\delta w_\delta(y) - \delta w_\delta(y')| = 0 \quad \text{uniformly in } y, y' \in \mathbb{R}^{N_2}.$$

This in turn implies that $\{w_\delta - w_\delta(0)\}$ is an equicontinuous and equibounded net. Therefore, up to subsequence, $w_\delta - w_\delta(0)$ converges uniformly to a limit χ. By stability, χ solves the true cell problem (1.2.3) with $\lambda = \bar{H}(\bar{t}, \bar{x}, \bar{p}_1, \bar{p}_2)$. Then H is ergodic at $(\bar{t}, \bar{x}, \bar{p}_1, \bar{p}_2)$ and

$$\delta w_\delta(y) \to -\bar{H}(\bar{t}, \bar{x}, \bar{p}_1, \bar{p}_2) \quad \text{as } \delta \to 0, \text{ uniformly in } y.$$

Thus, formula (1.2.6) can be proved as Proposition VII.1.3 in [19]. $\square$

1.2.2 A weak convergence result for HJB equation

Let us denote by $\mathbb{T}^{N_2}$ the N_2-dimensional torus $\mathbb{R}^{N_2}/\mathbb{Z}^{N_2}$ and by $\mathcal{P}(\mathbb{T}^{N_2} \times A_2)$ the set of Radon probability measures on $\mathbb{T}^{N_2} \times A_2$. For any $t > 0$, $y \in \mathbb{R}^{N_2}$ and $\alpha_2 \in \mathcal{A}_2$ consider the solution $y(s)$ of (1.2.5), and define a measure in $\mathcal{P}(\mathbb{T}^{N_2} \times A_2)$, called occupational measure, as

$$\mu_{t,x,y,\alpha_2} := \frac{1}{t}\int_0^t \delta_{(y(s),\alpha_2(s))}\mathrm{d}s,$$

where δ is the Dirac's delta.

We denote by $Z(x)$ the set of weak star limits of these measures, i.e. the set of measures $\mu \in \mathcal{P}(\mathbb{T}^{N_2} \times A_2)$ such that

$$\mu = \lim_{n\to\infty} \mu_{t_n,x,y,\alpha_2} \quad \text{weak-star}$$

for some $t_n \to +\infty$, $\alpha_2 \in \mathcal{A}_2$ and $y \in \mathbb{R}^{N_2}$. A measure in $Z(x)$ is called limiting occupational measure, or limiting relaxed control. See [2], [90], [156] for details.

Proposition 1.2.2. *If the system* (1.2.5) *is bounded time controllable then* $Z(x)$ *is a non-empty convex and compact subset of* $\mathcal{P}(\mathbb{T}^{N_2} \times A_2)$ *in the sense of weak star topology.*

Proof. See [156, Theorem 1.3]. □

Moreover, we have the following lemma whose proof is postponed to the appendix §1.6.

Lemma 1.2.2. *If the system* (1.2.5) *is bounded time controllable, then the multifunction* $x \rightsquigarrow Z(x)$ *is upper semicontinuous*[1]*. This holds in particular under the stronger controllability assumption* (1.3.1).

For any $x \in \mathbb{R}^N$, any $\mu \in Z(x)$ and any $\alpha_1 \in A_1$ we define the averaged vector field and running cost as follows:

$$\begin{aligned}
\bar{f}_1(x,\alpha_1,\mu) &:= \int_{\mathbb{R}^{N_2}\times A_2} f_1\big(x,y,\alpha_1,\alpha_2\big)\mathrm{d}\mu(y,\alpha_2);\\
\bar{f}_2(x,\mu) &:= \int_{\mathbb{R}^{N_2}\times A_2} f_2(x,y,\alpha_2)\mathrm{d}\mu(y,\alpha_2);\\
\bar{\ell}(s,x,\alpha_1,\mu) &:= \int_{\mathbb{R}^{N_2}\times A_2} \ell\big(s,x,y,\alpha_1,\alpha_2\big)\mathrm{d}\mu(y,\alpha_2).
\end{aligned}$$

The functions above inherit boundedness and uniform continuity from f_1, f_2 and ℓ respectively. Moreover $\bar{f}_1$ and $\bar{f}_2$ are Lipschitz–continuous in x uniformly with respect to (α_1,μ) and μ respectively; see [19].

[1]**Definition** (see [15, Definition 1.1.1]) A set valued map $\mathcal{F}$ from X to Y, Hausdorff topological spaces, is said to be *upper semicontinuous* at $x_\circ \in X$ if for any open N containing $\mathcal{F}(x_\circ)$, there exists a neighborhood M of $x_\circ$ such that $\mathcal{F}(M) \subset N$. We say that $\mathcal{F}$ is upper semicontinuous if it is so at every $x_\circ \in X$.

Take $t_n \to +\infty$, $\alpha_2 \in \mathcal{A}_2$ and $y \in \mathbb{T}^{N_2}$ as initial state for (1.2.5) and let $y(\cdot)$ be the corresponding solution. Observe that,

$$\begin{aligned}\bar{f}_2\left(x, \frac{1}{t_n}\int_0^{t_n} \delta_{(y(s),\alpha_2(s))}\mathrm{d}s\right) &= \frac{1}{t_n}\int_0^{t_n} \bar{f}_2(x, \delta_{(y(s),\alpha_2(s))})\mathrm{d}s \\ &= \frac{1}{t_n}\int_0^{t_n} f_2(x, y(s), \alpha_2(s))\mathrm{d}s = \frac{1}{t_n}(y(t_n) - y(0)) \quad (1.2.8)\end{aligned}$$

$y(t_n)$ being the projection on $\mathbb{T}^{N_2}$ of the solution to (1.2.5), and using the norm defined in (1.1.3), we have

$$\frac{1}{t_n}|y(t_n) - y(0)|_T \leq \frac{\sqrt{N_2}}{t_n}$$

which yields, after sending $n \to +\infty$,

$$\bar{f}_2(x, \mu) = 0 \qquad \text{for any } x \in \mathbb{R}^N,\ \mu \in Z(x). \tag{1.2.9}$$

In the following Theorem we establish a weak convergence result for the semilimits of solutions of the associated singular perturbation problem (1.2.2). It also provides an explicit expression for the effective Hamiltonian in terms of the averaged vector field and running cost.

Theorem 1.2.1. *Let $u^\varepsilon(t, x, y)$ be a solution of (1.2.2). If the fast subsystem (1.2.5) is bounded time controllable then the upper and lower semilimit of u^ε are respectively a supersolution and a subsolution of*

$$\begin{cases} \partial_t v + \max\limits_{\substack{\mu \in Z(x) \\ \alpha_1 \in A_1}} \left\{-D_{x_1}v \cdot \bar{f}_1(x, \alpha_1, \mu) - \bar{\ell}(t, x, \alpha_1, \mu)\right\} = 0 & \text{in } (0, +\infty) \times \mathbb{R}^N \\ v(0, x) = h(x) & \text{in } \mathbb{R}^N. \end{cases} \tag{1.2.10}$$

Proof. Put

$$F\big(t, x, y, p_1, p_2, \alpha_1, \alpha_2\big) := -p_1 \cdot f_1\big(x, y, \alpha_1, \alpha_2\big) - p_2 \cdot f_2(x, y, \alpha_2) - \ell\big(t, x, y, \alpha_1, \alpha_2\big).$$

Then, according with (1.1.6), we write

$$H(t, x, y, p) = \max_{(\alpha_1,\alpha_2)\in A} F\left(t, x, y, p, \alpha_1, \alpha_2\right).$$

Fix $\bar{t} \in (0, +\infty)$ and $\bar{x}, \bar{p} = (\bar{p}_1, \bar{p}_2) \in \mathbb{R}^N$. By Lemma 1.2.1 H is ergodic at $(\bar{t}, \bar{x}, \bar{p})$. This permits to uniquely define the value $\bar{H}(\bar{t}, \bar{x}, \bar{p})$ for the effective Hamiltonian $\bar{H}$: $(0, +\infty) \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$. By Proposition 1.2.1 the semilimits $\bar{u}^*$ and $\underline{u}_*$ are respectively a viscosity subsolutions and supersolution of

$$\begin{cases} \partial_t v + \bar{H}\left(t, x, D_x v\right) = 0 & \text{in } (0, +\infty) \times \mathbb{R}^N \\ v(0, x) = h(x) & \text{in } \mathbb{R}^N. \end{cases}$$

It remains to prove that

$$\bar{H}(t, x, p) = \bar{H}(t, x, p_1) = \max_{\substack{\mu \in Z(x) \\ \alpha_1 \in A_1}} \left\{ -p_1 \cdot \bar{f}_1(x, \alpha_1, \mu) - \bar{\ell}(t, x, \alpha_1, \mu) \right\}. \tag{1.2.11}$$

We adapt the argument of [2, Theorem 7]. Let us denote by $\tilde{H}(t, x, p_1)$ the right hand side of (1.2.11) and set

$$\tilde{F}(t, x, p, \alpha_1, \mu) := -p_1 \cdot \bar{f}_1(x, \alpha_1, \mu) - \bar{\ell}(t, x, \alpha_1, \mu);$$

thus

$$\tilde{H}(t, x, p_1) = \max_{\substack{\mu \in Z(x) \\ \alpha_1 \in A_1}} \tilde{F}(t, x, p, \alpha_1, \mu). \tag{1.2.12}$$

Finally set

$$\bar{F}_{\alpha_1}(t, x, p, \alpha_2) := \limsup_{T \to +\infty} \left\{ \frac{1}{T} \int_0^T F\big(t, x, y(s), p, \alpha_1, \alpha_2(s)\big) \mathrm{d}s \,\Big|\, y(\cdot) \text{ solves } (1.2.5) \right\}.$$

By (1.2.6),

$$\bar{H}(t, x, p) = \sup_{\substack{\alpha_1 \in A_1 \\ \alpha_2 \in \mathcal{A}_2}} \bar{F}_{\alpha_1}(x, p, \alpha_2). \tag{1.2.13}$$

Any $\bar{\mu} \in Z(x)$ is generated by some $\bar{y}(\cdot)$, solution of (1.2.5) corresponding to a certain $\bar{\alpha}_2 \in \mathcal{A}_2$, and some $t_n \to +\infty$. We compute, as in (1.2.8),

$$\begin{aligned} \frac{1}{t_n} \int_0^{t_n} F\big(t, x, \bar{y}(s), p, \alpha_1, \bar{\alpha}_2(s)\big) \mathrm{d}s &= \frac{1}{t_n} \int_0^{t_n} \tilde{F}\big(t, x, p_1, \alpha_1, \delta_{(\bar{y}(s), \bar{\alpha}_2(s))}\big) \mathrm{d}s \\ &= \tilde{F}\left(t, x, p_1, \alpha_1, \frac{1}{t_n} \int_0^{t_n} \delta_{(\bar{y}(s), \bar{\alpha}_2(s))} \mathrm{d}s\right). \end{aligned} \tag{1.2.14}$$

The right-hand side of the previous identity converges to $\tilde{F}(t, x, p_1, \alpha_1, \bar{\mu})$ as $n \to \infty$. Then, taking into account (1.2.13) we discover

$$\begin{aligned}\bar{H}(t,x,p) &= \sup_{\substack{\alpha_1\in A_1\\ \alpha_2\in\mathcal{A}_2}} \bar{F}_{\alpha_1}(t,x,p,\alpha_2) \geq \bar{F}_{\alpha_1}(t,x,p,\bar{\alpha}_2)\\ &\qquad\qquad \geq \lim_n \frac{1}{t_n}\int_0^{t_n} F\big(t,x,\bar{y}(s),p,\alpha_1,\bar{\alpha}_2(s)\big)\mathrm{d}s = \tilde{F}(t,x,p_1,\alpha_1,\bar{\mu}).\end{aligned}$$

The previous inequality is valid for any $\alpha_1 \in A_1$ and any $\bar{\mu} \in Z(x)$, then taking the supremum over these variables and recalling (1.2.12) we get $\bar{H}(t,x,p) \geq \tilde{H}(t,x,p_1)$.

We now establish the opposite inequality. Fix $\bar{\alpha}_1 \in A_1$, an initial state $y \in \mathbb{R}^{N_2}$, and a control $\bar{\alpha}_2 \in \mathcal{A}_2$ and consider the corresponding solution $\bar{y}(\cdot)$ of (1.2.5). Take also a sequence t_n diverging as n goes to $+\infty$, such that

$$\bar{F}_{\bar{\alpha}_1}(t,x,p,\bar{\alpha}_2) = \lim_n \frac{1}{t_n}\int_0^{t_n} F(t,x,\bar{y}(s),p,\bar{\alpha}_1,\bar{\alpha}_2(s))\mathrm{d}s.$$

The support of the sequence of occupational measures $\frac{1}{t_n}\int_0^{t_n}\delta_{(\bar{y}(s),\bar{\alpha}_2(s))}\mathrm{d}s$ is uniformly bounded, thanks to the periodicity assumption. Then it converges up to subsequences as $n\to\infty$, to some $\bar{\mu}\in Z(x)$. Thus, by (1.2.14),

$$\tilde{H}(t,x,p_1) \geq \tilde{F}(t,x,p_1,\bar{\alpha}_1,\bar{\mu}) = \bar{F}_{\bar{\alpha}_1}(t,x,p,\bar{\alpha}_2).$$

The previous inequality is valid for any $\bar{\alpha}_1 \in A_1$ and any $\bar{\alpha}_2 \in \mathcal{A}_2$. Then, by taking the supremum over these variables in both sides of the previous inequality and using (1.2.13) we conclude that $\tilde{H}(t,x,p_1) \geq \bar{H}(t,x,p)$. Formula (1.2.11) is then established and the proof is completed. □

1.3 Main results: Limiting Relaxed Control Problems

1.3.1 Convergence under a strong controllability condition

The following result provides a representation of the limiting dynamics in terms of limiting relaxed controls. It requires a stronger controllability assumption on the fast variables.

Theorem 1.3.1. *Assume that*

$$\begin{gathered}\textit{for any } x \textit{ there exists } \nu(x)>0 \textit{ s.t.}\\ B(0,\nu(x)) \subseteq \overline{\mathrm{co}} f_2(x,y,A_2) \textit{ for any } y.\end{gathered} \tag{1.3.1}$$

Then, as $\varepsilon \to 0$*, the sequence* $v^\varepsilon(t,x)$ *of solutions of* (1.1.5) *converges locally uniformly on* $(0,+\infty)\times\mathbb{R}^N$ *to*

$$\bar{v}(t,x) := \inf\left\{\int_0^t \bar{\ell}(s,x(s),\alpha_1(s),\mu(s))\mathrm{d}s + h(x(t))\right\}, \tag{1.3.2}$$

where the infimum is taken over trajectories $x(\cdot)$ *satisfying*

$$\begin{cases} \dot{x}_1(t) = \bar{f}_1\left(x(t),\alpha_1(t),\mu(t)\right) \\ \dot{x}_2(t) = 0 \\ \alpha_1 \in \mathcal{A}_1, \quad \mu(t) \in Z(x(t)) \\ x(0) = x. \end{cases} \tag{1.3.3}$$

Proof. By Theorem 1.2.1 the effective Hamiltonian is given by

$$\bar{H}(t,x,p_1) = \max_{\substack{\mu\in Z(x)\\ \alpha_1\in A_1}}\left\{-p_1\cdot\bar{f}_1(x,\alpha_1,\mu) - \bar{\ell}(t,x,\alpha_1,\mu)\right\}.$$

It is well known (see, e.g. [152, Corollary 3.7]) that under assumption (1.3.1) the fast subsystem (1.2.5) is small-time controllable, namely, the time t' in Definition 1.2.1 to reach y' with a trajectory of (1.2.5) starting at y can be bounded by $C|y-y'|/\nu(x)$. This implies that the true cell problem (1.2.3) is solved by a Lipschitz continuous corrector; then H is ergodic and $\bar{H}$ is Lipschitz continuous.

If we look for a solution of (1.1.5) of the form $v^\varepsilon(t,x) = u^\varepsilon(t,x,x_2/\varepsilon)$, then $u^\varepsilon(t,x,y)$ solves (1.2.2). By Theorem 1.2.1 the upper and lower semilimits of u^ε are respectively a viscosity subsolution and supersolution of (1.2.10). Moreover, since $\bar{H}(t,x,p)$ is Lipschitz continuous, the comparison principle holds for problem (1.2.10) and u^ε converges locally uniformly as $\varepsilon\to 0$ to $v(t,x)$, unique continuous viscosity solution of (1.2.10); then by Remark 1.2.1 v^ε also converges locally uniformly on $(0,+\infty)\times\mathbb{R}^N$ to the same function.

To prove that $v \equiv \bar{v}$, one has to show that $\bar{v}(t,x)$ solves the effective problem (1.2.10). This can be done, as in [156], by proving that the limiting dynamics (1.3.3) admits trajectories defined for any $t>0$, that the value function $\bar{v}(t,x)$ is continuous in $[0,+\infty)\times\mathbb{R}^N$ and that the set-valued map

$$x \mapsto \left\{\bar{f}_1(x,\alpha_1,\mu)\,\middle|\,\alpha_1\in A_1,\ \mu\in Z(x)\right\},$$

is Lipschitz continuous. □

Remark 1.3.1. *If the hypothesis* (1.3.1) *fails than homogenization may not hold. To see this, consider the example of Section 1.6.2, in which* $f_2(x, y, \alpha_2) = x_1 + \alpha_2$ *and* α_2 *varies in the interval* $A_2 \equiv [-1, 1]$*; then*

$$\overline{\text{co}} f_2(x, y, A_2) = [x_1 - 1, x_1 + 1] \quad \textit{for any } y.$$

If $|x_1| \geq 1$ *the condition* (1.3.1) *is not satisfied; we know, by Proposition 1.6.1, that there may be no homogenization in this case. If instead* $|x_1| < 1$*, the hypothesis* (1.3.1) *is readily verified and according with Proposition 1.6.1 homogenization holds. Notice that the Hamiltonian defined in* (1.6.5) *is coercive in this case and, using* (1.6.8)*, we can easily compute*

$$\bar{H}(x_1, p_1) = \sup\{-p_1 v \,|\, v \in [0, 2]\} = 2p_1^-, \tag{1.3.4}$$

which is Lipschitz continuous.

1.3.2 Convergence of singularly perturbed trajectories

We start with the general case considered for the convergence of the value function, where we assume the strong controllability condition (1.3.1) to be satisfied. We recall for the sake of clarity and convenience of the reader the system (1.2.1) which we will refer to as *the perturbed dynamics*

$$\begin{cases} \dot{x}_1(t) = f_1\left(x(t), y(t), \alpha_1(t), \alpha_2(t)\right) \\ \dot{x}_2(t) = f_2\left(x(t), y(t), \alpha_2(t)\right) \\ \dot{y}(t) = \frac{1}{\varepsilon} f_2\left(x(t), y(t), \alpha_2(t)\right) \\ x(0) = x, \quad y(0) = y \end{cases} \tag{1.3.5}$$

and also the system (1.3.3) which we will refer to as *the relaxed dynamics*

$$\begin{cases} \dot{x}_1(t) = \bar{f}_1\left(x(t), \alpha_1(t), \mu(t)\right) \\ \dot{x}_2(t) = 0 \\ \alpha_1 \in \mathcal{A}_1, \quad \mu(t) \in Z(x(t)) \\ x(0) = x \end{cases} \tag{1.3.6}$$

where the second equation is a consequence of (1.2.9).

Remark 1.3.2. *An averaged system analogous to* (1.3.6) *is considered in [88, equation*

(2.5)] for a different setting of singular perturbations. The latter considers the set of limit occupational measures as defined in [90, equation (2.5)], rather than $Z(x)$ as we do here. But under the standing assumptions and bounded time controllability of the fast subsystem, it is shown in [155, Theorem 1.10] (see also [156, Theorem 1.3]) that our set $Z(x)$ coincides with the one defined in [90, equation (2.5)] and used for the averaging in [88, equation (2.5)].

Theorem 1.3.2. *Under the same assumptions as in Theorem 1.3.1, every solution to (1.3.6) is an accumulation point to a sequence of x-components of trajectories (1.3.5) with respect to the uniform convergence topology.*

Proof. Fix an initial position $y \in \mathbb{R}^{N_2}$ and consider a fixed pair $(\bar{x}(\cdot), \bar{\mu}(\cdot)) : [0,t] \to \mathbb{R}^N \times \mathcal{P}(\mathbb{T}^{N_2} \times A_2)$ solution to the system of relaxed dynamics. We construct an optimal control problem where the cost functional is of the form (1.1.2) with the final cost $h \equiv 0$ and the running cost

$$\ell(s, x(s), y(s), \alpha_1(s), \alpha_2(s)) = \phi(|x(s) - \bar{x}(s)|^p)$$

with $p \in [1, +\infty[$, and where the function $\phi : \mathbb{R} \to \mathbb{R}$ is smooth and bounded with 0 as its unique minimizer and such that

$$\phi(0) = 0, \quad \phi(t) > 0, \ \forall\, t \neq 0.$$

We assume in addition that

$$|t| \leq \phi(t), \ \forall t \in [-A, A] \tag{1.3.7}$$

where $A > 0$ is a constant to be made precise.

Note that since ℓ defined as above does not depend on (y, α_2), we have $\ell = \bar{\ell}$. Now the value function is

$$v^\varepsilon(t, x, y) = \inf_{(\alpha_1, \alpha_2) \in \mathcal{A}} \int_0^t \phi(|x(s) - \bar{x}(s)|^p) ds. \tag{1.3.8}$$

By the previous theorem, $v^\varepsilon(t, x, y)$ converges locally uniformly to $\bar{v}(t, x)$ defined as in (1.3.2). Now observe that $\bar{v}(t, \cdot) \geq 0$ for all t, and in fact $\bar{v}(t, x) = 0$ since $(\bar{x}(\cdot), \bar{\mu}(\cdot))$ is an admissible solution. Therefore one has

$$|v^\varepsilon(t, x, y) - \bar{v}(t, x)| = |v^\varepsilon(t, x, y)| \to 0, \ \text{as } \varepsilon \to 0$$

which writes for an arbitrary small $\delta > 0$

$$\exists\, E > 0 \;\text{ s.t. }\; \forall \varepsilon \leq E,\; |v^{\varepsilon}(t,x,y)| \leq \frac{\delta}{2}.$$

Now let $x^{\varepsilon}(\cdot)$ be an $\varepsilon/2$-optimal solution to (1.3.8). Then one has

$$0 \leq \int_0^t \phi(|x^{\varepsilon}(s) - \bar{x}(s)|^p)ds \leq v^{\varepsilon}(t,x,y) + \frac{\varepsilon}{2} \leq \frac{\delta + \varepsilon}{2}. \tag{1.3.9}$$

And since ε can be chosen as small as we want, we consider $0 < \varepsilon \leq \delta$, and hence from the previous inequality one gets

$$\forall \delta > 0,\, \exists\, E > 0 \text{ s.t. } \forall\, \varepsilon \leq E,\; 0 \leq \int_0^t \phi(|x^{\varepsilon}(s) - \bar{x}(s)|^p)ds \leq \delta. \tag{1.3.10}$$

We claim that there exists a positive constant A such that for all $\varepsilon > 0$ and for all $s \in [0,t]$

$$|x^{\varepsilon}(s) - \bar{x}(s)|^p \leq A. \tag{1.3.11}$$

Therefore, (1.3.10) together with (1.3.7) yield

$$\forall \delta > 0,\, \exists\, E > 0 \text{ s.t. } \forall\, \varepsilon \leq E,\; 0 \leq \int_0^t |x^{\varepsilon}(s) - \bar{x}(s)|^p ds \leq \delta \tag{1.3.12}$$

and hence

$$\|x^{\varepsilon} - \bar{x}\|_{L^p(0,t)} \to 0, \;\text{ as }\; \varepsilon \to 0. \tag{1.3.13}$$

Proof of the claim (1.3.11):

Denote by $F := \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$. We have for $s \in [0,t]$

$$\begin{aligned}
|x^{\varepsilon}(s) - \bar{x}(s)| \leq\; & \int_0^s \int_{\mathbb{R}^{N_2} \times A_2} |F(x^{\varepsilon}(r), y^{\varepsilon}(r), \alpha_1^{\varepsilon}(r), \alpha_2^{\varepsilon}(r)) - F(\bar{x}(r), y, \bar{\alpha}_1(r), \alpha_2)| d\mu(y,\alpha_2) dr \\
\leq\; & \int_0^s \int_{\mathbb{R}^{N_2} \times A_2} |F(x^{\varepsilon}(r), y^{\varepsilon}(r), \alpha_1^{\varepsilon}(r), \alpha_2^{\varepsilon}(r)) - F(\bar{x}(r), y, \alpha_1^{\varepsilon}(r), \alpha_2^{\varepsilon}(r))| + \\
& \qquad |F(\bar{x}(r), y, \alpha_1^{\varepsilon}(r), \alpha_2^{\varepsilon}(r)) - F(\bar{x}(r), y, \bar{\alpha}_1(r), \alpha_2)|\; d\mu(y,\alpha_2) dr
\end{aligned}$$

Denote the first term by

$$FT(r) := |F(x^{\varepsilon}(r), y^{\varepsilon}(r), \alpha_1^{\varepsilon}(r), \alpha_2^{\varepsilon}(r)) - F(\bar{x}(r), y, \alpha_1^{\varepsilon}(r), \alpha_2^{\varepsilon}(r))|$$

and the second term by

$$ST(r) := |F(\bar{x}(r), y, \alpha_1^\varepsilon(r), \alpha_2^\varepsilon(r)) - F(\bar{x}(r), y, \bar{\alpha}_1(r), \alpha_2)|.$$

By the general assumptions on f_1, f_2, F is Lipschitz–continuous uniformly in α, with a Lipschitz constant denoted by K. It is also $\mathbb{Z}^{N_2}$ periodic in y-variable, so for each $r \in (0, s)$ and $y \in \mathbb{R}^{N_2}$, one can find $n \in \mathbb{Z}$ such that $|y^\varepsilon(r) + n - y| \leq N_2$. Therefore, one has

$$\begin{aligned} |FT(r)| &= |F(x^\varepsilon(r), y^\varepsilon(r) + n, \alpha_1^\varepsilon(r), \alpha_2^\varepsilon(r)) - F(\bar{x}(r), y, \alpha_1^\varepsilon(r), \alpha_2^\varepsilon(r))| \\ &\leq K(|y^\varepsilon(r) + n - y| + |x^\varepsilon(r) - \bar{x}(r)|) \\ &\leq KN_2 + K|x^\varepsilon(r) - \bar{x}(r)|. \end{aligned}$$

We also have F uniformly continuous in α, with α valued in a compact set $A_1 \times A_2$, so

$$\exists\, M > 0, \text{ s.t. } \forall\, r \in (0, t), \quad |ST(r)| \leq M$$

Hence,

$$\begin{aligned} |x^\varepsilon(s) - \bar{x}(s)| &\leq \int_0^s \int_{\mathbb{R}^{N_2} \times A_2} |FT(r)| + |ST(r)|\, d\mu(y, \alpha_2) dr \\ &\leq \int_0^s \int_{\mathbb{R}^{N_2} \times A_2} KN_2 + M + K|x^\varepsilon(r) - \bar{x}(r)|\, d\mu(y, \alpha_2) dr \\ &\leq Cs + \int_0^s |x^\varepsilon(r) - \bar{x}(r)| dr; \quad \text{where } \ C := KN_2 + M, \end{aligned}$$

and Grönwall inequality yields

$$\begin{aligned} |x^\varepsilon(s) - \bar{x}(s)| &\leq sCe^{sK}, \quad \forall\, s \in [0, t] \\ &\leq tCe^{tK} \end{aligned}$$

Choosing any $A \geq (tCe^{tK})^p$ is enough to prove the claim.

Finally, the uniform convergence is deduced using the pointwise convergence (1.3.13) combined with Ascoli-Arzelà theorem provided the sequence $(z_n)_n := (x^{\varepsilon_n} - \bar{x})_n$ with $\varepsilon_n := 1/n$, is uniformly bounded and equicontinuous on $[0, t]$. This is true thanks to (1.3.11) which insures uniform boundedness, and to the general assumptions on f_1, f_2

which guarantee the equicontinuity as follows: let $0 \leq s_1 \leq s_2 \leq t$, then

$$
\begin{aligned}
|z_n(s_2) - z_n(s_1)| \leq & \int_{s_1}^{s_2} \int_{\mathbb{R}^{N_2} \times A_2} \mid F(x^{\varepsilon_n}(r), y^{\varepsilon_n}(r), \alpha_1^{\varepsilon_n}(r), \alpha_2^{\varepsilon_n}(r)) - \\
& \qquad\qquad F(\bar{x}(r), y, \bar{\alpha}_1(r), \alpha_2) \mid d\mu(y, \alpha_2) dr \\
\leq & \int_{s_1}^{s_2} \int_{\mathbb{R}^{N_2} \times A_2} |FT(r)| + |ST(r)| \;\; d\mu(y, \alpha_2) dr. \\
\leq & \int_{s_1}^{s_2} C + \sup_{s \in [0,t]} |x^{\varepsilon_n} - \bar{x}(s)| \, dr = C'|s_2 - s_1|,
\end{aligned}
$$

where C' is a positive constant that only depends on F and on the time horizon t following the estimates used in the proof of the claim (1.3.11). $\square$

The converse of this result is in general not true since the set of solutions of (1.3.6) is not closed. Still, a similar result can be proved but for the following *convexified relaxed dynamics*

$$
\begin{cases}
\dot{x}_1(t) \in \overline{\mathrm{co}} \bar{f}_1 \left(x(t), A_1, Z(x(t)) \right) \\
\dot{x}_2(t) = 0 \\
x(0) = x.
\end{cases}
\tag{1.3.14}
$$

where $\overline{\mathrm{co}}$ denotes the closure of the convex hull.

Remark 1.3.3. *The system* (1.3.6) *can be written as a differential inclusion of the form*

$$
\begin{cases}
\dot{x}_1(t) \in \bar{f}_1 \left(x(t), A_1, Z(x(t)) \right) \\
\dot{x}_2(t) = 0 \\
x(0) = x.
\end{cases}
\tag{1.3.15}
$$

And the set of solutions of the latter is dense in the set of solutions of (1.3.14) *when* $\bar{f}_1(\cdot, A_1, Z(\cdot))$ *is Lipschitz–continuous with compact values, while it does not necessarily holds when it is only continuous (a well-known counterexample is due to Plis [145] and can also be found in [15, Example p. 127]). This is known as the Relaxation theorem and is due to Filippov and Ważewski (see [15, Theorem 2.4.2]).*

Before proving the converse of the previous corollary, we will need the following two results which are respectively a regularity result for the differential inclusion (1.3.14), and a weak version of the Relaxation theorem referred to in the above remark, since in our case $\bar{f}_1(\cdot, A_1, Z(\cdot))$ is not with compact values because its range is not closed.

In the sequel, we set

$$\mathcal{F}(x) := \overline{\mathrm{co}}\bar{f}_1(x, A_1, Z(x)) \times \{0\}, \tag{1.3.16}$$

and (1.3.14) writes as $\dot{x}(t) \in \mathcal{F}(x(t)),\ x(0) = x$. Then the following proposition is a direct consequence of Lemma 1.2.2.

Proposition 1.3.1. *If the system* (1.2.5) *is bounded time controllable, then* $\mathcal{F}$ *is proper*[2] *and upper semicontinuous.*

Proof. If $\mathcal{F}$ is not proper, then $\mathcal{F}$ is the empty set map, that is, for each $x \in \mathbb{R}^N$, $\mathcal{F}(x) = \emptyset$. This is not true, since $Z(x)$ is nonempty thanks to Proposition 1.2.2.
Upper semicontinuity follows directly from [16, Proposition 1.4.14] since $Z(x)$ is compact for every x by Proposition 1.2.2 and the upper semicontinuity of the set-valued map $x \rightsquigarrow Z(x)$. □

Remark 1.3.4. *Since* $\mathcal{F}$ *is upper semicontinuous, it is in particular upper hemicontinuous*[3] *(see [15, Proposition 1.4.1]). This property is a key ingredient for the Convergence Theorem used at the end of the proof of the next Proposition.*

Proposition 1.3.2. *Let* x_n *be a sequence of continuous functions solution to the differential inclusion* (1.3.15) *and converging locally uniformly to a continuous function* $\bar{x}$*. Then* $\bar{x}$ *solves* (1.3.14)*. In particular, the set of solutions of* (1.3.14) *is closed with respect to the uniform convergence topology.*

Proof. Note first that the second statement of the proposition is a direct consequence of the fact that if x_n is a solution to (1.3.15), then it is also a solution to (1.3.14), and closedeness of the set of solutions of the latter differential inclusion follows directly from the first statement. In particular, we can consider without loss of generality that x_n is a solution of (1.3.14), and we denote by $\mathcal{F}$ the latter differential inclusion, that is:

$$\dot{x}_n(s) \in \mathcal{F}(x_n(s)), \quad x_n(0) = x \in \mathbb{R}^N, \quad s \in [0, t] \tag{1.3.17}$$

where $\mathcal{F}(x) := \overline{\mathrm{co}}\bar{f}_1(x, A_1, Z(x)) \times \{0\}$.
With the general assumptions on f_1 together with proposition 1.2.2 and proposition 1.3.1, the set-valued function $\mathcal{F}$ is proper, hemicontinuous and such that $\mathcal{F}(x)$ is convex and compact for each $x \in \mathbb{R}^N$.

[2] The set-valued map $\mathcal{F}$ is *proper* if its domain is nonempty, that is, $\mathcal{F}$ is not the trivial map $x \rightsquigarrow \emptyset$.

[3] **Definition** (see [15, Definition 1.4.1]) Let $\mathcal{F}$ be a set-valued map from a Hausdorff locally convex space X to the closed convex subsets of a Banach space Y. We say that $\mathcal{F}$ is upper hemicontinuous at $x^\circ \in X$ if, for every $p \in Y^*$, the function $x \to \sigma(\mathcal{F}(x), p) := \sup_{y \in \mathcal{F}(x)} \langle p, y \rangle$ is upper semicontinuous at x°.

To prove the first statement, we follow almost the same process as for the first proof of [15, Theorem 2.1.3]. Mainly, we will use a compactness theorem together with a convergence result to show that, up to a subsequence, the limit of $(x_n, \dot{x}_n)$, in a sense to be made precise, does belong to the graph of $\mathcal{F}$.

Let x_n be a sequence of continuous functions solution to (1.3.17). With the same arguments as in the proof of the claim in (1.3.11), such a sequence satisfies the following

i) $\forall\, s \in [0,t]$, $\{x_n(s)\}_n$ is a relatively compact subset of $\mathbb{R}^N$,

ii) there exists a positive function $c(\cdot)$ in $L^1([0,t],\mathbb{R})$ such that for almost all $s \in [0,t]$, $\|\dot{x}_n(s)\| \leq c(s)$.

Indeed, the second statement is insured by the following inequalities

$$\begin{aligned}\|\dot{x}_n(s)\| &\leq \sup_{(\alpha,\mu)\in A_1\times Z(x_n(s))} |\bar{f}_1(x_n(s),\alpha,\mu)| \\ &\leq \sup_{(\alpha,\mu)\in A_1\times Z(x_n(s))} |\bar{f}_1(x_n(s),\alpha,\mu) - \bar{f}_1(\bar{x}(s),\alpha,\mu)| + |\bar{f}_1(\bar{x}(s),\alpha,\mu)|\end{aligned}$$

The Lipschitz-continuity uniformly w.r.t. (α,μ), together with the uniform boundedness proved in the claim (1.3.11), yield the following

$$|\bar{f}_1(x_n(s),\alpha,\mu) - \bar{f}_1(\bar{x}(s),\alpha,\mu)| \leq K|x_n(s) - \bar{x}(s)| \leq sCe^{sK}$$

where C is a positive constant independent of n. Thus, one has

$$\|\dot{x}_n(s)\| \leq sCe^{sK} + \sup_{(\alpha,\mu)\in A_1\times Z(x_n(s))} |\bar{f}_1(\bar{x}(s),\alpha,\mu)|$$

and the right-hand side of the inequality is positive and in $L^1([0,t],\mathbb{R})$.
Therefore by a Compactness Theorem [15, Theorem 0.3.4, p.13], we can extract a subsequence (again denoted by) $x_n(\cdot)$ converging to an absolutely continuous function $x(\cdot) : [0,t] \to \mathbb{R}^N$ in the sense that

i) $x_n(\cdot)$ converges uniformly to $x(\cdot)$ over compact subsets of $[0,t]$,

ii) $\dot{x}_n(\cdot)$ converges weakly to $\dot{x}(\cdot)$ in $L^1([0,t],\mathbb{R}^N)$.

And since x_n already converges locally uniformly to $\bar{x}$ by assumption, then $\bar{x} \equiv x$ and $\dot{\bar{x}} \equiv \dot{x}$. Moreover

$$\mathrm{dist}((x_n(s),\dot{x}_n(s)),\mathrm{graph}(\mathcal{F})) = 0$$

that is

for almost all $s \in [0,t]$, for any neighborhood $\mathcal{N}$ of $0 \in \mathbb{R}^N \times \mathbb{R}^N$, there exists $n_0 := n_0(s, \mathcal{N})$ such that $\forall\, n \geq n_0$, $(x_n(s), \dot{x}_n(s)) \in \mathrm{graph}(\mathcal{F}) \subset \mathrm{graph}(\mathcal{F}) + \mathcal{N}$.

Finally, using a Convergence Theorem [15, Theorem 1.4.1, p. 60], we have for almost all $s \in [0,t]$, $(\bar{x}(s), \dot{\bar{x}}(s)) \in \mathrm{graph}(\mathcal{F})$, i.e. $\dot{\bar{x}}(s) \in \mathcal{F}(\bar{x}(s))$. The uniform convergence follows easily as in the proof of Theorem (1.3.2). □

We are now able to state and prove the following convergence result.

Theorem 1.3.3. *Under the same assumptions as in Theorem 1.3.1 , every accumulation point with respect to the uniform convergence topology of a sequence of x-components of trajectories (1.3.5) as $\varepsilon \to 0$ is an almost everywhere solution to* (1.3.14).

Proof. Fix an initial position $x \in \mathbb{R}^N$ and assume that a solution $x^\varepsilon(\cdot)$ to the perturbed dynamics converges locally uniformly as $\varepsilon \to 0$ to some $\bar{x}(\cdot)$. We shall prove that $\bar{x}(\cdot)$ is a solution to the convexified relaxed dynamics (1.3.14).
Using the same notation as in the proof of Theorem 1.3.2 and following its same spirit, we have $v^\varepsilon(t,x,y)$ that converges locally uniformly to $\bar{v}(t,x)$. In addition, one has

$$0 \leq v^\varepsilon(t,x,y) \leq \int_0^t \phi(|x^\varepsilon(s) - \bar{x}(s)|^p)ds$$

since x^ε is an admissible solution (where we recall that $p \geq 1$ is arbitrary chosen and fixed). This implies that as $\varepsilon \to 0$, $v^\varepsilon(t,x,y) \to 0$ locally uniformly, and hence $\bar{v}(t,x) = 0$.
Now set x_r^δ a δ-optimal solution for the relaxed optimal control problem. This writes

$$0 \leq \int_0^t \phi(|x_r^\delta(s) - \bar{x}(s)|^p)ds \leq \delta$$

which implies that $\phi(|x_r^\delta(s) - \bar{x}(s)|^p) \to 0$ as $\delta \to 0$ for almost all $s \in [0,t]$. And by the assumptions of ϕ (as in the proof of Theorem 1.3.2), one deduces $|x_r^\delta(s) - \bar{x}(s)|^p \to 0$ as $\delta \to 0$ for almost every $s \in [0,t]$.
Now note that since x_r^δ is a solution to (1.3.15), it is in particular a solution to (1.3.14), and the value function of the same relaxed optimal control problem but with the dynamics (1.3.14) equals the value function of the one with the dynamics (1.3.15) which is in fact $\bar{v}(t,x)$. The advantage of choosing $(x_r^\delta)_{\delta>0}$ in the set of solutions to (1.3.14) is that we can now extract a subsequence that converges within the same set of solutions since it is closed thanks to the proposition 1.3.2, which means that $\bar{x}$ is a solution to (1.3.14) in an almost everywhere sense. □

Remark 1.3.5. *The previous two theorems hold true in particular for optimal trajectories solution of an optimal control problem. Indeed, it suffices to choose in the previous proof for the perturbed and relaxed dynamics, the optimal ones yielded by the optimal control problem in question.*

1.4 Convergence results in particular cases

1.4.1 Fast variables independent of the slow variables

Consider now the situation in which the function f_2 driving the x_2 variables depends on x_2/ε but not on x. We rewrite the system (1.1.1) for the reader's convenience:

$$\begin{cases} \dot{x}_1(t) = f_1\left(x(t), \frac{x_2(t)}{\varepsilon}, \alpha_1(t), \alpha_2(t)\right) \\ \dot{x}_2(t) = f_2\left(\frac{x_2(t)}{\varepsilon}, \alpha_2(t)\right) \\ x(0) = x. \end{cases} \tag{1.4.1}$$

This simplification permits to prove homogenization without the controllability assumption (1.3.1) required in Theorem 1.3.1. The value function $v^\varepsilon(t,x)$, defined in (1.1.2)–(1.1.4), solves

$$\begin{cases} \partial_t v^\varepsilon + \max\limits_{(\alpha_1,\alpha_2)\in A} \Big\{ -D_{x_1} v^\varepsilon \cdot f_1\left(x, \dfrac{x_2}{\varepsilon}, \alpha_1, \alpha_2\right) - \ell\left(t, x, \dfrac{x_2}{\varepsilon}, \alpha_1, \alpha_2\right) \\ \qquad\qquad -D_{x_2} v^\varepsilon \cdot f_2\left(\dfrac{x_2}{\varepsilon}, \alpha_2\right)\Big\} = 0 \qquad \text{in } (0,+\infty)\times\mathbb{R}^N \\ v^\varepsilon(0,x) = h(x) \qquad\qquad \text{in } \mathbb{R}^N. \end{cases} \tag{1.4.2}$$

Observe that in this case the fast subsystem

$$\dot{y}(t) = f_2\left(y(t), \alpha_2(t)\right), \qquad y(0) = y \tag{1.4.3}$$

and consequently the set of limiting relaxed control $Z(x) \equiv Z$ is independent of x. Arguing as in (1.2.8) we discover

$$\bar{f}_2(\mu) := \int_{\mathbb{R}^{N_2}\times A_2} f_2(y,\alpha_2)\mathrm{d}\mu(y,\alpha_2) = 0 \qquad \text{for any } \mu \in Z.$$

Theorem 1.4.1. *Assume that* (1.4.3) *is bounded time controllable. Then, as* $\varepsilon \to 0$, *the sequence* $v^\varepsilon(t,x)$ *of solutions of* (1.4.2) *converges locally uniformly on* $(0,+\infty) \times \mathbb{R}^N$ *to*

$$\bar{v}(t,x) := \inf \left\{ \int_0^t \bar{\ell}(s, x(s), \alpha_1(s), \mu(s)) \mathrm{d}s + h(x(t)) \right\},$$

where $x(\cdot)$ *is subject to the dynamics*

$$\begin{cases} \dot{x}_1(t) = \bar{f}_1\left(x(t), \alpha_1(t), \mu(t)\right) \\ \dot{x}_2(t) = 0 \\ \alpha_1 \in \mathcal{A}_1, \quad \mu(t) \in Z \\ x(0) = x. \end{cases} \tag{1.4.4}$$

Proof. By Theorem 1.2.1 the upper and lower semilimit of the sequence $u^\varepsilon(t,x,y)$ of solutions of (1.2.2) are respectively a subsolution and a supersolution of

$$\begin{cases} \partial_t v + \max\limits_{\substack{\mu \in Z \\ \alpha_1 \in A_1}} \left\{ -D_{x_1} v \cdot \bar{f}_1(x, \alpha_1, \mu) - \bar{\ell}(t, x, \alpha_1, \mu) \right\} = 0 & \text{in } (0,+\infty) \times \mathbb{R}^N \\ v(0,x) = h(x) & \text{in } \mathbb{R}^N. \end{cases} \tag{1.4.5}$$

Since f_2 does not depend on x, then the effective Hamiltonian inherits from H regularity properties that guarantee the comparison principle for the effective Cauchy problem; therefore u^ε converges uniformly with respect to y to the unique solution of (1.4.5) (see [3, Proposition 2]); by Remark 1.2.1, the same conclusion holds for the sequence v^ε.

The set of limiting relaxed controls Z is independent of x; moreover, by Proposition 1.2.2, it is convex and compact in the weak star topology. Then, by standard results in optimal control theory, the value function $\bar{v}(t,x)$ solves (1.4.5).

□

In this configuration, where fast variables are independent of the slow variables, we can again deduce convergence of the singularly perturbed trajectories as in 1.3.2. Indeed, as it has been pointed out above, the limiting relaxed control $Z(x) \equiv Z$ is independent of x. This implies in particular that the regularity result in Lemma 1.2.2 holds automatically and hence is not needed in the following convergence results of trajectories whose proofs are *mutatis mutandis* the same as for Theorem 1.3.2 and Theorem 1.3.3.

Corollary 1.4.1. *Under the same assumptions as in Theorem 1.4.1, every solution to (1.4.4) is an accumulation point to a sequence of trajectories (1.4.1) with respect to the uniform convergence topology.*

Corollary 1.4.2. *Under the same assumptions as in Theorem 1.4.1, every accumulation point with respect to the uniform convergence topology of a sequence of trajectories (1.4.1) as $\varepsilon \to 0$, is a solution to the convexification of (1.4.4), that is, when $\bar{f}_1$ is replaced with $\overline{co}\bar{f}_1$.*

1.4.2 Explicit limit system for a partially decoupled problem

In this section we assume that x_1 and x_2 are governed respectively by the components α_1 and α_2 of the control variable. We further assume that the dynamics depends on the variable x_2 only through the oscillating term x_2/ε:

$$\begin{cases} \dot{x}_1(t) = f_1\left(x_1(t), \frac{x_2(t)}{\varepsilon}, \alpha_1(t)\right) \\ \dot{x}_2(t) = f_2\left(x_1(t), \frac{x_2(t)}{\varepsilon}, \alpha_2(t)\right) \\ x(0) = x \equiv (x_1, x_2). \end{cases} \tag{1.4.6}$$

We consider a running cost ℓ with the same dependence as f_1, and a terminal cost which only depends on x_1. Thus the cost associated to any solution of (1.4.6) is

$$J^\varepsilon\big(t, x_1, x_2, \alpha_1, \alpha_2\big) := \int_0^t \ell\left(s, x_1(s), \frac{x_2(s)}{\varepsilon}, \alpha_1(s)\right) \mathrm{d}s + h(x_1(t)). \tag{1.4.7}$$

The value function $v^\varepsilon\big(t, x_1, x_2\big) := \inf\limits_{(\alpha_1,\alpha_2)\in\mathcal{A}} J^\varepsilon\big(t, x_1, x_2, \alpha_1, \alpha_2\big)$ solves

$$\begin{cases} \partial_t v^\varepsilon + \max\limits_{\alpha_1\in A_1} \left\{-D_{x_1}v^\varepsilon \cdot f_1\left(x_1, \dfrac{x_2}{\varepsilon}, \alpha_1\right) - \ell\left(t, x_1, \dfrac{x_2}{\varepsilon}, \alpha_1\right)\right\} & \\ \qquad\qquad + \max\limits_{\alpha_2\in A_2} \left\{-D_{x_2}v^\varepsilon \cdot f_2\left(x_1, \dfrac{x_2}{\varepsilon}, \alpha_2\right)\right\} = 0 & \text{in } (0,+\infty)\times\mathbb{R}^N \\ v^\varepsilon(0, x_1, x_2) = h(x_1) & \text{in } \mathbb{R}^N. \end{cases} \tag{1.4.8}$$

Let us introduce the set

$$\tilde{\mathcal{A}} := \{(y, \alpha_1) : [0,\infty) \to \mathbb{T}^{N_2} \times A_1 \text{ measurable}\}$$

and the following dynamics controlled by $(y, \alpha_1) \in \tilde{\mathcal{A}}$:

$$\begin{cases} \dot{x}_1(t) = f_1(x_1(t), y(t), \alpha_1(t)) \\ x_1(0) = x_1, \\ (y, \alpha_1) \in \tilde{\mathcal{A}}; \end{cases} \tag{1.4.9}$$

the associated cost functional is

$$\tilde{J}(t, x_1, \alpha_1, y) = \int_0^t \ell(s, x_1(s), y(s), \alpha_1(s)) \mathrm{d}s + h(x_1(t)), \qquad (\alpha_1, y) \in \tilde{\mathcal{A}}. \tag{1.4.10}$$

In the next Theorem we will show that problem (1.4.9)-(1.4.10) is the appropriate limit of problem (1.4.6)-(1.4.7). In this case the assumption to be made on f_2 is weaker than the controllability condition (1.3.1).

Theorem 1.4.2. *Assume that for any $x_1 \in \mathbb{R}^{N_1}$ the system*

$$\dot{y}(t) = f_2\left(x_1, y(t), \alpha_2(t)\right), \qquad y(0) = y \tag{1.4.11}$$

is bounded time controllable. Assume further that

$$\max_{\alpha_2 \in A_2} \{-q \cdot f_2(x_1, y, \alpha_2)\} \geq 0 \quad \text{for any } x_1 \in \mathbb{R}^{N_1} \text{ any } y, q \in \mathbb{R}^{N_2}. \tag{1.4.12}$$

Then, as $\varepsilon \to 0$, the sequence $v^\varepsilon(t, x_1, x_2)$ of solutions of (1.4.8) *converges locally uniformly in $[0, +\infty) \times \mathbb{R}^N$ to*

$$\tilde{v}(t, x_1) := \inf_{(\alpha_1, y) \in \tilde{\mathcal{A}}} \tilde{J}(t, x_1, \alpha_1, y).$$

where $x_1(\cdot)$ is subject to (1.4.9).

Proof. As usual we denote by H the Hamiltonian in (1.4.8):

$$H(t, x_1, y, p_1, p_2) = \max_{\alpha_1 \in A_1} \{-p_1 \cdot f_1(x_1, y, \alpha_1) - \ell(t, x_1, y, \alpha_1)\} + \max_{\alpha_2 \in A_2} \{-p_2 \cdot f_2(x_1, y, \alpha_2)\}.$$

This expression can be obtained by developing (1.1.6) taking into account the decoupled structure of system (1.4.6).

We look for a solution of (1.4.8) of the form $v^\varepsilon\big(t, x_1, x_2\big) = u^\varepsilon(t, x_1, \frac{x_2}{\varepsilon})$. Then $u^\varepsilon(t, x_1, y)$ solves

$$\begin{cases} \partial_t u^\varepsilon + H\left(t, x_1, y, D_{x_1} u^\varepsilon, \dfrac{1}{\varepsilon} D_y u^\varepsilon\right) = 0 & \text{in } (0, +\infty) \times \mathbb{R}^N \\ u^\varepsilon(0, x_1, y) = h(x_1) & \text{in } \mathbb{R}^N. \end{cases} \tag{1.4.13}$$

Observe that since f_1 and f_2 do not depend on x_2, but only on x_2/ε, the dynamics of x_2 can be ignored in the singularly perturbed system obtained from (1.4.6) by introducing an extra variable $y = x_2/\varepsilon$:

$$\begin{cases} \dot{x}_1(t) = f_1\left(x_1(t), y(t), \alpha_1(t)\right) \\ \dot{y}(t) = \frac{1}{\varepsilon} f_2\left(x_1(t), y(t), \alpha_2(t)\right) \\ x(0) = x, \quad y(0) = y. \end{cases} \tag{1.4.14}$$

The unique solution of (1.4.13) is the value function

$$u^\varepsilon(t, x_1, y) = \inf_{(\alpha_1, \alpha_2) \in \mathcal{A}} J\big(t, x_1, y, \alpha_1, \alpha_2\big)$$

where the cost functional is given by

$$J\big(t, x_1, y, \alpha_1, \alpha_2\big) := \int_0^t \ell\left(s, x_1(s), y(s), \alpha_1(s)\right) \mathrm{d}s + h(x_1(t)).$$

Thanks to the assumed bounded time controllability, by Lemma 1.2.1, for any $t \in (0, +\infty)$ and $x_1, p_1 \in \mathbb{R}^{N_1}$, H is ergodic and $\bar{H}(t, x_1, p_1)$ exists. Furthermore, by Proposition 1.2.1 the semilimits of u^ε are respectively a supersolution and a subsolution of

$$\begin{cases} \partial_t v + \bar{H}\left(t, x_1, D_{x_1} v\right) = 0 & \text{in } (0, +\infty) \times \mathbb{R}^{N_1} \\ v(0, x_1) = h(x_1) & \text{in } \mathbb{R}^{N_1}. \end{cases} \tag{1.4.15}$$

We have in this case the following explicit formula for the effective Hamiltonian:

$$\bar{H}(t, x_1, p_1) = \max_{y \in \mathbb{R}^{N_2}} H(t, x_1, y, p_1, 0) = \max_{\substack{\alpha_1 \in A_1 \\ y \in \mathbb{R}^{N_2}}} \left\{-p \cdot f_1(x_1, y, \alpha_1) - \ell(t, x_1, y, \alpha_1)\right\}. \tag{1.4.16}$$

To prove (1.4.16), we argue as in [4, Proposition 6.6]. Let $\bar{t}$, $\bar{x}_1$ and $\bar{p}_1$ be fixed, and assume by contradiction that $\bar{H}(\bar{t}, \bar{x}_1, \bar{p}_1) < H(\bar{t}, \bar{x}_1, y, \bar{p}_1, 0)$ for any y in a neighborhood of a maximum point of $H(\bar{t}, \bar{x}_1, y, \bar{p}_1, 0)$. Then let $w_\delta(y)$ be the solution of the δ-cell problem (1.2.7) (with $\bar{p}_2 = 0$); we have

$$\begin{aligned} &\max_{\alpha_2 \in A_2} \left\{-Dw_\delta \cdot f_2(\bar{x}_1, y, \alpha_2)\right\} = H(\bar{t}, \bar{x}_1, y, \bar{p}_1, Dw_\delta) - H(\bar{t}, \bar{x}_1, y, \bar{p}_1, 0) \\ &= -\delta w_\delta(y) - H(\bar{t}, \bar{x}_1, y, \bar{p}_1, 0) = \bar{H}(\bar{t}, \bar{x}_1, \bar{p}_1) - H(\bar{t}, \bar{x}_1, y, \bar{p}_1, 0) + o(1) < 0 \quad \text{as } \delta \to 0 \end{aligned}$$

in an open set. A contradiction with (1.4.12) that proves (1.4.16).

Thus, problem (1.4.15) explicitly reads

$$\begin{cases} \partial_t v + \max\limits_{\substack{\alpha_1 \in A_1 \\ y \in \mathbb{R}^{N_2}}} \{-D_{x_1} v \cdot f_1(x_1, y, \alpha_1) - \ell(t, x_1, y, \alpha_1)\} = 0 & \text{in } (0, +\infty) \times \mathbb{R}^{N_1} \\ v(0, x_1) = h(x_1) & \text{in } \mathbb{R}^{N_1}. \end{cases} \tag{1.4.17}$$

Formula (1.4.16) also shows that $\bar{H}$ has the same regularity in x_1 as H, in particular it satisfies the comparison principle. Then u^ε - and a fortiori, by Remark 1.2.1, v^ε - converges locally uniformly in $[0, +\infty) \times \mathbb{R}^N$ as $\varepsilon \to 0$, to the unique viscosity solution of (1.4.17).

Since $f_1(x_1, \cdot, \alpha_1)$ and $\ell(t, x_1, \cdot, \alpha_1)$ are $\mathbb{Z}^{N_2}$–periodic, the maximum in y in the formula (1.4.16) for $\bar{H}$ is achieved on the N_2-dimensional torus $\mathbb{T}^{N_2}$. Moreover the set $A_1 \times \mathbb{T}^{N_2}$ of values of controls in $\tilde{\mathcal{A}}$ is compact. Then, by classical results in optimal control theory, the unique solution of the problem (1.4.17) is $\tilde{v}$. □

Remark 1.4.1. *If the hypothesis* (1.4.12) *is not satisfied then homogenization may not hold. The counterexample of Section 1.6.2 fits the setting of problem* (1.4.6) *with* $f_1(x_1, y, \alpha_1) = \cos y + 1$, $f_2(x_1, y, \alpha_2) = x_1 + \alpha_2$, $\ell \equiv 0$ *and* $h = h(x_1)$. *Condition* (1.4.12) *fails in this example because*

$$\max_{|\alpha_2| \leq 1} -q \cdot f_2(x_1, y, \alpha_2) = -q x_1 + |q|$$

can be negative for some $q \in \mathbb{R}$ *if* $|x_1| > 1$. *By Proposition 1.6.1, homogenization is not guaranteed in this case. If instead* $|x_1| \leq 1$, *then* $-q x_1 + |q| \geq 0$; *thus* (1.4.12) *is satisfied and homogenization holds. By using* (1.4.16) *we can compute*

$$\bar{H}(x_1, p_1) = \max_{y \in \mathbb{R}} -p_1(\cos y + 1) = 2p_1^-,$$

according with formula (1.3.4).

Here again we can deduce convergence of the singularly perturbed dynamics. Indeed, the limiting relaxed control set $Z(x) \equiv Z = \{\delta_{y(\cdot)} : y(\cdot) : [0, \infty) \to \mathbb{T}^{N_2}\}$ is independent of x, and Lemma 1.2.2 automatically holds. We can hence prove the same convergence result for trajectories as proved in Theorem 1.3.2 and Theorem 1.3.3.

Corollary 1.4.3. *Under the same assumptions as in Theorem 1.4.2, every solution to (1.4.9) is an accumulation point to a sequence of trajectories (1.4.6) with respect to the uniform convergence topology.*

Corollary 1.4.4. *Under the same assumptions as in Theorem 1.4.2, every accumulation point with respect to the uniform convergence topology of a sequence of trajectories (1.4.6) as $\varepsilon \to 0$, is a solution to the convexification of (1.4.9).*

Finally let us conclude these two sections by the next remark on the dependency of the limiting dynamics on the slow variables.

Remark 1.4.2. *Notice that in the three cases considered in Theorem 1.3.1, Theorem 1.4.1 and Theorem 1.4.2 the limiting dynamics is always in dimension N_1 being stationary with respect to x_2. Moreover, in the latter case, since the dynamics depend on x_2 only through oscillations x_2/ε, the limiting dynamics* (1.4.9) *in not influenced by the initial state of x_2.*

1.5 A different relaxation for uncontrolled oscillations

We now consider the case in which the dynamics for x_2 in (1.1.1) is not controlled, that is $f_2 = f_2\left(x, \frac{x_2}{\varepsilon}\right)$. We are dealing with

$$\begin{cases} \dot{x}_1(t) = f_1\left(x(t), \frac{x_2(t)}{\varepsilon}, \alpha(t)\right) \\ \dot{x}_2(t) = f_2\left(x(t), \frac{x_2(t)}{\varepsilon}\right) \\ x(0) = x. \end{cases} \tag{1.5.1}$$

Since only the dynamics for x_1 is controlled, we use the notation $\alpha \in \mathcal{A}$ for the controls. The value function $v^\varepsilon(t,x) := \inf_{\alpha\in\mathcal{A}} J^\varepsilon(t,x,\alpha)$, with J^ε as in (1.1.2), solves

$$\begin{cases} \partial_t v^\varepsilon - D_{x_2} v^\varepsilon \cdot f_2\left(x, \dfrac{x_2}{\varepsilon}\right) + \max\limits_{\alpha\in A}\left\{-D_{x_1} v^\varepsilon \cdot f_1\left(x, \dfrac{x_2}{\varepsilon}, \alpha\right) - \ell\left(t, x, \dfrac{x_2}{\varepsilon}, \alpha\right)\right\} = 0 \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{in } (0,+\infty)\times\mathbb{R}^N \\ v^\varepsilon(0,x) = h(x) \qquad\qquad\qquad\qquad\qquad\qquad \text{in } \mathbb{R}^N. \end{cases} \tag{1.5.2}$$

As before, we look for a solution of (1.5.2) of the form $v^\varepsilon(t,x) = u^\varepsilon(t,x,\frac{x_2}{\varepsilon})$. Then $u^\varepsilon(t,x,y)$ solves

$$\begin{cases} \partial_t u^\varepsilon - \left(D_{x_2} u^\varepsilon + \dfrac{1}{\varepsilon} D_y u^\varepsilon \right) \cdot f_2(x,y) \\ \quad + \max\limits_{\alpha \in A} \left\{ -D_{x_1} u^\varepsilon \cdot f_1(x,y,\alpha) - \ell(t,x,y,\alpha) \right\} = 0 \qquad \text{in } (0,+\infty) \times \mathbb{R}^N \times \mathbb{R}^{N_2} \\ u^\varepsilon(0,x,y) = h(x) \qquad \text{in } \mathbb{R}^N \times \mathbb{R}^{N_2}. \end{cases} \tag{1.5.3}$$

The unique solution of the problem above is the value function

$$u^\varepsilon(t,x_1,x_2,y) = \inf_{\alpha \in \mathcal{A}} \left\{ \int_0^t \ell(s, x_1(s), x_2(s), y(s), \alpha(s)) + h(x_1(t), x_2(t)) \right\}$$

associated to the singularly perturbed dynamics obtained from (1.5.1) by introducing the additional variable $y = x_2/\varepsilon$:

$$\begin{cases} \dot{x}_1(t) = f_1(x_1(t), x_2(t), y(t), \alpha(t)) \\ \dot{x}_2(t) = f_2(x_1(t), x_2(t), y(t)) \\ \dot{y}(t) = \frac{1}{\varepsilon} f_2(x_1(t), x_2(t), y(t)) \\ x(0) = (x_1, x_2), \quad y(0) = y. \end{cases} \tag{1.5.4}$$

1.5.1 A weak convergence result

Fix $x \in \mathbb{R}^N$ and consider the fast subsystem of (1.5.4):

$$\dot{y}(t) = f_2(x, y(t)), \qquad y(0) = y. \tag{1.5.5}$$

Given a measure μ in the set of periodic Radon probability measures on $\mathbb{R}^{N_2}$ (which we can identify with the set of Radon probability measure on the torus $\mathbb{T}^{N_2}$) and a function φ, continuous and $\mathbb{Z}^{N_2}$-periodic in $\mathbb{R}^{N_2}$, we define

$$\mu(\varphi) := \int_{\mathbb{T}^{N_2}} \varphi(y) \mathrm{d}\mu(y). \tag{1.5.6}$$

We denote by S_t the semigroup associated to the dynamics:

$$S_t \varphi(y) := \varphi(y(t)), \qquad y(\cdot) \text{ solving (1.5.5)}.$$

Definition 1.5.1. *A periodic Radon probability measure on* $\mathbb{R}^{N_2}$, μ, *is said to be invariant for the dynamics* (1.5.5) *if*

$$\mu(S_t\varphi) = \mu(\varphi), \qquad \textit{for any } t \geq 0 \textit{ and } \varphi \in C_{per}(\mathbb{R}^{N_2}). \tag{1.5.7}$$

The connection between ergodicity and invariant measure is stated in the following Proposition quoted from [4] (see also [69] and [159]).

Proposition 1.5.1. *The dynamics* (1.5.5) *has an invariant probability measure. The invariant probability measure is unique if and only if*

$$\textit{for every } \varphi \in C(\mathbb{T}^{N_2}), \quad \lim_{t\to+\infty} \frac{1}{t}\int_0^t S_s\varphi(y)\mathrm{d}s = \mathrm{const} \quad \textit{uniformly w.r.t. } y. \tag{1.5.8}$$

In this case the constant in (1.5.8) *is* $\mu(\varphi)$.

Proof. See Proposition 3.1 in [4]. □

To stress the fact that invariant measures of (1.5.5) may depend on the choice of x we denote it by μ_x.

We expect from [4] that the effective Hamiltonian be the average of the Hamiltonian in the y variables with respect to an ergodic measure μ_x. We will rewrite it as a Bellman Hamiltonian by means of the following extended set of controls

$$A^{ex}(x) := L^1\left((\mathbb{T}^{N_2}, \mu_x); A\right).$$

Note that $A^{ex}(x)$, the set of integrable functions from the measure space $(\mathbb{T}^{N_2}, \mu_x)$ to A, contains a copy of A, given by the constant functions. Define now the average of the vector field f_1 and the running cost ℓ as follows

$$\hat{f}_1(x,\alpha) := \int_{\mathbb{T}^{N_2}} f_1(x,y,\alpha(y))\, d\mu_x(y), \quad \hat{\ell}(s,x,\alpha) := \int_{\mathbb{T}^{N_2}} \ell(s,x,y,\alpha(y))\, d\mu_x(y), \quad \alpha \in A^{ex}(x).$$

Remark 1.5.1. *For any* $t > 0$ *and* $y \in \mathbb{R}^{N_2}$ *consider the solution* $y(s)$ *of* (1.5.5) *and define the occupational measure (in* $\mathcal{P}(\mathbb{T}^{N_2})$*)*

$$\mu_{t,x,y} := \frac{1}{t}\int_0^t \delta_{y(s)}\mathrm{d}s.$$

Unlike the controlled case of Section 1.3, since the dynamics is now uncontrolled, occupational measures are now operating just on state space. We denote by $Z_0(x)$ *the set of* *measures* $\mu_x \in \mathcal{P}(\mathbb{T}^{N_2})$ *such that*

$$\mu_x = \lim_{n\to\infty} \mu_{t_n,x,y} \quad \textit{weak-star}$$

for some $t_n \to +\infty$ *and* $y \in \mathbb{R}^{N_2}$*. Although the dynamics* (1.5.5) *is not controlled, in analogy with the set* $Z(x)$ *of Section 1.3, the set* $Z_0(x)$ *can be interpreted as a set of limiting relaxed controls. Observe that any* $\mu_x \in Z_0(x)$ *is invariant in the sense of Definition 1.5.1. Then, if* μ_x *is the unique invariant measure for* (1.5.5)*, then* $Z_0(x) = \{\mu_x\}$ *and, arguing as in* (1.2.8) *we discover*

$$\int_{\mathbb{T}^{N_2}} f_2(x,y)\, d\mu_x(y) = 0. \tag{1.5.9}$$

The following Lemma is inspired by a time-averaging result in [28].

Lemma 1.5.1. *Fix* $t > 0$*,* $x, p \in \mathbb{R}^N$ *and let* $H_1(t,x,y,p) := \max_{\alpha\in A}\{-p\cdot f_1(x,y,\alpha) - \ell(t,x,y,\alpha)\}$*. Then*

$$\int_{\mathbb{T}^{N_2}} H_1(t,x,y,p)\, d\mu_x(y) = \sup_{\alpha\in A^{ex}(x)} \{-p\cdot \hat{f}_1(x,\alpha) - \hat{\ell}(t,x,\alpha)\}. \tag{1.5.10}$$

Proof. We name I the left hand side and G the right hand side of (1.5.10). For any $\varepsilon > 0$ there is $\alpha_\varepsilon \in A^{ex}(x)$ such that

$$\begin{aligned}
G &\le -p\cdot \hat{f}_1(x,\alpha_\varepsilon) - \hat{\ell}(t,x,\alpha_\varepsilon) + \varepsilon \\
&= -\int_{\mathbb{T}^{N_2}} [p\cdot f_1(x,y,\alpha_\varepsilon(y)) + \ell(t,x,y,\alpha_\varepsilon(y))]\, d\mu_x(y) + \varepsilon \\
&\le I + \varepsilon
\end{aligned}$$

which proves the inequality $I \ge G$.

For the opposite inequality freeze t, x, p, let $F(y,\alpha) := -p\cdot f_1(x,y,\alpha) - \ell(t,x,y,\alpha)$, and observe that $H_1(y) \in F(y, A)$ for all $y \in \mathbb{T}^{N_2}$. Since H_1 and F are continuous, μ_x is finite, and A is compact, a classical selection theorem (e.g., [94, Theorem 7.1, p. 66]) implies the existence of a measurable selector, i.e., $\gamma \in A^{ex}(x)$ such that $H_1(y) = F(y,\gamma(y))$ for all $y \in \mathbb{T}^{N_2}$. Then

$$I = -\int_{\mathbb{T}^{N_2}} [p\cdot f_1(x,y,\gamma(y)) + \ell(t,x,y,\gamma(y))]\, d\mu_x(y) = -p\cdot \hat{f}_1(x,\gamma) - \hat{\ell}(t,x,\gamma) \le G,$$

which completes the proof. $\square$

In the next Theorem we establish an explicit expression for the effective Hamiltonian through invariant probability measures and a weak convergence result for the semilimits of solutions of the associated singular perturbation problem (1.5.3).

Theorem 1.5.1. *Let $u^\varepsilon(t,x,y)$ be a solution of* (1.5.3). *Assume that for any $x \in \mathbb{R}^N$ the dynamics* (1.5.5) *has a unique invariant probability measure μ_x. Then the upper and lower semilimit of $u^\varepsilon(t,x,y)$ are respectively a subsolution and a supersolution of*

$$\begin{cases} \partial_t v + \sup\limits_{\alpha \in A^{ex}(x)} \Big\{ -D_{x_1} v \cdot \hat{f}_1(x,\alpha) - \hat{\ell}(t,x,\alpha) \Big\} = 0 & \text{in } (0,+\infty) \times \mathbb{R}^N \\ v(0,x) = h(x) & \text{in } \mathbb{R}^N. \end{cases} \tag{1.5.11}$$

Proof. Let us denote the Hamiltonian in (1.5.3) as

$$G\big(t,x,y,p_1,p_2,q\big) = -q \cdot f_2(x,y) + G_1\big(t,x,y,p_1,p_2\big)$$

with

$$G_1\big(t,x,y,p_1,p_2\big) = -p_2 \cdot f_2(x,y) + \max_{\alpha \in A} \left\{ -p_1 \cdot f_1(x,y,\alpha) - \ell(t,x,y,\alpha) \right\}.$$

We claim that the effective Hamiltonian is

$$\bar{H}\big(\bar{t},\bar{x},\bar{p}_1\big) = \int_{\mathbb{T}^{N_2}} \max_{\alpha \in A} \Big\{ -\bar{p}_1 \cdot f_1(\bar{x},y,\alpha) - \ell(\bar{t},\bar{x},y,\alpha) \Big\} \mathrm{d}\mu_x(y). \tag{1.5.12}$$

To prove the claim (1.5.12) we fix $\bar{t} > 0$, $\bar{x} \in \mathbb{R}^N$, $\bar{p}_1 \in \mathbb{R}^{N_1}$ and $\bar{p}_2 \in \mathbb{R}^{N_2}$ and consider the evolutive cell problem

$$\partial_t w - D_y w \cdot f_2(\bar{x},y) + G_1\big(\bar{t},\bar{x},y,\bar{p}_1,\bar{p}_2\big) = 0 \quad \text{in } (0,+\infty) \times \mathbb{R}^{N_2}$$

with initial condition $w(0,y) = 0$. Since the equation above is linear, its unique viscosity solution is

$$w(t,y) = \int_0^t G_1\big(\bar{t},\bar{x},y(s),\bar{p}_1,\bar{p}_2\big) \mathrm{d}s$$

with $y(\cdot)$ solving (1.5.5). By hypothesis, the dynamics (1.5.5) admits a unique invariant measure μ_x. Then, by Proposition 1.5.1, we have

$$\lim_{t \to +\infty} \frac{w(t,y)}{t} = \lim_{t \to +\infty} \frac{1}{t} \int_0^t S_s G_1\big(\bar{t},\bar{x},y,\bar{p}_1,\bar{p}_2\big) \mathrm{d}s = \text{const}$$

uniformly in y. The fact that the ratio $w(t,y)/t$ converges to a constant in a uniform way with respect to y, as $t \to +\infty$, guarantees that G is ergodic at $(\bar{t},\bar{x},\bar{p}_1,\bar{p}_2)$. Moreover the constant above gives the value of the effective Hamiltonian at $(\bar{t},\bar{x},\bar{p}_1,\bar{p}_2)$ (see [4, Sect. 2.1]); we have

$$\begin{aligned}\mu_x(G_1) &= \int_{\mathbb{T}^{N_2}} G_1\big(\bar{t}, \bar{x}, y, \bar{p}_1, \bar{p}_2\big) \mathrm{d}\mu_x(y) \\ &= \int_{\mathbb{T}^{N_2}} \Big[-\bar{p}_2 \cdot f_2(\bar{x}, y) + \max_{\alpha \in A} \Big\{-\bar{p}_1 \cdot f_1(\bar{x}, y, \alpha) - \ell(\bar{t}, \bar{x}, y, \alpha)\Big\}\Big] \mathrm{d}\mu_x(y).\end{aligned}$$

Taking also into account (1.5.9) we obtain (1.5.12). Then the conclusion follows from Proposition 1.2.1 and Lemma 1.5.1. □

1.5.2 Convergence for ergodic measure independent of the slow variables

In this section we assume in addition that

$$\text{the unique invariant measure } \mu \text{ of the system (1.5.5) is independent of } x. \tag{1.5.13}$$

Of course this assumption is satisfied if $f_2 = f_2(y)$ is independent of $x = (x_1, x_2)$. Another interesting case is the following.

Example 1.1. *Assume $f_2 = f_2(x)$ is independent of y and satisfies the non-resonance condition*

$$f_2(x) \cdot k \neq 0 \quad \forall\, k \in \mathbb{Z}^{N_2} \backslash \{0\},\ x \in \mathbb{R}^N.$$

Then the unique invariant probability measure of $\dot{y} = f_2(x)$ is the Lebesgue measure on $\mathbb{T}^{N_2}$, for all x, by a classical result in ergodic theory.

Under the current assumption the extended control set $A^{ex}(x) = A^{ex}$ is independent of x and we set $\mathcal{A}^{ex} := \{\alpha : [0, +\infty) \to A^{ex} \text{ measurable}\}$.

Corollary 1.5.1. *Assume* (1.5.13). *Then the sequence $v^\varepsilon(t, x)$ of solutions of* (1.5.2) *converges locally uniformly in $(0, +\infty) \times \mathbb{R}^N$, as $\varepsilon \to 0$, to the unique viscosity solution v of* (1.5.11). *Moreover*

$$v(t, x) = \inf_{\alpha \in \mathcal{A}^{ex}} \left\{ \int_0^t \hat{\ell}\,(s, x(s), \alpha(s))\, \mathrm{d}s + h(x(t)) \right\}, \tag{1.5.14}$$

where $x(\cdot)$ is subject to

$$\begin{cases} \dot{x}_1(t) = \hat{f}_1\,(x(t), \alpha(t)) \\ \dot{x}_2(t) = 0 \\ x(0) = x. \end{cases} \tag{1.5.15}$$

Proof. We look for solutions of (1.5.2) of the form $v^\varepsilon(t,x) = u^\varepsilon(t,x,\frac{x_2}{\varepsilon})$. Then $u^\varepsilon(t,x,y)$ solves (1.5.3) and by Theorem 1.5.1 the semilimits $\bar{u}^*$ and $\underline{u}_*$ of u^ε are respectively a supersolution and a subsolution of (1.5.11). Thanks to formula (1.5.12) we observe that $\bar{H}$ satisfies the assumptions of the comparison principle, because it is an average of H with respect to a measure which is independent of x. Thus $\bar{u}^*$ and $\underline{u}_*$ coincide and u^ε converges locally uniformly in $(0,+\infty) \times \mathbb{R}^N \times \mathbb{R}^{N_2}$ to the unique solution of (1.5.11). By Remark 1.2.1 the upper and lower semilimits of v^ε also coincide and we conclude that the convergence of $v^\varepsilon(t,x) = u^\varepsilon(t,x,\frac{x_2}{\varepsilon})$ to the unique solution of (1.5.11) is uniform.

To complete the proof we observe that (1.5.11) is the Bellman equation associated to the value function $V(x,t)$ on the right hand side of (1.5.14). Therefore it is enough to check that the control problem (1.5.14), (1.5.15) satisfies some basic structure conditions that imply V continuous and solving (1.5.11) in viscosity sense. For instance, the assumptions of Chapter III in [19] are easily verified. In particular, the continuity of $\hat{\ell}$ and $\hat{f}_1$ from $[0,+\infty) \times \mathbb{R}^N \times A^{ex}$ to $\mathbb{R}$ and from $\mathbb{R}^N \times A^{ex}$ to $\mathbb{R}^{N_1}$, respectively, follows from the following argument by contradiction. Assume $t_n \to t$, $x_n \to x$ and $\alpha_n \to \alpha$ in $L^1\left((\mathbb{T}^{N_2},\mu);A\right)$, $\hat{\ell}(t_n,x_n,\alpha_n) \to L$, but $L \neq \hat{\ell}(t,x,\alpha)$. We can extract a subsequence n_k such that $\alpha_{n_k} \to \alpha$ $\mu-$almost everywhere. Then by the Dominated Convergence Theorem $\hat{\ell}(t_{n_k},x_{n_k},\alpha_{n_k}) \to \hat{\ell}(t,x,\alpha)$, a contradiction that achieves the proof. □

The convergence of the trajectories can be deduced again from the convergence of the value function as it was proved previously. The difference now is that instead of the limiting relaxed control set $Z(x)$ which is now nothing but the singleton $\{\mu\}$, we have to consider the extended set of controls $\mathcal{A}^{ex}(x)$ for the slow dynamics. But since we are assuming (1.5.13), the latter is independent of x and no regularity assumption of the invariant measure is needed. This case is reminiscent of the one in section 1.4.1 and we have the following results whose proofs are *mutatis mutandis* the same as for Theorem 1.3.2 and Theorem 1.3.3.

Corollary 1.5.2. *Under the same assumptions as in Corollary 1.5.1, every solution to* (1.5.15) *is an accumulation point with respect to the uniform convergence topology to a sequence of trajectories of the singular perturbation system associated to* (1.5.4).

Corollary 1.5.3. *Under the same assumptions as in Corollary 1.5.1, every accumulation point with respect to the uniform convergence topology of a sequence of trajectories satisfying the singular perturbation system associated to* (1.5.4) *as* $\varepsilon \to 0$*, is a solution to the convexification of 1.5.15.*

Remark 1.5.2. *The same remark as in Remark (1.3.5) stands in this context.*

1.5.3 An example of simplified limit under a decoupling condition

In this Section we give a representation of the solution to the effective Cauchy problem (1.5.11) as value function of an optimal control problem, assuming that in (1.5.1)–(1.1.2) the control variable and the oscillating term are decoupled in the dynamics for x_1 and in the running cost. This is inspired by some singular perturbation results in stochastic control obtained in [116] and [21]. More precisely we consider

$$\begin{cases} \dot{x}_1(t) = \xi\left(x(t), \alpha(t)\right) + \eta\left(x(t), \frac{x_2(t)}{\varepsilon}\right) \\ \dot{x}_2(t) = f_2\left(x(t), \frac{x_2(t)}{\varepsilon}\right) \\ x(0) = x, \end{cases} \tag{1.5.16}$$

$$J^\varepsilon\left(t, x, \alpha\right) := \int_0^t \left[\lambda\left(s, x(s), \alpha(s)\right) + \gamma\left(x(s), \frac{x_2(s)}{\varepsilon}\right)\right] \mathrm{d}s + h(x(t)), \tag{1.5.17}$$

with $\xi = \xi(x,\alpha) : \mathbb{R}^N \times A \to \mathbb{R}^{N_1}$ bounded and uniformly continuous, Lipschitz-continuous uniformly with respect to α; $\eta = \eta(x,y) : \mathbb{R}^N \times \mathbb{R}^{N_2} \to \mathbb{R}^{N_1}$ and $\gamma = \gamma(x,y) : \mathbb{R}^N \times \mathbb{R}^{N_2} \to \mathbb{R}$ bounded and uniformly continuous and $\mathbb{Z}^{N_2}$-periodic with respect to y; $\lambda : [0,+\infty) \times \mathbb{R}^N \times A \to \mathbb{R}$ bounded and uniformly continuous.

The value function $v^\varepsilon(t,x) := \inf_{\alpha\in\mathcal{A}} J^\varepsilon(t,x,\alpha)$ solves

$$\begin{cases} \partial_t v^\varepsilon + \max\limits_{\alpha\in A}\{-D_{x_1}v^\varepsilon \cdot \xi(x,\alpha) - \lambda(t,x,\alpha)\} \\ \qquad = D_{x_1}v^\varepsilon \cdot \eta\left(x, \dfrac{x_2}{\varepsilon}\right) + D_{x_2}v^\varepsilon \cdot f_2\left(x, \dfrac{x_2}{\varepsilon}\right) + \gamma\left(x, \dfrac{x_2}{\varepsilon}\right) & \text{in } (0,+\infty)\times\mathbb{R}^N \\ v^\varepsilon(0,x) = h(x) & \text{in } \mathbb{R}^N. \end{cases} \tag{1.5.18}$$

In this case the limit problem satisfied by the limit of v^ε is explicit; the following Corollary is an immediate application of Theorem 1.5.1 and Corollary 1.5.1.

Corollary 1.5.4. *Assume* (1.5.13)*. Put*

$$\bar{f}_1(x,\alpha) = \xi(x,\alpha) + \int_{\mathbb{T}^{N_2}} \eta(x,y)\mathrm{d}\mu(y), \qquad \bar{\ell}(t,x,\alpha) = \lambda(t,x,\alpha) + \int_{\mathbb{T}^{N_2}} \gamma(x,y)\mathrm{d}\mu(y). \tag{1.5.19}$$

Then, as $\varepsilon \to 0$ *the sequence* $v^\varepsilon(t,x)$ *of solutions of* (1.5.18) *converges, locally uniformly in* $(0,+\infty) \times \mathbb{R}^N$ *to*

$$\bar{v}(t,x) = \inf_{\alpha \in \mathcal{A}} \left\{ \int_0^t \bar{\ell}\,(s, x(s), \alpha(s)) + h(x(t)) \right\}, \tag{1.5.20}$$

where $x(\cdot)$ *is subject to*

$$\begin{cases} \dot{x}_1(t) = \bar{f}_1\,(x(t), \alpha(t)) \\ \dot{x}_2(t) = 0 \\ x(0) = x. \end{cases} \tag{1.5.21}$$

Proof. By Theorem 1.5.1, and Corollary 1.5.1 $v^\varepsilon(t,x)$ converges locally uniformly in $(0,\infty) \times \mathbb{R}^N$ to the unique solution of problem

$$\begin{cases} \partial_t v + \max\limits_{\alpha \in A} \{ -D_{x_1} v \cdot \xi(x,\alpha) - \lambda(t,x,\alpha) \} \\ \qquad\qquad = \displaystyle\int_{\mathbb{T}^{N_2}} \Big[D_{x_1} v \cdot \eta\,(x,y) + \gamma\,(x,y) \Big] \mathrm{d}\mu(y) & \text{in } (0,+\infty) \times \mathbb{R}^N \\ v(0,x) = h(x) & \text{in } \mathbb{R}^N. \end{cases} \tag{1.5.22}$$

Thanks to (1.5.19), problem (1.5.22) reads as

$$\begin{cases} \partial_t v + \max\limits_{\alpha \in A} \Big\{ -D_{x_1} v \cdot \bar{f}_1(x,\alpha) - \bar{\ell}(t,x,\alpha) \Big\} = 0 & \text{in } (0,+\infty) \times \mathbb{R}^N \\ v(0,x) = h(x) & \text{in } \mathbb{R}^N. \end{cases} \tag{1.5.23}$$

Then, by standard results in optimal control theory, the unique solution of the previous problem is the value function $\bar{v}(t,x)$ defined in (1.5.20). □

Remark 1.5.3. *Note that Corollary 1.5.1 and 1.5.4 are in strict connection with Theorem 1.4.1. In fact, as observed in Remark 1.5.1, since any limiting relaxed control is invariant for the dynamics* (1.5.5)*, if hypothesis* (1.5.13) *holds, the set* Z_0 *is independent of* x *and coincides with the singleton* $\{\mu\}$.

Remark 1.5.4. *The convergence of the trajectories can be deduced from the convergence of the value function similarly to Corollary 1.5.2 and Corollary 1.5.3 with no changes in the proofs.*

1.6 Appendix

1.6.1 Proof of Lemma 1.2.2

To prove the upper semicontinuity of the set-valued map $x \rightsquigarrow Z(x)$ we will follow several steps. First we will show that, thanks to Proposition 1.2.2, it is equivalent to prove that the latter map is outer semicontinuous[4] for which a sequential characterization exists and will be useful. Then assuming the time bounded controllability (see Definition 1.2.1), we can represent the set $Z(x)$ in a different way that will turn out to be handy for using the sequential characterization proven in the previous step.

Lemma 1.6.1. *The set-valued map $x \rightsquigarrow Z(x)$ is upper semicontinuous if and only if it is outer semicontinuous.*

Proof. Thanks to Proposition 1.2.2, $Z(x)$ is a compact subset of $\mathcal{P}(\mathbb{T}^{N_2} \times A_2)$ in the weak star topology, and hence is locally bounded. Therefore, a result in [148, Theorem 5.19, p.160] insures that the outer semicontinuity is equivalent to the upper semicontinuity defined as in [15, Definition 1.1.1.] (see the footnote in Lemma 1.2.2). □

The next Lemma follows from a result in [89, Theorem 2.1], that we recall here for consistency.

Lemma 1.6.2. *Consider the control system* (1.2.5) *for an arbitrary fixed $x \in \mathbb{R}^N$*

$$\dot{y}(t) = f_2(x, y(t), \alpha_2(t)), \quad y(0) = y$$

where the function $f_2(x, \cdot, \cdot) : (y, \alpha_2) \in \mathbb{T}^{N_2} \times A_2 \to \mathbb{R}^{N_2}$ is continuous in (y, α_2), Lipschitz in y, A_2 is a compact metric space, and the controls are Lebesgue measurable functions. If this control system is bounded time controllable (see Definition 1.2.1), then the corresponding limit occupational measure set $Z(x)$ coincides with

$$W(x) := \left\{ \mu \;\middle|\; \mu \in \mathcal{P}(\mathbb{T}^{N_2} \times A_2); \int_{\mathbb{T}^{N_2} \times A_2} \nabla\phi(y) \cdot f_2(x, y, u)\mu(dy, du) = 0 \; \forall \, \phi \in C^1(\mathbb{T}^{N_2}) \right\}, \tag{1.6.1}$$

i.e., $Z(x) = W(x)$.

Before presenting the next Lemma, we shall introduce some notation and definitions which will be used in the sequel. For a fixed x, we shall denote by $\Gamma(t, y_0)$ the set of all occupational measures defined as in §1.2.2

[4]**Definition** (see [148, Definition 5.4, p.152]) A set-valued map $x \rightsquigarrow \mathcal{F}(x)$ is outer semicontinuous at x_o if $\limsup_{x \to x_o} \mathcal{F}(x) \subset \mathcal{F}(x_o)$.

$$\begin{aligned}
\Gamma(t,y_0) \qquad &:= \bigcup_{(y(\cdot),\alpha_2(\cdot))} \left\{ \mu_{t,x,y_0,\alpha_2} \right\} \\
\text{where} \quad \mu_{t,x,y_0,\alpha_2}(Q) \quad &= \frac{1}{t}\int_0^t \delta_{(y(s),\alpha_2(s))}(Q)\mathrm{d}s \\
&= \frac{1}{t}\Big|\{s \in [0,t] \mid (y(s),\alpha_2(s)) \in Q\}\Big|
\end{aligned}$$

where $y(0) = y_0$, Q is any Borel set of $\mathbb{T}^{N_2} \times A_2$ and $|\cdot|$ is the Lebesgue measure. Following [89], we shall treat the set $\mathcal{P}(\mathbb{T}^{N_2} \times A_2)$ as a compact metric space with a metric ρ, which is consistent with its weak convergence topology (see, e.g., [35]) . A sequence $\mu^k \in \mathcal{P}(\mathbb{T}^{N_2} \times A_2)$ converges to $\mu \in \mathcal{P}(\mathbb{T}^{N_2} \times A_2)$ in this metric if and only if

$$\lim_{k\to\infty} \int_{\mathbb{T}^{N_2}\times A_2} q(y,u)d\mu^k(y,u) = \int_{\mathbb{T}^{N_2}\times A_2} q(y,u)d\mu(y,u)$$

for any continuous $q(u,y) : \mathbb{T}^{N_2} \times A_2 \to \mathbb{R}$. A possible choice for such a metric ρ is the one adopted in [89] defined as follows: $\forall\, \mu', \mu'' \in \mathcal{P}(\mathbb{T}^{N_2} \times A_2)$,

$$\rho(\mu',\mu'') := \sum_{l=1}^{\infty} \frac{1}{2^l} \left| \int_{\mathbb{T}^{N_2}\times A_2} q_l(y,u)d\mu'(y,u) - \int_{\mathbb{T}^{N_2}\times A_2} q_l(y,u)d\mu''(y,u) \right|,$$

where $q_l(\cdot),\ l = 1,2,...$ is a sequence of Lipschitz continuous functions which is dense in the unit ball of the space of continuous functions $C(\mathbb{T}^{N_2} \times A_2)$. Therefore, one can define the Hausdorff metric ρ_H on the set of subsets of $\mathcal{P}(\mathbb{T}^{N_2} \times A_2)$ as follows: $\forall\, \Gamma_i \subset \mathcal{P}(\mathbb{T}^{N_2} \times A_2),\ i = 1,2;$

$$\rho_H(\Gamma_1,\Gamma_2) := \max\left\{ \sup_{\mu\in\Gamma_1} \rho(\mu,\Gamma_2),\ \sup_{\mu\in\Gamma_2} \rho(\mu,\Gamma_1) \right\},$$

where $\rho(\mu,\Gamma_i) := \inf_{\mu'\in\Gamma_i} \rho(\mu,\mu')$.
Armed with these tools, we are now ready to prove Lemma 1.6.2.

Proof. Fix an element x. Thanks to [89, Theorem 2.1 (iii)], it is sufficient to prove that

$$\rho_H(\Gamma(S,y'),\Gamma(S,y'')) \le \omega(S),\ \forall\, y',y'' \in \mathbb{T}^{N_2} \tag{1.6.2}$$

for some $\omega(S)$, $\lim_{S\to\infty} \omega(S) = 0$. Indeed, this insures the existence of a limit occupational measure set $\Gamma \equiv Z(x)$, as the limit of sets $\Gamma(S,y)$, $\forall\, y \in \mathbb{T}^{N_2}$, in the Hausdorff metric, and which will turn out to be equal to W as defined in the Lemma 1.6.2 thanks to [89, Theorem 2.1 (ii)]. To show that (1.6.2) holds, we will adapt the proof in [156, 1.5].
Fix two initial conditions $y_{1,o}, y_{2,o}$, and set $\Gamma_i := \Gamma(S, y_{i,o}),\ i = 1,2$. We will prove that

there exists a positive constant C such that $\forall\ \mu'' \in \Gamma_2,\ \rho(\mu'', \Gamma_1) \leq CS^{-1}$ for S large enough. If $\mu'' \in \Gamma_2$, then there exists a control $\alpha_2 \in \mathcal{A}_2$ such that $\mu'' = \mu_{S,y_{2,o},\alpha_2}$. Since the fast subsystem is bounded time controllable, $\exists\ T > 0$ and a control $u^o \in \mathcal{A}_2$ such that the corresponding trajectory starting from $y_{1,o}$ meets the initial position $y_{2,o}$ at some $t_o \leq T$, i.e. $y^{u^o}(t_o; y_{1,o}) = y_{2,o}$. Define a control $u \in \mathcal{A}_2$ as

$$u(t) = \begin{cases} u^o(t), & \text{if } t \leq t_o \\ \alpha_2(t - t_o), & \text{if } t > t_o \end{cases}$$

We use the corresponding trajectory $y^u(\cdot; y_{1,o})$ to define $\mu' := \mu_{S,y_{1,o},u}$. Since $y^u(\cdot; y_{1,o})$ and $y^{\alpha_2}(\cdot; y_{2,o})$ agree $\forall\ S \geq t_o$, we have

$$\begin{aligned}
\rho(\mu'', \Gamma_1) &\leq \rho(\mu'', \mu') \\
&= \sum_{l=1}^{\infty} \frac{1}{2^l} \left| \int_{\mathbb{T}^{N_2} \times A_2} q_l(y,u)d\mu'(y,u) - \int_{\mathbb{T}^{N_2} \times A_2} q_l(y,u)d\mu''(y,u) \right| \\
&= \frac{1}{S} \sum_{l=1}^{\infty} \frac{1}{2^l} \left| \int_0^S q_l\Big(y^u(t; y_{1,o}), u(t)\Big)dt - \int_0^S q_l\Big(y^{\alpha_2}(t; y_{2,o}), \alpha_2(t)\Big)dt \right| \\
&\leq \frac{1}{S} \sum_{l=1}^{\infty} \frac{1}{2^l} \left(\int_0^{t_o} \left| q_l\Big(y^u(t; y_{1,o}), u(t)\Big) \right| dt + \int_{S-t_o}^{S} \left| q_l\Big(y^{\alpha_2}(t; y_{2,o}), \alpha_2(t)\Big) \right| dt \right) \\
&\leq \frac{2t_o L}{S} \leq \frac{2TL}{S}
\end{aligned}$$

where $L = \sum\limits_{l=1}^{\infty} \frac{1}{2^l} \|q_l\|_\infty$, which is finite since q_l is a sequence of Lipschitz continuous functions dense in the unit ball of $C(\mathbb{T}^{N_2} \times A_2)$. Hence, (1.6.2) follows immediately. $\square$

Finally, to prove the Lemma (1.2.2) it suffices, thanks to Lemma 1.6.1 and Lemma 1.6.2, to show that the set-valued map $x \rightsquigarrow W(x)$ is indeed outer semicontinuous.

Proof. We want to show that $\limsup\limits_{x \to \bar{x}} W(x) \subset W(\bar{x})$, for all $\bar{x}$, that is

$$\forall\ x_n \to \bar{x},\ \forall \mu_n \in W(x_n),\ \ \mu_n \overset{*}{\rightharpoonup} \bar{\mu} \Rightarrow \bar{\mu} \in W(\bar{x})$$

Let ϕ be in $C^1(\mathbb{T}^{N_2})$, and $\mu_n \in W(x_n)$ such that $x_n \to \bar{x}$ and $\mu_n \overset{*}{\rightharpoonup} \bar{\mu}$. We have

$$\int_{\mathbb{T}^{N_2} \times A_2} \nabla\phi(y) \cdot f_2(x_n, y, u)d\mu_n(y,u) = 0,\ \forall\ n, \forall\ \phi \in C^1.$$

We need to show that

$$M(\phi) := \int_{\mathbb{T}^{N_2}\times A_2} \nabla\phi(y)\cdot f_2(\bar{x},y,u)d\bar{\mu}(y,u) = 0,\ \forall\phi\in C^1$$

We omit $\mathbb{T}^{N_2}\times A_2$ in the integral for clarity of notation, and denote by L_{f_2} the Lipschitz constant of f_2. We have

$$\begin{aligned}
|M(\phi)| &= \left|\int \nabla\phi(y)\cdot(f_2(\bar{x},y,u)-f_2(x_n,y,u))d\bar{\mu}(y,u) + \int \nabla\phi(y)\cdot f_2(x_n,y,u)d(\bar{\mu}-\mu_n)(y,u)\right| \\
&\leq \underbrace{\int |\nabla\phi(y)|d\bar{\mu}(y,u)L_{f_2}}_{=:C(\phi)}|\bar{x}-x_n| + \left|\int \nabla\phi(y)\cdot f_2(x_n,y,u)d(\bar{\mu}-\mu_n)(y,u)\right| \\
&\leq C(\phi)|\bar{x}-x_n| + \left|\int \nabla\phi(y)\cdot(f_2(x_n,y,u)-f_2(\bar{x},y,u))d(\bar{\mu}-\mu_n)(y,u)\right| + \\
&\qquad\qquad + \left|\int |\nabla\phi(y)\cdot f_2(\bar{x},y,u)|d(\bar{\mu}-\mu_n)(y,u)\right| \\
&\leq C(\phi)|\bar{x}-x_n| + \left|\int |\nabla\phi(y)|d(\bar{\mu}-\bar{\mu}_n)(y,u)L_f|\bar{x}-x_n|\right| + \\
&\qquad\qquad + \left|\int |\nabla\phi(y)\cdot f_2(\bar{x},y,u)|d(\bar{\mu}-\mu_n)(y,u)\right| \\
&\leq \underbrace{\left(C(\phi) + L_{f_2}\left|\int |\nabla\phi(y)|d(\bar{\mu}-\bar{\mu}_n)(y,u)\right|\right)}_{=:C(\phi)}|\bar{x}-x_n| + \\
&\qquad\qquad + \left|\int |\nabla\phi(y)\cdot f_2(\bar{x},y,u)|d(\bar{\mu}-\mu_n)(y,u)\right|
\end{aligned}$$

Now, as $n\to\infty$, the first term goes to 0 since the coefficient (again denoted by) $C(\phi)$ is constant, and in the second term, the function $(y,u)\mapsto|\nabla\phi(y)\cdot f_2(\bar{x},y,u)|$ defines a bounded and continuous function on $\mathbb{T}^{N_2}\times A_2$ and $\mu_n \overset{*}{\rightharpoonup} \bar{\mu}$. This insures that $M(\phi)=0$, for all $\phi\in C^1$, that is $\bar{\mu}\in W(\bar{x})$.

Therefore, the set-valued map $x\rightsquigarrow W(x)$ is outer semicontinuous. And thanks to Lemma 1.6.2, $x\rightsquigarrow Z(x)$ is outer semicontinuous. Finally, using Lemma 1.6.1, $x\rightsquigarrow Z(x)$ is upper semicontinuous, which is the desired result. $\square$

1.6.2 A counterexample to homogenization

We revisit a counterexample to uniform convergence of solutions of singular perturbation problems presented in [4, Section 8]. Consider the following dynamics in $\mathbb{R}^2$

$$\begin{cases} \dot{x}_1(t) = \cos\left(\frac{x_2(t)}{\varepsilon}\right) + 1, & x_1(0) = x_1 \\ \dot{x}_2(t) = x_1(t) + \alpha_2(t), & x_2(0) = x_2 \end{cases} \qquad |\alpha_2| \leq 1 \tag{1.6.3}$$

and the terminal cost functional given by

$$J(t, x_1) = h(x_1(t))$$

where $h : \mathbb{R} \to \mathbb{R}$ is a strictly increasing continuous function. The value function is then

$$v^\varepsilon\big(t, x_1, x_2\big) = \inf\left\{ h(x_1(t)) \,\middle|\, x_1(\cdot) \text{ solves } (1.6.3) \right\}.$$

Proposition 1.6.1. *The family* $v^\varepsilon\big(t, x_1, x_2\big)$ *of solutions of*

$$\begin{cases} \partial_t v^\varepsilon - v^\varepsilon_{x_1}\left(\cos\left(\frac{x_2}{\varepsilon}\right) + 1\right) + \left(|v^\varepsilon_{x_2}| - x_1 v^\varepsilon_{x_2}\right) = 0 & \textit{in } (0, +\infty) \times \mathbb{R}^2 \\ v^\varepsilon\big(0, (x_1, x_2)\big) = h(x_1) & \textit{in } \mathbb{R}^2. \end{cases} \tag{1.6.4}$$

converges uniformly in any compact subset of $[0, +\infty) \times \mathbb{R} \backslash \{1\} \times \mathbb{R}$*, but not on compact subsets of* $[0, +\infty) \times \mathbb{R} \times \mathbb{R}$.

Proof. The Hamiltonian in (1.6.4) is

$$H\left(x_1, \frac{x_2}{\varepsilon}, p_1, p_2\right) = -p_1\left(\cos\left(\frac{x_2}{\varepsilon}\right) + 1\right) + \left(|p_2| - x_1 p_2\right). \tag{1.6.5}$$

We look for a solution of (1.6.4) of the form $v^\varepsilon\big(t, x_1, x_2\big) = u^\varepsilon\big(t, x_1, \frac{x_2}{\varepsilon}\big)$. Hence $u^\varepsilon(t, x_1, y)$ solves the re-scaled equation

$$\begin{cases} \partial_t u^\varepsilon + H\left(x_1, y, u^\varepsilon_{x_1}, \dfrac{1}{\varepsilon} u^\varepsilon_y\right) = 0 & \text{in } (0, +\infty) \times \mathbb{R}^2 \\ u^\varepsilon(0, x_1, y) = h(x_1) & \text{in } \mathbb{R}^2, \end{cases} \tag{1.6.6}$$

whose unique solution is the value function

$$u^\varepsilon(t, x_1, y) = \inf\left\{h(x_1(t))\right\}$$

for the singularly perturbed control system

$$\begin{cases} \dot{x}_1(t) = \cos y(t) + 1 & x_1(0) = x_1 \\ \dot{y}(t) = \frac{1}{\varepsilon}\left[x_1(t) + \alpha_2(t)\right], & y(0) = y \end{cases} \qquad |\alpha_2| \leq 1. \tag{1.6.7}$$

System (1.6.7) is obtained from (1.6.3) by introducing the fast variable $y = x_2/\varepsilon$ and observing that - since in (1.6.3) the drift for x_2, i.e. $f_2(x_1, \alpha_2) = x_1 + \alpha_2$, does not depend on x_2 - we can ignore the dynamics for x_2. Consequently we expect the limit of v^ε to be independent of x_2.

For any $x_1 \in \mathbb{R}$ the fast dynamics $\dot{y}(t) = x_1 + \alpha_2(t)$ is bounded time controllable (see Definition 1.2.1), thus H is ergodic with effective Hamiltonian $\bar{H}(x_1, p_1)$. Notice that H is not coercive with respect to p_2 when $|x_1| > 1$ because, in this case, the quantity $-x_1 p_2 + |p_2|$ can be negative for some p_2.

Taking into account formula (1.2.6) and proceeding along the lines of [4, Lemma 8.2], we come to the following expression for $\bar{H}$:

$$\bar{H}(x_1, p_1) = \sup\left\{ -p_1 v \;\middle|\; v \in \left[\tilde{f}(x_1), 2 - \tilde{f}(x_1)\right]\right\} \tag{1.6.8}$$

where $\tilde{f}(x_1) := 1 - \cos\theta$, and θ is the unique solution of the equation $\tan\theta - \theta = \frac{\pi}{2}\left(|x_1| - 1\right)^+$. Note that $\tilde{f}(x_1)$ is not Lipschitz at $x_1 = 1$.

Arguing again as in [4, Section 8], and taking into account that h is strictly increasing, we observe that the effective problem

$$\begin{cases} \partial_t u + \bar{H}(x_1, u_{x_1}) = 0 & \text{in } (0, +\infty) \times \mathbb{R} \\ u(0, x_1) = h(x_1) & \text{in } \mathbb{R}, \end{cases} \tag{1.6.9}$$

is solved by the value function $\bar{u}(t, x_1) = \inf\{h(x_1(t))\}$, where the infimum is taken over solutions of the dynamics

$$\dot{x}_1(t) = \tilde{f}(x_1(t)), \qquad x_1(0) = x_1. \tag{1.6.10}$$

The dynamics above has exactly one solution if and only if $x_1 \neq 1$; a flow $g_t x_1$ can be consequently associated in this case. If instead $x_1 = 1$ there are infinitely many solutions, because the drift $\tilde{f}$ is not Lipschitz continuous at that point. In this case we denote by $x_1^+(t)$ and $x_1^-(t)$ the largest and the smallest solution of (1.6.10) respectively; both x_1^+ and x_1^- can be computed explicitly in this example. Lemma 8.3 in [4] provides an explicit representation of the maximal supersolution and minimal subsolution of (1.6.9), denoted by $u^\sharp(t,x_1)$ and $u_\sharp(t,x_1)$ respectively, in terms of solutions of (1.6.10):

$$u^\sharp(t,x_1) = u_\sharp(t,x_1) = h(g_t x_1), \qquad \text{in } [0,\infty)\times\mathbb{R}\backslash\{1\}, \tag{1.6.11}$$

$$u_\sharp(t,1) = h(x_1^-(t)) > h(x_1^+(t)) = u^\sharp(t,1) \qquad \text{for any } t>0. \tag{1.6.12}$$

Using the formulas above we observe that $u_\sharp$ coincides with $(u^\sharp)_*$, the l.s.c. envelop of $u^\sharp$, which is the larger l.s.c. less than or equal to $u^\sharp$. Since both $u_\sharp$ and $\underline{v}_*$ are l.s.c. and $\underline{v}_* \le u^\sharp$, we conclude that $u_\sharp \ge \underline{v}_*$ and then, by (1.2.4), $u_\sharp = \underline{v}_*$; similarly, $\bar{v}^* = u^\sharp$. In particular, as noticed before, $\bar{v}^*$ and $\underline{v}_*$ do not depend on x_2. The information gathered, together with (1.2.4), give

$$u_\sharp = \underline{v}_* \le \bar{v}^* = u^\sharp \quad \text{in } [0,+\infty)\times\mathbb{R}. \tag{1.6.13}$$

Now, by (1.6.11), the maximal supersolution and the minimal subsolution agree everywhere except for $x_1 = 1$. Then $v^\varepsilon(t,x_1,x_2)$ converges uniformly on compact subsets of $[0,+\infty)\times\mathbb{R}\backslash\{1\}\times\mathbb{R}$. On the other hand, by (1.6.12) and (1.6.13), $\underline{v}_*(t,1) < \bar{v}^*(t,1)$ for any t, thus v^ε cannot converge in any compact neighborhood of any point in $[0,+\infty)\times\{1\}\times\mathbb{R}$. $\square$

Chapter 2

Deep relaxation via singular perturbations of stochastic control problems

2.1 Introduction

We are interested in studying the asymptotic behavior as $\varepsilon \to 0$ of a system of controlled and singularly perturbed stochastic differential equations

$$
\begin{aligned}
\mathrm{d}X_t &= f(X_t, Y_t, u_t)\,\mathrm{d}t + \sqrt{2}\,\sigma^{\varepsilon}(X_t, Y_t, u_t)\,\mathrm{d}W_t, \\
\mathrm{d}Y_t &= \frac{1}{\varepsilon}\,b(X_t, Y_t)\,\mathrm{d}t + \sqrt{\frac{2}{\varepsilon}}\,\varrho(X_t, Y_t)\,\mathrm{d}W_t,
\end{aligned}
\tag{$SDE\left(\frac{1}{\varepsilon}\right)$}
$$

where $X_t \in \mathbb{R}^n$ is the *slow* dynamics, $Y_t \in \mathbb{R}^m$ is the *fast* dynamics, u_t is the control taking values in a given compact set U and W_t is a multidimensional Brownian motion. We will allow the components of the drift and the diffusion of the slow dynamics to depend at most linearly on the fast process Y. And while the diffusion coefficient of the process X can be degenerate (i.e. $\sigma^{\varepsilon} = 0$ is possible), the diffusion coefficient of the process Y is required to be nondegenerate. The precise assumptions are give in Section 2.2. We carry our analysis in the context of stochastic optimal control problems of the form

$$
\sup_u \; J(t,x,y,u) := \mathbb{E}\left[e^{\lambda(t-T)} g(X_T, Y_T) + \int_t^T \ell(s, X_s, Y_s, u_s) e^{\lambda(s-T)} \mathrm{d}s \,\middle|\, X_t = x, Y_t = y \right].
$$

Such a quantity is denoted by $V^{\varepsilon}(t,x,y)$ and refers to the value function which solves in the viscosity sense a fully nonlinear parabolic degenerate PDE in $(0,T) \times \mathbb{R}^n \times \mathbb{R}^m$.

Our motivation is the Stochastic Gradient Descent algorithm in the context of Deep Learning and Big Data analysis, where one needs to consider the possible unboundedness of the data and the state space. In particular, such a kind of singularly perturbed system of SDEs has been considered (with no control) in [64] and where it is used to build an algorithm for a Stochastic Gradient Descent. In this chapter, we show how our results allow us to prove such a convergence with or without a control under rather general assumptions. This also captures the previous results in [21, 23] where the coefficients in the slow variable are assumed to be bounded with respect to the fast variables. Moreover, we rely in our analysis on arguments and methods sometimes different from those in [21, 23] and borrowed from probability theory, which were key ingredients for handling unboundedness of the data and the state domain.

Let us also mention that our results also recover a large range of applications in finance, e.g. models of pricing and trading derivative securities in financial markets with stochastic volatility as it has been done in [23], or applications in economics and advertising theory as it is the case in [21].

There is a wide literature on singular perturbations for control systems that goes back to the late 60's [112], and also for diffusion processes, with and without control, and different models with fast variables have been studied since then both in deterministic and stochastic settings and using methods of probability, analysis, measure theory, or control. We refer the reader to the introduction in [21, 23] where a large but non-exhaustive list of references on these topics are provided.
We will however mention two types of results concerning the singular perturbations problem for stochastic differential equations

without control: In a series of papers [141, 142, 143] Pardoux and Veretennikov tackled the problem of approximation of diffusions from the point of view of Poisson equation where the differential operator is the generator of the singularly perturbed stochastic differential equations (without control). And hence they proved the convergence in distribution (in [141]) under quite general assumptions using the generator of the latter. Other convergence results are in [142, 143]

with control: In [47] (see also [48] and the references therein) Borkar and Gaitsgory studied the convergence of the singularly perturbed stochastic differential equations with control both in the slow and in the fast variables. They relied on the *Limit Occupational Measure Set* to prove convergence in law to the averaged system. They also recovered stronger convergence under further assumptions.

In the present chapter, we analyse the convergence of singular perturbations in the framework of stochastic control and we rely on the associated Hamilton-Jacobi-Bellman equation. We also insist on making assumptions that can be easily checked and which do comply with the applications we are interested in.

Overview.

We are interested in the behavior as $\varepsilon \to 0$ of the controlled and singularly perturbed system of stochastic differential equations ($SDE\left(\frac{1}{\varepsilon}\right)$). We start first by embedding this system in a family of control problems that we identify through there value function $V^\varepsilon(t,x,y)$. The latter is characterized (**Proposition 2.2.1**) as the unique viscosity solution to a Hamilton-Jacobi-Bellman equation. Then we rely on ergodicity of the fast process to construct the effective Hamiltonian (**Proposition 2.3.2**) and initial data (**Proposition 2.3.3**) that allow us to set the limit Cauchy problem. Using methods from homogenization and viscosity theory, we prove (**Theorem 2.4.1**) the convergence of V^ε to the unique viscosity solution of the limit Hamilton-Jacobi equation. And only after proving the effective Hamiltonian is of Bellman type (**Proposition 2.5.1**), by means of a selection argument, we can consider the limit PDE as a Hamilton-Jacobi-Bellman equation and we can identify (**Theorem 2.5.2**) its unique viscosity solution with the value function of an optimal control problem with a stochastic differential inclusion. At this stage we have the convergence of the value function of a family of optimal control problems to a value function of another family of control problems. We use the latter result to finally make precise (**Theorem 2.5.3** and **Theorem 2.5.4**) the limit system of $SDE\left(\frac{1}{\varepsilon}\right)$.

Organization of the chapter.

This chapter is organized as follows. In **Section 2.2** we present the two scale stochastic control problem and the assumptions that will stand all along this chapter, together with the associated Hamilton-Jacobi-Bellman equation. **Section 2.3** is devoted to the study of the fast variables Y. We will state and prove some useful lemmas that are of their own interest, using probabilistic arguments. Then we will construct the effective Hamiltonian and initial data in a new way, different from what it has been done previously in the literature. This will be an important step for the convergence result of the value function that we next show in **Section 2.4**. Indeed, we will rely in this section on viscosity methods and will provide a new adaptation of the perturbed test function first introduced in [78], in order to fit our unbounded context. Finally, in **Section 2.5**, We show how our result applies to the control of stochastic gradient descent. In order to do so, we show that the effective Hamiltonian enjoys an exchange property, using a selection argument borrowed from measure theory before we construct an effective optimal control problem with a stochastic differential inclusion whose value function solves in the viscosity sense the effective Cauchy problem of section 2.4. And only then, we can provide the convergence results of the controlled and singularly perturbed trajectories.

2.2 The two scale stochastic control problem

2.2.1 The stochastic system

Let $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ be a complete filtered probability space and let $(W_t)_t$ be an $\mathcal{F}_t$-adapted standard r-dimensional Brownian motion. We consider the following stochastic control system

$$\begin{cases} \mathrm{d}X_t = f(X_t, Y_t, u_t)\,\mathrm{d}t + \sqrt{2}\,\sigma^\varepsilon(X_t, Y_t, u_t)\,\mathrm{d}W_t, \quad X_0 = x \in \mathbb{R}^n \\ \mathrm{d}Y_t = \dfrac{1}{\varepsilon}\,b(X_t, Y_t)\,\mathrm{d}t + \sqrt{\dfrac{2}{\varepsilon}}\,\varrho(X_t, Y_t)\,\mathrm{d}W_t, \quad Y_0 = y \in \mathbb{R}^m \end{cases} \tag{2.2.1}$$

For a given compact set U, $f : \mathbb{R}^n \times \mathbb{R}^m \times U \to \mathbb{R}^n$, $\sigma^\varepsilon : \mathbb{R}^n \times \mathbb{R}^m \times U \to \mathbb{M}^{n,r}$, $b : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ and $\varrho : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^{m,r}$ are continuous functions, Lipschitz continuous in (x, y) uniformly w.r.t. $u \in U$ and $\varepsilon > 0$ and with linear growth in both x and y, that is

$$\text{for some } \ C > 0, \quad |f(x, y, u)|, \|\sigma^\varepsilon(x, y, u)\| \le C(1 + |x| + |y|), \quad \forall\, x, y,\ \forall \varepsilon > 0 \tag{2.2.2}$$

$$\text{for some } \ C > 0, \quad |b(x, y|, \|\varrho(x, y)\| \le C(1 + |x| + |y|), \quad \forall\, x, y, \tag{2.2.3}$$

Moreover we assume that

$$\lim_{\varepsilon \to 0} \sigma^\varepsilon(x, y, u) = \sigma(x, y, u) \quad \text{locally uniformly,} \tag{2.2.4}$$

where $\sigma : \mathbb{R}^n \times \mathbb{R}^m \times U \to \mathbb{M}^{n,r}$ satisfies the same conditions as σ^ε. Finally, we assume that a stronger[1] version of the *recurrence condition*, introduced by Pardoux and Veretennikov in [141] and usually called Khasminskii's assumption, holds for the fast variables $Y_\cdot$, that is the drift satisfies

$$\exists\, A, R > 0 \text{ s.t. } \quad \langle y, b(x, y)\rangle < -A\|y\|, \quad \forall\, \|y\| \ge R,\ \forall\, x \in \mathbb{R}^n \tag{2.2.5}$$

[1]The assumption on the drift in the recurrence condition in [141] is: $\lim\limits_{|y| \to \infty} \sup\limits_{x \in \mathbb{R}^n} \langle y, b(x, y)\rangle = -\infty$ uniformly in x. This is satisfied as soon as (2.2.5) holds.

and the diffusion ϱ driving the fast variables Y_t is such that $\varrho\varrho^\top$ is uniformly bounded and non degenerate, i.e.

$$\exists\ \underline{\Lambda},\ \overline{\Lambda}\ > 0, \ \text{ s.t. } \ \underline{\Lambda}\|\xi\|^2 \leq \xi\varrho(x,y)\varrho^\top(x,y)\cdot\xi = |\xi\varrho(x,y)|^2 \leq \overline{\Lambda}\|\xi\|^2,\ \forall\, x,y,\xi. \tag{2.2.6}$$

And we will not make any nondegeneracy assumption on the matrix $\sigma^\varepsilon, \sigma$, so the case $\sigma \equiv 0$ is allowed.

2.2.2 The optimal control problem

We define the following pay off functional for a finite horizon optimal control problem associated to system (2.2.1) for $t \in [0,T]$

$$J(t,x,y,u) := \mathbb{E}\left[e^{\lambda(t-T)}g(X_T,Y_T) + \int_t^T \ell(s,X_s,Y_s,u_s)e^{\lambda(s-T)}\mathrm{d}s \,\middle|\, X_t = x,\ Y_t = y\right], \tag{2.2.7}$$

The associated value function is

$$V^\varepsilon(t,x,y) := \sup_{u\in\mathcal{U}} J(t,x,y,u), \quad \text{subject to } \ (2.2.1)\,. \tag{ $OCP(\varepsilon)$ }$$

The discount factor is $\lambda \geq 0$. The utility function $g : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ and the running cost $\ell : [0,T] \times \mathbb{R}^n \times \mathbb{R}^m \times U \to \mathbb{R}$ are continuous functions and satisfy

$$\exists\ K > 0 \ \text{ such that } \ |g(x,y)|, |\ell(s,x,y,u)| \leq K(1 + |x|^2 + |y|^2),\ \forall x,y. \tag{2.2.8}$$

The set of admissible control functions $\mathcal{U}$ is the standard one in stochastic control problems, i.e. it is the set of $\mathcal{F}_t$-progressively measurable processes taking values in U.

2.2.3 The HJB equation

The HJB equation associated via Dynamic Programming to the value function V^ε is

$$-V^\varepsilon_t + F^\varepsilon\left(t,x,y,V^\varepsilon,D_xV^\varepsilon,\frac{D_yV^\varepsilon}{\varepsilon},D^2_{xx}V^\varepsilon,\frac{D^2_{yy}V^\varepsilon}{\varepsilon},\frac{D^2_{x,y}V^\varepsilon}{\sqrt{\varepsilon}}\right) = 0, \quad \text{in } \ (0,T)\times\mathbb{R}^n\times\mathbb{R}^m, \tag{2.2.9}$$

complemented with the obvious terminal condition

$$V^\varepsilon(T,x,y) = g(x,y) \tag{2.2.10}$$

This is a fully nonlinear degenerate parabolic equation (strictly parabolic in the y variables by the assumption (2.2.6)). We denote by $\mathbb{M}^{n,m}$ (respec. $\mathbb{S}^n$) the set of matrices of n rows and m columns (respec. the subset of n-dimensional squared symmetric matrices). The Hamiltonian $F^\varepsilon : [0,T] \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{S}^n \times \mathbb{S}^m \times \mathbb{M}^{n,m} \to \mathbb{R}$ is defined as

$$F^\varepsilon(t,x,y,r,p,q,M,N,Z) := H^\varepsilon(t,x,y,p,M,Z) - \mathcal{L}(x,y,q,N) + \lambda r, \tag{2.2.11}$$

where

$$H^\varepsilon(t,x,y,p,M,Z) := \min_{u\in U} \left\{ -\mathrm{trace}(\sigma^\varepsilon \sigma^{\varepsilon\top} M) - f\cdot p - 2\mathrm{trace}(\sigma^\varepsilon \varrho^\top Z^\top) - \ell \right\} \tag{2.2.12}$$

where σ^ε, f are computed at (x,y,u), $\ell = \ell(t,x,y,u)$ and $\varrho = \varrho(x,y)$, and

$$\mathcal{L}(x,y,q,N) := b(x,y)\cdot q + \mathrm{trace}(\varrho(x,y)\varrho^\top(x,y)N) \tag{2.2.13}$$

We define also the Hamiltonian H which is as H^ε where σ^ε is replaced by σ

$$H(t,x,y,p,M,Z) := \min_{u\in U} \left\{ -\mathrm{trace}(\sigma\sigma^\top M) - f\cdot p - 2\mathrm{trace}(\sigma\varrho^\top Z^\top) - \ell \right\} \tag{2.2.14}$$

Our first result is the following proposition whose proof is, mutatis mutandis, the same as [23, Proposition 3.1] or in [21].

Proposition 2.2.1. *([21, Proposition 2.1]) For any $\varepsilon > 0$, the function V^ε in ($OCP(\varepsilon)$) is the unique continuous viscosity solution to the Cauchy problem* (2.2.9)-(2.2.10) *with at most quadratic growth in x and y, i.e.,*

$$\exists\, K > 0 \ \text{ such that } \ |V^\varepsilon(t,x,y)| \le K(1+|x|^2+|y|^2), \quad \forall\, t\in[0,T],\ x\in\mathbb{R}^n,\ y\in\mathbb{R}^m. \tag{2.2.15}$$

Moreover the functions V^ε are locally equibounded.

It is important to note that V^ε is not bounded in y, but has a quadratic growth. This comes from the assumption (2.2.2) together with (2.2.8).

2.3 Ergodicity of the fast variables and the effective data

Consider the diffusion processes in $\mathbb{R}^m$ obtained by putting $\varepsilon = 1$ in (2.2.1) and fixing $x \in \mathbb{R}^n$

$$\mathrm{d}Y_t = b(x, Y_t)\,\mathrm{d}t + \sqrt{2}\varrho(x, Y_t)\,\mathrm{d}W_t, \quad Y_0 = y \in \mathbb{R}^m \tag{2.3.1}$$

called *fast subsystem.* To recall the dependence on the parameter x, we will denote the process in (2.3.1) as $Y^x_\cdot$. Observe that its infinitesimal generator is $\mathcal{L}_x w := \mathcal{L}(x, y, D_y w, D^2_{yy} w)$ with $\mathcal{L}$ defined by (2.2.13). In this section, we will rely on the invariant measure for the process $Y^x_\cdot$ in (2.3.1) (see for example [129])

Definition 2.3.1. *We say that μ_x is an invariant measure for the process* (2.3.1)*, if μ_x is a probability measure satisfying*

$$\int_{\mathbb{R}^m} \mathbb{E}[f(Y_t)\,|Y_0 = y]\,d\mu_x(y) = \int_{\mathbb{R}^m} f(y)\,d\mu_x(y), \quad \forall\, t > 0 \tag{2.3.2}$$

for all bounded Borel function f in $\mathbb{R}^m$

Other characterizations of the invariant measure are possible, for instance as being the stationary solution of the Fokker-Planck equation $\mathcal{L}^*_x \mu_x = 0$, where $\mathcal{L}^*_x$ is the adjoint operator to $\mathcal{L}_x$.

It is well known that the assumption (2.2.5) on the drift insures the existence of an invariant measure, and its uniqueness follows from the non degeneracy assumption (2.2.6) on the diffusion ϱ. This is proven for instance in [157] (see also [141, 142, 143]). Another proof of existence and uniqueness of the invariant measure can be found in [23] assuming a Lyapunov-type condition. This is also compatible with our setting noticing that the recurrence condition implies the latter assumption (see [21]). We denote by μ_x its unique invariant probability measure, and we say that the process $Y^x_\cdot$ is ergodic.

For simplicity of notation only, we will drop in this section the dependency on x in the fast subsystem and its coefficients, and we recall instead its dependency on its initial position. Therefore we write

$$\mathrm{d}Y_y(t) = b(Y_y(t))\,\mathrm{d}t + \sqrt{2}\varrho(Y_y(t))\,\mathrm{d}W_t, \quad Y_y(0) = y \in \mathbb{R}^m \tag{2.3.3}$$

It should be understood here that $x \in \mathbb{R}^n$ is an arbitrary fixed parameter on which all the coefficients still depend. Similarly, we denote by μ its unique invariant measure. And when there is no confusion, we drop the dependency on the initial position and simply write Y_t (instead of $Y_y(t)$).

This section will be devoted to the fast variables Y, and we will make use of the index n to refer to a sequence; not to be confused with the dimension of the slow variables $x \in \mathbb{R}^n$.

2.3.1 Useful Proposition and Lemmas

The first result we need is the following Lipschitz regularity of the invariant measure μ_x, which guarantees local Lipschitz regularity in x of the effective Hamiltonian (in Proposition 2.3.2) and the effective initial data (in Proposition 2.3.3). This is crucial for the convergence result in §2.4.

Proposition 2.3.1. *([21, Proposition 2.3]) Besides the standing assumptions, assume that*

$$b \in C^1(\mathbb{R}^n \times \mathbb{R}^m, \mathbb{R}^m) \quad \text{and} \quad \varrho \in C^1(\mathbb{R}^n \times \mathbb{R}^m, \mathbb{M}^{m,r}) \tag{2.3.4}$$

with all their derivatives bounded and Hölder continuous in y uniformly in x. Then the invariant measure μ_x of the process $Y^x_\cdot$ in (2.3.1) *has a density $\varphi_x(y)$ and there exist $k > 1$, $C > 0$, such that*

$$|\varphi_{x_1}(y) - \varphi_{x_2}(y)| \leq C\frac{1}{1+|y|^k}|x_1 - x_2|, \quad \forall\, x_1, x_2 \in \mathbb{R}^n,\ y \in \mathbb{R}^m. \tag{2.3.5}$$

***Proof*.** The proof can be found in [142, Theorem 6]. We refer also to [21, Remark 2.2] for a connection with a similar result in [47, Proposition 5.2]. □

In what follows, we will also need a result stronger than just ergodicity of the fast subsystem, that is a convergence result of the probability law of $Y^x_\cdot$ towards its unique invariant probability measure.

Lemma 2.3.1. *Under assumption* (2.2.5)*, there exists $C > 0$ and for some $d, k > 0$, one has*

$$\|\mathbb{P}_{Y_y(t)}(\cdot) - \mu(\cdot)\|_{TV} \leq C(1 + |y|^d)(1+t)^{-(1+k)}$$

In particular, the invariant measure μ does exist and has finite moments of any order.

We use $\|\mu - \nu\|_{TV}$ for the total variation distance between two probability measures μ, ν defined by

$$\|\mu - \nu\|_{TV} = \sup_{A \in \mathcal{B}} |\mu(A) - \nu(A)|$$

where $\mathcal{B}$ is the class of Borel sets. In particular, $\|\mu\|_{TV} = \int_{\mathbb{R}^m} \mathrm{d}|\mu|$ and is equal to $\mu(\mathbb{R}^m) = \int_{\mathbb{R}^m} \mathrm{d}\mu$ when μ is positive.

Proof *of Lemma 2.3.1.* This is a particular case of the more general result in [157, Theorem 6]. Indeed, the main assumption in [157] is

$$\exists\, M_0 \geq 0,\ r \geq 0 \ \text{ s.t.} \quad \langle b(y), y\rangle \leq -r, \quad \forall\, |y| \geq M_0 \tag{2.3.6}$$

Then introduce the following constants

$$\Lambda_- := \inf_{y\neq 0} \langle \varrho\varrho^*(y)\frac{y}{|y|}, \frac{y}{|y|}\rangle, \qquad \Lambda_+ := \sup_{y\neq 0} \langle \varrho\varrho^*(y)\frac{y}{|y|}, \frac{y}{|y|}\rangle$$
$$\tilde{\Lambda} := \sup_{y} \frac{\operatorname{trace}(\varrho\varrho^*(y))}{m}, \qquad r_0 := [r - (m\tilde{\Lambda} - \Lambda_-)/2]\Lambda_+^{-1}$$

Now Theorem 6 in [157] states that under assumption (2.3.6), with $r_0 > \frac{3}{2}$, one has $\forall\, k \in (0, r_0 - \frac{3}{2})$, $\forall\, d \in (2k+2, 2r_0 - 1)$

$$\|\mathbb{P}_{Y_y(t)}(\cdot) - \mu(\cdot)\|_{TV} \leq C(1 + |y|^d)(1+t)^{-(1+k)}$$

In our case, assumption (2.2.5) guarantees a constant r (in (2.3.6)) as large as we want.

For the finite moments, see [157, eq. (28) in §6] where it is shown that the invariant measure has finite moments of order $d \in (2k+2, 2r_0 - 1)$, where again $k \in (0, r_0 - \frac{3}{2})$. It is enough to use Hölder inequality together with the fact that $\mu(\mathbb{R}^m) = 1$ to prove finite moments of any order $d \geq 1$. □

Remark 2.3.1. *Lemma 2.3.1 holds again when the diffusion coefficient ϱ is degenerate but satisfies a Hörmander type condition. The proof is the same but relies instead on the results in [158].*

The following result describes the behavior of the exit time of the process $Y_y(\cdot)$ in (2.3.3). This will be needed together with the previous Lemma for constructing the limit PDE in the next section. Let $\tau_n^Y := \inf\{t \geq 0 \mid \|Y_y(t)\| \geq n\}$ denote the first exit time of $Y_y(\cdot)$ from the ball centered in 0 with radius n, and where $y = Y_y(0)$ is the initial position.

Lemma 2.3.2. *Under assumption* (2.2.5)*, there exist η, C positive constants and ℓ a positive function such that for any $\delta > 0$ one has*

$$\mathbb{E}\left[e^{-\delta\tau_n^Y}\right] \leq C\frac{\ell(\delta)}{\delta}e^{-n\eta}, \qquad \textit{locally uniformly in } y, \tag{2.3.7}$$

where $\ell(\delta) = 1 + O(\delta)$ when $\delta \to 0^+$. In particular for any $\alpha \geq 0$ and $\beta > 0$, one has

$$\mathbb{E}\left[n^{\alpha} e^{-\frac{1}{n^{\beta}}\tau_n^Y}\right] \leq C n^{\alpha+\beta} e^{-n\eta} \longrightarrow 0 \text{ as } n \to +\infty. \tag{2.3.8}$$

Proof *of Lemma 2.3.2.* In what follows, we will not distinguish between the exit time of the process $Y_\cdot \in \mathbb{R}^m$ from the ball centered in 0 with radius n, and the exit time of the process $\|Y_\cdot\| \in \mathbb{R}$ from the interval $[-n, n]$, and define it by $\tau_n^Y := \inf\{t \geq 0 \mid \|Y(t)\| \geq n\}$. Moreover, the notation τ_n^Y refers to the exit time (of the process $Y_\cdot$ or similarly $\|Y_\cdot\|$) seen as a random variable in the probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ where $\mathbb{P}$ is the reference probability measure. On the other hand, the notation τ_n is a random variable defined on the space of continuous paths on a probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P}_X)$ where X is some stochastic process. Hence, comparing τ_n^Y and τ_n^Z is the same as comparing τ_n when evaluated in probability spaces where the law is respectively $\mathbb{P}_Y$ and $\mathbb{P}_Z$.

Step 0. (A comparison observation for exit times)
Assuming one can find a process $Z_t \in \mathbb{R}$ such that $\|Y(t)\| \leq Z_t$ a.s., then one has $\tau_n^Y \geq \tau_n^Z$ a.s., where τ_n^Y, τ_n^Z are the exit times of Y and Z respectively, and hence

$$\begin{aligned} \|Y_t\| \leq Z_t \ a.s. \ &\Rightarrow \ \tau_n^Y \geq \tau_n^Z, \ a.s. \\ &\Rightarrow \ \mathbb{E}\left[e^{-\delta\tau_n^Y}\right] \leq \mathbb{E}\left[e^{-\delta\tau_n^Z}\right], \quad \forall\, \delta > 0 \end{aligned} \tag{2.3.9}$$

that is,

$$\|Y_t\| \leq Z_t \ a.s. \ \Rightarrow \ \mathbb{E}_{\|Y\|}\left[e^{-\delta\tau_n}\right] \leq \mathbb{E}_Z\left[e^{-\delta\tau_n}\right], \quad \forall\, \delta > 0 \tag{2.3.10}$$

where the expectation now is taken in the probability space defined with the law of $\|Y\|$ and the law of Z respectively. Armed with this observation, we can tackle our problem by first constructing such a process Z, and then by giving an upper-bound for the second inequality in (2.3.10).

Step 1. (Construct the process Z such that $\|Y_t\| \leq Z_t$ a.s.)
We follow, *mutatis mutandis*, the construction of Z in the proof of [93, Proposition 1.4]. Fix A and R positive constants such that (2.2.5) is satisfied, i.e.

$$\forall\, \|y\| \geq R, \quad \langle y, b(y)\rangle \leq -A\|y\|. \tag{2.3.11}$$

Define $h : \mathbb{R}^m \to \mathbb{R}$ as a C^2 function such that $h(y) = \|y\|$ when $\|y\| \geq R$, and $h(y) < R$ otherwise.

Using the notation "$a \vee b = \max\{a; b\}$", we set

$$Z_t := R \vee \|y_o\| + \sqrt{2}M_t - \eta\xi_t + L_t \tag{2.3.12}$$

where

- $y_o \in \mathbb{R}^m$ is the initial condition of Y_t, i.e. $Y_0 = y_o$,
- $M_t = \int_0^t \nabla h(Y_s)^\top \varrho(Y_s) dW_s$, for $t \geq 0$,
- $\xi_t = \int_0^t \|\nabla h(Y_s)^\top \varrho(Y_s)\|^2 ds$ is the quadratic variation of the continuous local martingale M_t,
- η is a positive constant to be made precise,
- L_t is an increasing process (of finite variation) which increases only at times t for which $Z_t = R$, and is of zero value when $Z > R$; such pair (Z, L) is the unique pair of continuous adapted process giving by Skorokhod's lemma (see e.g. [146, chap.VI, §2]): Z is a reflected process and L its compensator.

Notice that when $\|y\| \geq R$, $\nabla h(y) = \frac{y}{\|y\|}$ so $\left\|\nabla h(y)^\top \varrho(y)\right\|^2 = \frac{1}{\|y\|^2} y^\top \varrho(y)\varrho(y)^\top y \leq \overline{\Lambda}$ and hence $d\xi_t \leq \overline{\Lambda}\, dt$ on $\{\|Y_t\| \geq R\}$. On the other hand, define $\tilde{K} := \sup_{\|y\| \leq R} \|\nabla h(y)\|^2$. Then we have $\xi_t \leq (1 \vee \tilde{K})\, \overline{\Lambda}\, t$ for all $t \geq 0$, where $\overline{\Lambda}$ is defined in (2.2.6). We set $K := (1 \vee \tilde{K})\overline{\Lambda}$, and we have

$$0 \leq \xi_t \leq Kt, \quad \forall\, t \geq 0 \tag{2.3.13}$$

Now in order to prove that

$$\|Y_t\| \leq Z_t \quad a.s. \,\forall\, t \geq 0 \tag{2.3.14}$$

we will use the same procedure as in [93] that we adapt to our context. We choose $f \in C^2(\mathbb{R})$ such that

$$\begin{aligned} f(x) > 0 \quad &\text{and} \quad f'(x) > 0, \quad \forall\, x > 0 \\ f(x) = 0, \quad & \qquad\qquad\qquad\quad \forall\, x \leq 0 \end{aligned}$$

We set $a(y) := \varrho(y)\varrho(y)^\top$. According to Itô's formula, for $t \geq 0$,

$$\begin{aligned} \mathrm{d}h(Y_s) &= \left(\nabla h(Y_s)^\top b(Y_s) + \operatorname{trace}\left(a(Y_s) D^2 h(Y_s)\right)\right) \mathrm{d}s + \sqrt{2}\nabla h(Y_s)^\top \varrho(Y_s) \mathrm{d}W_s \\ \mathrm{d}Z_s &= -\eta d\xi_s + \mathrm{d}L_s + \sqrt{2}\mathrm{d}M_s \\ &= -\eta \left\|\nabla h(Y_s)^\top \varrho(Y_s)\right\|^2 \mathrm{d}s + \mathrm{d}L_s + \sqrt{2}\nabla h(Y_s)^\top \varrho(Y_s) \mathrm{d}W_s \end{aligned}$$

that is

$$\mathrm{d}\left(h(Y)-Z\right)_s = \Big(\nabla h(Y_s)^\top b(Y_s) + \mathrm{trace}\left(a(Y_s)D^2h(Y_s)\right) + \eta\left\|\nabla h(Y_s)^\top \varrho(Y_s)\right\|^2\Big)\,\mathrm{d}s - \mathrm{d}L_s \tag{2.3.15}$$

and again by Itô's formula (see e.g. [108, Theorem 3.3, Chap.3, p.149]) one gets

$$\begin{aligned} f(h(Y_t)-Z_t) =& f(h(y_o) - R\vee\|y_o\|) + \int_0^t f'(h(Y_s)-Z_s)\mathrm{d}(h(Y)-Z)_s + \\ &+ \frac{1}{2}\int_0^t f''(h(Y_s)-Z_s)\mathrm{d}\langle h(Y)-Z\rangle_s, \end{aligned}$$

where we recall, for a stochastic process $\mathrm{d}\zeta_t = f(\zeta_t)\mathrm{d}t + \sigma(\zeta_t)\mathrm{d}W_t$, its quadratic variation is defined by $\langle\zeta\rangle_t = \int_0^t \sigma(\zeta_s)\sigma^\top(\zeta_s)\mathrm{d}s$.

And we have $f(h(y_o) - R\vee\|y_o\|) = 0$ by definition of h and f. Moreover, $h(Y_\cdot) - Z_\cdot$ is a continuous process with no Wiener process term, and hence it has zero quadratic variation, i.e. $\mathrm{d}\langle h(Y)-Z\rangle_s = 0$. Now, again by definition of h and Z, we have $h(Y_t)\le Z_t$ on $\{\|Y_t\|\le R\}$, so $\{h(Y_t) > Z_t\} = \{\|Y_t\| > Z_t\}$ is a subset of $\{\|Y_t\| > R\}$. And when $\|y\|\ge R$, we have $\nabla h(y) = \frac{y}{\|y\|}$ and $D^2h(y) = \frac{1}{\|y\|}\left(\mathbb{I}_m - \frac{y\otimes y}{\|y\|^2}\right)$.
Setting $a(y) = (a_{ij}(y))_{1\le i,j\le m}$, we have

$$\begin{aligned} \mathrm{trace}\left(a(y)D^2h(y)\right) &= \sum_{i,j=1}^m a_{ij}(y)(D^2h(y))_{ji} \\ &= \frac{1}{\|y\|}\sum_{i=1}^m a_{ii}(y)\left(1-\frac{y_i^2}{\|y\|^2}\right) - \frac{1}{\|y\|}\sum_{i=1}^m\sum_{\substack{j=1\\ j\ne i}}^m a_{ij}(y)\frac{y_iy_j}{\|y\|^2} \\ &\le \frac{\overline{\Lambda}(m-1)}{\|y\|} - \frac{1}{\|y\|}\sum_{i,j=1}^m a_{ij}(y)\frac{y_iy_j}{\|y\|^2} + \frac{1}{\|y\|}\sum_{i=1}^m a_{ii}(y)\frac{y_i^2}{\|y\|^2} \\ &\le \frac{\overline{\Lambda}\,m}{\|y\|} + \frac{1}{\|y\|}\sum_{i,j=1}^m a_{ij}(y)\frac{|y_i|\,|y_j|}{\|y\|^2} \ \le \frac{\overline{\Lambda}\,m(m+1)}{\|y\|} \end{aligned} \tag{2.3.16}$$

where in the last inequality we have used $|y_i|\le\|y\|$ and $\sum\limits_{i,j=1}^m a_{ij}(y)\le\overline{\Lambda}\,m^2$.

Hence the expression

$$\int_0^t f'(\|Y_s\|-Z_s)\left\{\frac{1}{\|Y_s\|}\langle Y_s, b(Y_s)\rangle + \frac{m(m+1)}{\|Y_s\|}\overline{\Lambda} + \eta\overline{\Lambda}\right\}\mathrm{d}s - \int_0^t f'(\|Y_s\|-Z_s)\mathrm{d}L_s$$

which is obtained for $\|Y_s\| > R$, is an upper-bound of $f(h(Y_t) - Z_t)$. Furthermore, $\mathrm{d}L_s = 0$ for $\|Y_s\| > R$, and therefore one has

$$f(h(Y_t) - Z_t) \leq \int_0^t f'(\|Y_s\| - Z_s) \left\{ \frac{1}{\|Y_s\|} \langle Y_s, b(Y_s) \rangle + \frac{m(m+1)}{\|Y_s\|} \overline{\Lambda} + \eta \overline{\Lambda} \right\} \mathrm{d}s \quad (2.3.17)$$

Now using (2.3.11) yields

$$\frac{1}{\|Y_s\|} \langle Y_s, b(Y_s) \rangle + \frac{m(m+1)}{\|Y_s\|} \overline{\Lambda} + \eta \overline{\Lambda} \leq -A + \frac{m(m+1)}{\|Y_s\|} \overline{\Lambda} + \eta \overline{\Lambda}$$

and since this upper-bound is obtained for $\|Y_s\| > R$, then

$$-A + \frac{m(m+1)}{\|Y_s\|} \overline{\Lambda} + \eta \overline{\Lambda} \leq -A + \frac{m(m+1)}{R} \overline{\Lambda} + \eta \overline{\Lambda} \,.$$

It suffices then to choose $\eta > 0$ and such that the r.h.s. of the above inequality is negative, i.e.

$$\eta \leq \frac{A}{\overline{\Lambda}} - \frac{m(m+1)}{R} \quad (2.3.18)$$

which therefore insures $f(h(Y_t) - Z_t) \leq 0$ and implies $\|Y_t\| \leq Z_t$ a.s. by definition of f. And this is possible since the ball radius R in (2.3.11) can be chosen as large as we want; it suffices indeed to notice that if (2.3.11) is satisfied outside the ball of radius R, then it is in particular true outside any ball of radius greater or equal than R. So by choosing any $\widetilde{R}$ such that $\widetilde{R} > \max \left\{ R \,;\, \left(\frac{m(m+1)}{A} \overline{\Lambda} \right) \right\}$, there can exist such $\eta > 0$. And it suffices then to write (2.3.11) with such $\widetilde{R}$ instead of R.

Step 2. (upper-bound $\mathbb{E}_Z \left[e^{-\delta \tau_n} \right]$*)*
Fix $\delta > 0$ and let us set $\gamma := \frac{\delta}{K}$ where K is the constant in (2.3.13).
By Itô's formula, we have for any $\Phi \in C^2(\mathbb{R})$,

$$\begin{aligned} \mathrm{d}(\Phi(Z_t) e^{-\gamma \xi_t}) &= \sqrt{2} \Phi'(Z_t) e^{-\gamma \xi_t} \mathrm{d}M_t + \Phi'(Z_t) e^{-\gamma \xi_t} \mathrm{d}L_t \\ &\quad + e^{-\gamma \xi_t} \left\{ \Phi''(Z_t) - \eta \Phi'(Z_t) - \gamma \Phi(Z_t) \right\} \mathrm{d}\xi_t. \end{aligned}$$

Since we are interested in the limit as $n \to \infty$, we can assume without loss of generality that $n > R$. We choose Φ such that

$$\begin{cases} \Phi''(z) - \eta \Phi'(z) - \gamma \Phi(z) = 0, & \text{for } z \in [R, n] \\ \Phi'(R) = 0 \quad \text{and} \quad \Phi(n) = 1 \end{cases} \quad (2.3.19)$$

then $\Phi(Z_t)e^{-\gamma\xi_t}$ is a local martingale which is bounded up to time τ_n. Hence we are allowed to apply Doob's stopping theorem to obtain

$$\Phi(R \vee \|y_o\|) = \mathbb{E}_Z\left[\Phi(Z_{\tau_n})e^{-\gamma\xi_{\tau_n}}\right] \tag{2.3.20}$$

and since $Z_{\tau_n} = n$, $\Phi(n) = 1$, and $\xi_t \leq Kt$ for all $t \geq 0$, we have

$$\mathbb{E}_Z\left[e^{-\gamma K\tau_n}\right] \leq \mathbb{E}_Z\left[e^{-\gamma\xi_{\tau_n}}\right] = \Phi(R \vee \|y_o\|) \tag{2.3.21}$$

which yields

$$\mathbb{E}_Z\left[e^{-\delta\tau_n}\right] \leq \Phi(R \vee \|y_o\|) \tag{2.3.22}$$

Now solving the differential equation (2.3.19) yields

$$\Phi(z) = \frac{-\lambda_2 e^{\lambda_1(z-R)} + \lambda_1 e^{\lambda_2(z-R)}}{-\lambda_2 e^{\lambda_1(n-R)} + \lambda_1 e^{\lambda_2(n-R)}} \tag{2.3.23}$$

where $\lambda_2 < 0 < \lambda_1$, and are given by

$$\lambda_1 = \frac{1}{2}\left(\eta + \sqrt{\eta^2 + 4\gamma}\right), \qquad \lambda_2 = \frac{1}{2}\left(\eta - \sqrt{\eta^2 + 4\gamma}\right)$$

Hence,

$$\begin{aligned}\Phi(z) &\leq \frac{(\lambda_1 - \lambda_2)e^{\lambda_1(z-R)}}{-\lambda_2 e^{\lambda_1(n-R)}} \\ &\leq \frac{\lambda_1 - \lambda_2}{-\lambda_2} e^{\lambda_1(z-n)}\end{aligned}$$

Set $r := R \vee \|y_o\|$, then one gets

$$\begin{aligned}\Phi(r) &\leq \frac{\lambda_1 - \lambda_2}{-\lambda_2} e^{\lambda_1(r-n)} \\ &\leq 2\frac{\sqrt{1+\frac{4\gamma}{\eta^2}}}{\sqrt{1+\frac{4\gamma}{\eta^2}} - 1}\exp\left[r\frac{\eta}{2}\left(1+\sqrt{1+\frac{4\gamma}{\eta^2}}\right)\right]\exp\left[-n\frac{\eta}{2}\left(1+\sqrt{1+\frac{4\gamma}{\eta^2}}\right)\right] \\ &\leq 2\frac{\eta^2}{\gamma}\frac{1+\frac{2\gamma}{\eta^2}}{1-\frac{\gamma}{\eta^2}}\exp\left[r\eta\left(1+\frac{\gamma}{\eta^2}\right)\right]e^{-n\frac{\eta}{2}}\end{aligned}$$

Recall $\gamma := \frac{\delta}{K}$ where K is the constant in (2.3.13) and set

$$\ell(\delta) = \frac{1+\frac{2\delta}{K\eta^2}}{1-\frac{\delta}{K\eta^2}}\exp\left[r\frac{\delta}{K\eta}\right] \quad \text{and} \quad C = 2\eta^2 K e^{r\eta},$$

then the right-hand side in the last inequality writes as $C\frac{\ell(\delta)}{\delta}e^{-n\frac{\eta}{2}}$ and it is easy to see that $\ell(\delta) = 1 + O(\delta)$ when $\delta \to 0^+$. Together with (2.3.22), this finally yields

$$\mathbb{E}_Z\left[e^{-\delta\tau_n}\right] \leq C\frac{\ell(\delta)}{\delta}e^{-n\frac{\eta}{2}} \tag{2.3.24}$$

Finally, we have thanks to (2.3.10)

$$\mathbb{E}_{\|Y\|}\left[e^{-\delta\tau_n}\right] \leq C\frac{\ell(\delta)}{\delta}e^{-n\frac{\eta}{2}},$$

which concludes the proof of the first statement.
Now multiplying the last inequality by n^α for $\alpha \geq 0$ and choosing $\delta = n^{-\beta}$ for $\beta > 0$, one recovers the second statement of the desired lemma. $\square$

Lemma 2.3.3. *Under assumptions* (2.2.3), (2.2.5) *and* (2.2.6), *there exist* C_1, C_2 *and* $\kappa > 0$ *such that*

$$C_2\,(n^2 - |y|^2) \leq \mathbb{E}[\,\tau_n\,] \leq C_1\, e^{\kappa\, n^2} \quad \textit{locally uniformly in } y, \tag{2.3.25}$$

where $\tau_n = \inf\{s \in [0,T] \,:\, \|Y_y(s)\| \geq n\}$ *and* $Y_y(0) = y \in \mathbb{R}^m$.
In particular, for any n, $\tau_n < +\infty$ *almost surely.*

Proof *of Lemma 2.3.3.*
Step 1. (The upper-bound)
Let $y \in D_n := \{y \in \mathbb{R}^m \,:\, |y| < n\}$. Recall that $\bar{x}$ is kept fixed in all this section and all the constants depend on it. For simplicity of notation only, we drop the dependency on $\overline{x}$ and consider again the process $Y_y(\cdot)$ as defined in (2.3.3), and its infinitesimal generator $\mathcal{L}$ defined by (2.2.13) where we dropped the dependency on $\overline{x}$, i.e.

$$\mathcal{L}(y, DV, D^2V) = b(y)\cdot\nabla V(y) + \mathrm{trace}(a(y)D^2V(y))$$

where $a(y) = \varrho(y)\varrho(y)^\top$.
We proceed now in a similar way as in [133, page 80]. We define for $y \in \mathbb{R}^m$ the function $V(y) = -e^{-\alpha y_1}$ for some $\alpha > 0$ that we will choose in a suitable way. Indeed, from the non-degeneracy assumption (2.2.6), we have $a_{11}(y) \geq \underline{\Lambda} > 0$. Therefore, we have

$$\begin{aligned}
\mathcal{L}V(y) &= b(y)\cdot\nabla V(y) + \mathrm{trace}(a(y)D^2V(y)) \\
&= \alpha\, e^{-\alpha y_1}\,(\,b_1(y) - \alpha\, a_{11}(y)\,) \\
&\leq \alpha\, e^{-\alpha y_1}\,(\,b_1(y) - \alpha\,\underline{\Lambda}\,) \\
&\leq \alpha\, e^{-\alpha y_1}\,(\,C(1 + |\bar{x}| + n) - \alpha\,\underline{\Lambda}\,)
\end{aligned}$$

where in the last inequality we have used the linear growth assumption (2.2.3) together with $|y| < n$. Denote by $\mathcal{C}_n := \frac{1}{\underline{\Lambda}} C(1 + |\overline{x}| + n)$. So we have

$$\mathcal{L}V(y) \leq \underline{\Lambda}\, \alpha\, e^{-\alpha y_1} \left(\mathcal{C}_n - \alpha \right) \tag{2.3.26}$$

We need now to choose α such that the right hand side in the last inequality is negative. Let us now choose $\alpha = \mathcal{C}_n + \frac{1}{n} =: \alpha_n^* > 0$. Substituting in (2.3.26) we have

$$\mathcal{L}V(y) \leq -\frac{1}{n}\, \underline{\Lambda}\, \alpha_n^*\, e^{-\alpha_n^* y_1} \leq -\frac{1}{n}\, \underline{\Lambda}\, \alpha_n^*\, e^{-\alpha_n^*\, n} =: -K(n) < 0$$

By Itô's formula, we have

$$\begin{aligned} \mathbb{E}[\, V(Y_y(\tau_n \wedge T))\,] - V(y) &= \mathbb{E}\left[\int_0^{\tau \wedge T} \mathcal{L}V(Y_y(s))\, \mathrm{d}s \right] \\ &\leq -K(n)\, \mathbb{E}[\tau_n \wedge T]. \end{aligned}$$

On the other hand, denoting by y_1^n the first component of $Y_y(\tau_n \wedge T)$, we have

$$\begin{aligned} \mathbb{E}[\, V(Y_y(\tau_n \wedge T))\,] - V(y) &= e^{-\alpha_n^*\, y_1} - \mathbb{E}[\, e^{-\alpha_n^*\, y_1^n}\,] \\ &\geq e^{-\alpha_n^*\, n} - e^{\alpha_n^*\, n} \geq -e^{\alpha_n^*\, n}. \end{aligned}$$

Hence, the following holds

$$\mathbb{E}[\tau_n \wedge T] \leq \frac{e^{\alpha_n^*\, n}}{K(n)}.$$

And we have

$$\frac{e^{\alpha_n^*\, n}}{K(n)} = \frac{n}{C(1 + |\bar{x}| + n) + \frac{\underline{\Lambda}}{n}} \exp\left(\frac{2C}{\underline{\Lambda}}\, n^2 + \frac{2C}{\underline{\Lambda}}(1 + |\bar{x}|)\, n + 2 \right)$$

whose dominant term, as $n \to +\infty$, is $\frac{e^2}{C(1+|\bar{x}|)}\, e^{\frac{2C}{\underline{\Lambda}}\, n^2}$.
Finally, setting $\kappa := 1 + \frac{C}{\underline{\Lambda}} > 0$ and $C_1 := 1 + \frac{e^2}{C(1+|\bar{x}|)} > 0$ yields $\mathbb{E}[\tau_n \wedge T] \leq C_1 e^{\kappa\, n^2}$. Now take $T \to \infty$ and using the monotone convergence theorem we get $\mathbb{E}[\tau_n] \leq C_1 e^{\kappa\, n^2}$ which is finite for any n, and hence implies that $\tau_n < +\infty$ almost surely.

Step 2. (The lower-bound)
Using the same procedure as in Step 1, we choose now $V(y) = \frac{1}{2}|y|^2$ and we have

$\mathcal{L}V(y) = b(y) \cdot y + \text{trace}(a(y)) \leq b(y) \cdot y + m\,\overline{\Lambda}$. Hence, we have the following

$$\begin{aligned} \mathbb{E}[V(Y_y(\tau_n))] - \mathbb{E}[V(y)] &= \mathbb{E}\left[\int_0^{\tau_n} \mathcal{L}V(Y_y(s))\,\mathrm{d}s\right] \\ &\leq \mathbb{E}\left[\int_0^{\tau_n} \langle b(Y_y(s)), Y_y(s)\rangle + m\,\overline{\Lambda}\,\mathrm{d}s\right] \end{aligned} \tag{2.3.27}$$

And using the recurrence condition on the drift (2.2.5) and the growth assumption[2] (2.2.3), we have

$$\begin{aligned} \langle b(y), y\rangle &\leq -A\,|y|, \qquad \text{if } |y| > R \\ \text{or } \langle b(y), y\rangle &\leq |b(y)|\,|y|, \\ &\leq C(1+R)R \quad \text{if } |y| \leq R \end{aligned}$$

So for any $y \in \mathbb{R}^m$, we have $\langle b(y), y\rangle \leq -A|y| + C(1+R)R \leq C(1+R) + R$. This yields together with (2.3.27)

$$\begin{aligned} \mathbb{E}[V(Y_y(\tau_n))] - \mathbb{E}[V(y)] &\leq \mathbb{E}\left[\int_0^{\tau_n} C(1+R)R + m\,\overline{\Lambda}\,\mathrm{d}s\right] \\ &\leq (C(1+R)R + m\,\overline{\Lambda})\mathbb{E}[\tau_n] \end{aligned}$$

and hence, noticing that $\mathbb{E}[V(Y_y(\tau_n))] - \mathbb{E}[V(y)] = \frac{1}{2}(n^2 - |y|^2)$, we have

$$C_2(n^2 - |y|^2) \leq \mathbb{E}[\tau_n]$$

where we have set $C_2 := (2C(1+R)R + 2m\,\overline{\Lambda})^{-1}$. □

Comment. Lemma 2.3.3 provides a qualitative result on the growth of the exit time of the process $Y_y(\cdot)$. Indeed, the upper-bound shows that, although it can grow exponentially fast in expectation, the exit time remains finite almost surely. That is, the process $Y_y(\cdot)$ will almost surely leave all bounded domains, but in (exponentially) large time. Or, in other words, the process $Y_y(\cdot)$ will remain for most of its life time in bounded domains. On the other hand, the lower-bound insures a growth at least quadratic for the exit time in expectation. This complements the other inequality by telling us how much can we hope to make the exit time larger in expectation by enlarging the bounded domain, or analogously, how much can we enlarge a bounded domain in order to make the process $Y_y(\cdot)$ not quickly leaving it.

[2] where we omit the dependency on the slow variable x in (2.2.3) for simplicity of notation only.

2.3.2 The effective Hamiltonian

We will show the existence of an effective Hamiltonian that will characterize the limit PDE in the convergence theorem. In principle, for each $(\bar{t}, \overline{x}, \overline{p}, \overline{P})$ one expects the effective Hamiltonian $\overline{H}(\bar{t}, \overline{x}, \overline{p}, \overline{P})$ to be the unique constant $c \in \mathbb{R}$ such that the *cell problem*

$$- \mathcal{L}(\overline{x}, y, D\chi, D^2\chi) + H(\bar{t}, \overline{x}, y, \overline{p}, \overline{P}, 0) = c \quad \text{in } \mathbb{R}^m, \tag{2.3.28}$$

where H is defined in (2.2.14), has a viscosity solution χ, called corrector (see [2, 78, 127]). And in many cases (see [3]), it turns out that it is sufficient to consider a δ-*cell problem*

$$\delta\omega_\delta - \mathcal{L}(\overline{x}, y, D\omega_\delta, D^2\omega_\delta) + H(\bar{t}, \overline{x}, y, \overline{p}, \overline{P}, 0) = 0 \quad \text{in } \mathbb{R}^m \tag{2.3.29}$$

whose solution ω_δ is called *approximate corrector*, and then the effective Hamiltonian will be given as the limit of $-\delta\omega_\delta$ as $\delta \to 0$. In our setting, we don't show the existence of a (viscosity) solution to (2.3.28) (this will later be addressed in Chapter 4). However, in the convergence result of the next section, and which is the main contribution of this chapter, it is enough to show the existence of an effective Hamiltonian as a limit of a *truncated* δ-*cell problem.* Fix $(\overline{x}, \overline{p}, \overline{P})$, and let us denote for simplicity

$$\mathcal{L}(\overline{x}, y, D\omega, D^2\omega) := \mathcal{L}\omega(y) \tag{2.3.30}$$

and

$$h(y) := H(\bar{t}, \overline{x}, y, \overline{p}, \overline{P}, 0) \quad \text{in } \mathbb{R}^m \tag{2.3.31}$$

Under the assumptions (2.2.2) and (2.2.8), the Hamiltonian has at most a quadratic growth in y, i.e.

$$\exists\, K_h > 0, \quad |h(y)| \leq K_h(1 + |y|^2), \quad \forall\, y \in \mathbb{R}^m \tag{2.3.32}$$

where K_h is a constant that depends on the slow dynamics data (f, σ) and the running cost ℓ.

We consider the PDE (2.3.29) on a sequence of bounded and open domains D_n s.t.: $y \in D_n \subset D_{n+1} \subset \mathbb{R}^m$ and $D_n \underset{n\to\infty}{\longrightarrow} \mathbb{R}^m$ (in the sense of Hausdorff metric on the euclidean space $\mathbb{R}^m$ for instance) with C^2 boundaries. Set in addition $D_n \subseteq B(0, n) := \{y \in \mathbb{R}^m \mid \|y\| < n\}$ the euclidean open ball centered in 0 with radius n. Consider now the Dirichlet-Poisson problem

$$\begin{cases} \delta u(y) - \mathcal{L}u(y) = -h(y), \text{ in } D_n \\ u(y) = \phi(y), \text{ on } \partial D_n \end{cases} \tag{2.3.33}$$

It has a unique solution $u^{\delta,n}(\cdot)$ (see, e.g., [133, Theorem 8.1, page 79]) given by

$$u^{\delta,n}(y) = \mathbb{E}\left[\phi(Y_y(\tau_n))e^{-\delta\tau_n}\right] + \mathbb{E}\left[-\int_0^{\tau_n} h(Y_y(t))e^{-\delta t}\mathrm{d}t\right] \tag{2.3.34}$$

where τ_n is the first exist time of $Y_y(\cdot)$ from D_n.

For the sake of generality, we consider ϕ with a polynomial growth, that is $\exists\ K_\phi > 0$, and for some $\kappa \geq 0$ such that for all $y \in \mathbb{R}^m$

$$|\phi(y)| \leq K_\phi(1 + |y|^\kappa). \tag{2.3.35}$$

And we are interested in the limit as $(\delta, n) \to (0, \infty)$. The following proposition provides the existence of the effective Hamiltonian, using the above *truncated δ-cell problem* (2.3.33).

Proposition 2.3.2. *Let $u^{\delta,n}(\cdot)$ be the solution to* (2.3.33)*. Under the standing assumptions of section §2.2.1, and assuming* (2.3.32),(2.3.35)*, and $\delta = \delta(n) = O\left(\frac{1}{n^{4+\alpha}}\right)$, where $\alpha > 0$ is arbitrary, one has*

$$\lim_{n\to\infty} \left|\delta(n)u^{\delta(n),n}(y) + \mu(h)\right| = 0, \qquad \textit{locally uniformly in } y, \tag{2.3.36}$$

where $\mu(h) = \int_{\mathbb{R}^m} h(y)d\mu(y)$ and μ is the unique invariant probability measure for the process (2.3.1).

Remark 2.3.2. *This result still holds true if we consider, instead of assumption* (2.3.32)*, h such that*

$$\exists\ K_h > 0, \quad |h(y)| \leq K_h(1 + |y|^\gamma), \quad \forall\ y \in \mathbb{R}^m$$

where $\gamma \geq 0$ is as large as we want, provided we set $\delta = O\left(\frac{1}{n^{2\gamma+\alpha}}\right)$ in Proposition 2.3.2. This means that the slow dynamics is allowed to have a polynomial growth w.r.t. the fast variables.

Proof *of Proposition 2.3.2.* From (2.3.34), one has

$$u^{\delta,n}(x) + \frac{\mu(h)}{\delta} = \mathbb{E}\left[-\int_0^{\tau_n} h(Y_y(t))e^{-\delta t}\mathrm{d}t\right] + \int_0^\infty \int_{\mathbb{R}^m} h(y)e^{-\delta t}d\mu(y)\mathrm{d}t + \mathbb{E}\left[\phi(Y_y(\tau_n))e^{-\delta\tau_n}\right]$$

Exchanging the integrals in the first term (the expectation and the time integral), one gets

$$
\begin{aligned}
u^{\delta,n}(x)+\frac{\mu(h)}{\delta} &= \mathbb{E}\left[-\int_0^\infty \mathbb{1}_{D_n}(Y_y(t))h(Y_y(t))e^{-\delta t}\mathrm{d}t\right]+\int_0^\infty\int_{\mathbb{R}^m} h(y)e^{-\delta t}d\mu(y)\mathrm{d}t\\
&\quad +\mathbb{E}\left[\int_{\tau_n}^\infty \mathbb{1}_{D_n}(Y_y(t))h(Y_y(t))e^{-\delta t}\mathrm{d}t\right]+\mathbb{E}\left[\phi(Y_y(\tau_n))e^{-\delta\tau_n}\right]\\
&= -\int_0^\infty\int_{\mathbb{R}^m}\mathbb{1}_{D_n}(y)h(y)d\mathbb{P}_{Y_y(t)}(y)e^{-\delta t}\mathrm{d}t+\int_0^\infty\int_{\mathbb{R}^m} h(y)e^{-\delta t}d\mu(y)\mathrm{d}t\\
&\quad +\mathbb{E}\left[\int_{\tau_n}^\infty \mathbb{1}_{D_n}(Y_y(t))h(Y_y(t))e^{-\delta t}\mathrm{d}t+\phi(Y_y(\tau_n))e^{-\delta\tau_n}\right]\\
&= \int_0^\infty\int_{D_n} h(y)d\Big(\mu(y)-\mathbb{P}_{Y_y(t)}(y)\Big)e^{-\delta t}\mathrm{d}t+\frac{1}{\delta}\int_{D_n^c} h(y)d\mu(y)\\
&\quad +\mathbb{E}\left[\int_{\tau_n}^\infty \mathbb{1}_{D_n}(Y_y(t))h(Y_y(t))e^{-\delta t}\mathrm{d}t+\phi(Y_y(\tau_n))e^{-\delta\tau_n}\right]
\end{aligned}
$$

where D_n^c refers to the complement of D_n, that is $\mathbb{R}^m\setminus D_n$.

The first term:
Applying Hölder inequality to the first term yields

$$
\begin{aligned}
&\left|\int_0^\infty\int_{D_n} h(y)\mathrm{d}\Big(\mu(y)-\mathbb{P}_{Y_y(t)}(y)\Big)e^{-\delta t}\mathrm{d}t\right|\leq\\
&\qquad\leq\left(\int_0^\infty\left(\int_{D_n} h(y)\mathrm{d}(\mu(y)-\mathbb{P}_{Y_y(t)}(y))\right)^2\mathrm{d}t\right)^{1/2}\left(\int_0^\infty e^{-2\delta t}\mathrm{d}t\right)^{1/2}\\
&\qquad=\frac{1}{\sqrt{2\delta}}\left(\int_0^\infty\left(\int_{D_n} h(y)\mathrm{d}(\mu(y)-\mathbb{P}_{Y_y(t)}(y))\right)^2\mathrm{d}t\right)^{1/2}
\end{aligned}
$$

Now using Lemma 2.3.1, we can bound the term in the right-hand side of the latter inequality as follows

$$
\begin{aligned}
\int_0^\infty\left(\int_{D_n} h(y)\mathrm{d}(\mu(y)-\mathbb{P}_{Y_y(t)}(y))\right)^2\mathrm{d}t &\leq \int_0^\infty\left(\sup_{D_n}|h|\int_{D_n}\mathrm{d}(\mu(y)-\mathbb{P}_{Y_y(t)}(y))\right)^2\mathrm{d}t\\
&\leq \sup_{D_n}|h|^2\int_0^\infty\left\|\mathbb{P}_{Y_y(t)}(\cdot)-\mu(\cdot)\right\|_{TV}^2\mathrm{d}t\\
&\leq \frac{C(1+|y|^d)}{k}\sup_{D_n}|h|^2\leq\frac{C(1+|y|^d)}{k}K_h^2(1+n^2)^2
\end{aligned}
$$

Finally, we have the following upper-bound

$$\left|\int_0^\infty \int_{D_n} h(y)\mathrm{d}\Big(\mu(y) - \mathbb{P}_{Y_y(t)}(y)\Big)e^{-\delta t}\mathrm{d}t\right| \leq K_h \left(\frac{C(1+|y|^d)}{k}\right)^{1/2} \frac{(1+n^2)}{\sqrt{2\delta}} \tag{2.3.37}$$

The second term:
It is nothing but

$$\frac{1}{\delta}\int_{D_n^c} h(y)\mathrm{d}\mu(y) = \frac{1}{\delta}\left(\mu(h) - \int_{D_n} h(y)\mathrm{d}\mu(y)\right) \tag{2.3.38}$$

The third term:
Using the definition of D_n and the growth condition of h and ϕ, one has

$$\begin{aligned}
&\left|\mathbb{E}\left[\int_{\tau_n}^\infty \mathbb{1}_{D_n}(Y_y(t))h(Y_y(t))e^{-\delta t}\mathrm{d}t + \phi(Y_y(\tau_n))e^{-\delta\tau_n}\right]\right| \leq \\
&\qquad \leq \mathbb{E}\left[\int_{\tau_n}^\infty \mathbb{1}_{D_n}(Y_y(t))|h(Y_y(t))|e^{-\delta t}\mathrm{d}t\right] + \mathbb{E}\left[|\phi(Y_y(\tau_n))|e^{-\delta\tau_n}\right] \\
&\qquad \leq K_h\mathbb{E}\left[\int_{\tau_n}^\infty \mathbb{1}_{D_n}(Y_y(t))(1+|Y_y(t)|^2)e^{-\delta t}\mathrm{d}t\right] + K_\phi\mathbb{E}\left[(1+|Y_y(\tau_n)|^\kappa)e^{-\delta\tau_n}\right] \\
&\qquad \leq K_h\mathbb{E}\left[\int_{\tau_n}^\infty (1+n^2)e^{-\delta t}\mathrm{d}t\right] + K_\phi\mathbb{E}\left[(1+n^\kappa)e^{-\delta\tau_n}\right]
\end{aligned}$$

which finally yields the following upper-bound

$$\left|\mathbb{E}\left[\int_{\tau_n}^\infty \mathbb{1}_{D_n}(Y_y(t))h(Y_y(t))e^{-\delta t}\mathrm{d}t + \phi(Y_y(\tau_n))e^{-\delta\tau_n}\right]\right| \leq \left(K_h\frac{1+n^2}{\delta} + K_\phi(1+n^\kappa)\right)\mathbb{E}\left[e^{-\delta\tau_n}\right] \tag{2.3.39}$$

Conclusion:
Therefore, from (2.3.37), (2.3.38) and (2.3.39) one has

$$\begin{aligned}
\left|u^{\delta,n}(y) + \frac{\mu(h)}{\delta}\right| \leq K_h\left(\frac{C(1+|y|^d)}{k}\right)^{1/2}\frac{1+n^2}{\sqrt{2\delta}} + \frac{1}{\delta}\left|\mu(h) - \int_{D_n} h(y)\mathrm{d}\mu(y)\right| + \\
+ \left(K_h\frac{1+n^2}{\delta} + K_\phi(1+n^\kappa)\right)\mathbb{E}\left[e^{-\delta\tau_n}\right]
\end{aligned}$$

Hence

$$\left|\delta u^{\delta,n}(y)+\mu(h)\right| \le K_h\left(\frac{C(1+|h|^d)}{k}\right)^2\frac{1+n^2}{\sqrt{2}}\sqrt{\delta}+\left|\mu(f)-\int_{D_n}h(y)\mathrm{d}\mu(y)\right|+ \\ +\delta\left(K_h\frac{1+n^2}{\delta}+K_\phi(1+n^\kappa)\right)\mathbb{E}\left[e^{-\delta\tau_n}\right]$$

Now setting $\delta=\delta(n)=O(\frac{1}{n^{4+\alpha}})$, where $\alpha>0$ is arbitrary fixed, one has

$$\lim_{n\to\infty}\left|\delta(n)u^{\delta(n),n}(x)+\mu(f)\right|=0$$

since the last term converges to zero thanks to Lemma 2.3.2. □

Therefore, the effective Hamiltonian is defined by

$$\overline{H}(\bar{t},\bar{x},\bar{p},\overline{P}):=-\mu_{\bar{x}}(h)=-\int_{\mathbb{R}^m}H(\bar{t},\bar{x},y,\bar{p},\overline{P},0)d\mu_{\bar{x}}(y) \tag{2.3.40}$$

2.3.3 The effective initial data

In this section we construct the effective terminal value for the limit as $\varepsilon\to 0$ of the singular perturbations problem (2.2.9)-(2.2.10). We fix $\bar{x}$ and consider the following Cauchy initial problem:

$$\begin{cases}\omega_t-\mathcal{L}(y,D\omega,D^2\omega)=0 & (0,+\infty)\times\mathbb{R}^m,\\ \omega(0,y)=g(\bar{x},y),\end{cases} \tag{2.3.41}$$

where g satisfies assumption (2.2.8) and $\mathcal{L}$ is as defined in (2.2.13).

If g was a bounded function of the y variable[3] then one can use the result in [23, Proposition 4.4]: The Cauchy problem (2.3.41) admits a unique classical solution $(t,y)\mapsto\omega(t,y;\bar{x})$ and the effective initial data that we denote by $\overline{g}(\bar{x})$ is given by

$$\lim_{t\to+\infty}\omega(t,y;\bar{x})=\int_{\mathbb{R}^m}g(\bar{x},y)\mathrm{d}\mu(y)=:\overline{g}(\bar{x})\quad\text{locally uniformly in } y. \tag{2.3.42}$$

In our setting, the assumption on g, having a linear growth in y, makes the Cauchy problem (2.3.41) more difficult to solve and hence we cannot directly use the result (2.3.42). Indeed, very few results exist for such Cauchy problems with unbounded

[3]Note that although we may assume g to be uniformly bounded in y, the function $(x,y)\mapsto\mathbb{E}[g(X_t,Y_t)\,|\,X_0=x,Y_0=y]$, for any $t>0$, has at most a quadratic growth in y, since the process $X_\cdot$ depends linearly on y and g has at most a quadratic growth in x.

data in unbounded space domain (see in particular [36, Theorem 3]). But here we are interested in constructing the effective initial data, rather than solving the Cauchy problem with unbounded data. And to do so, we will proceed in a similar way as we did for the effective Hamiltonian in the previous section.

We consider an increasing sequence of bounded and open domains D_n with C^2 boundaries exactly as in §2.3.2, and such that $D_n \subseteq B(0,n) := \{y \in \mathbb{R}^m \mid \|y\| < n\}$ the euclidean open ball centered in 0 with radius n (for example $D_n = B(0,n)$). Now instead of (2.3.41), we set the Cauchy problem in the bounded domain $[0,T] \times D_n$ for some $T > 0$ that we will later made precise, that is, we consider the following problem

$$\begin{cases} \dfrac{\partial}{\partial t}\omega^{T,n} - \mathcal{L}(y, D\omega^{T,n}, D^2\omega^{T,n}) = 0 \quad \text{in } (0,T] \times D_n, \\ \omega^{T,n}(0,y) = g(\bar{x},y) \quad \text{in } D_n, \\ \omega^{T,n}(t,y) = 0 \quad \text{in } [0,T] \times \partial D_n, \end{cases} \tag{2.3.43}$$

and if we set $u^{T,n}(t,y) = \omega^{T,n}(T-t,y)$, then $u^{T,n}(\cdot,\cdot)$ solves the Cauchy problem (*Initial-Boundary Value Problem*)

$$\begin{cases} \dfrac{\partial}{\partial t}u^{T,n} + \mathcal{L}(y, Du^{T,n}, D^2u^{T,n}) = 0 \quad \text{in } [0,T) \times D_n, \\ u^{T,n}(T,y) = g(\bar{x},y) \quad \text{in } D_n, \\ u^{t,n}(t,y) = 0 \quad \text{in } [0,T] \times \partial D_n, \end{cases} \tag{2.3.44}$$

It is known (See [133, Theorem 8.2, page 81]) that the problem (2.3.44) admits a unique solution given by

$$u^{T,n}(t,y) = \mathbb{E}[\, \mathbb{1}_{\{\tau_n \wedge T = T\}}\, g(Y_{y,t}(T))\,] \tag{2.3.45}$$

where $Y_{y,t}(\cdot)$ is the fast process defined by (2.3.1) and such that $Y_{y,t}(t) = y \in \mathbb{R}^m$, and $\tau_n = \inf\{s \in [t,T] \,:\, Y_{y,t}(s) \notin D_n\}$ is the first exit time from D_n.

To construct the effective initial data as in (2.3.42), we will study the limit as T goes to $+\infty$ of $\omega^{T,n}(T,y) = u^{T,n}(0,y)$ but where $T = T(n)$ will be depending on n (i.e. on the increasing sequence of domains D_n) such that $T(n) \to +\infty$ as $n \to +\infty$.

Proposition 2.3.3. *Let $u^{T,n}(\cdot,\cdot)$ be as defined in* (2.3.45) *and assume* (2.2.3), (2.2.5) *and* (2.2.6) *hold true. Then for any increasing sequence $\{T(n)\}_{n>0}$ such that $T(n) \geq n^2$, we have the following*

$$\lim_{n \to +\infty} \left| u^{T(n),n}(0,y) - \overline{g} \right| = 0, \quad \textit{locally uniformly in } y, \tag{2.3.46}$$

were $\overline{g} := \int_{\mathbb{R}^m} g(y)d\mu(y)$ and μ is the unique invariant probability measure of the process

(2.3.1). *In particular* $\lim_{n\to+\infty} \omega^{T(n),n}(T(n),y) = \overline{g}$ *locally uniformly in* y.
Moreover $\overline{g}$ *is continuous and satisfy the quadratic growth condition* (2.2.8).

Remark 2.3.3. *Recall that here (as in all section §2.3), we are freezing the slow variable* $X_.$ *at some fixed value that we denote in all §2.3.3 by* $\overline{x}$. *For the sake of simplicity of notation only, we did not make explicit the dependency on* $\overline{x}$ *and wrote simply* $\omega^{T,n}(t,y)$, $g(y)$, $\overline{g}$ *and* $\mu(\cdot)$ *instead of* $\omega^{T,n}(t,y;\overline{x})$, $g(\overline{x},y)$, $\overline{g}(\overline{x})$ *and* $\mu_{\overline{x}}(\cdot)$ *respectively. Also, the fast process* $Y_{y,0}(\cdot)$ *which is the one taking the value* y *at time* 0, *is simply denoted by* $Y_y(\cdot)$.

Remark 2.3.4. *This result still holds true if we consider, instead of the growth assumption* (2.2.8), g *such that*

$$\exists\, K_g > 0, \quad |g(y)| \le K_g(1+|y|^{\gamma}), \quad \forall\, y \in \mathbb{R}^m$$

where $\gamma \ge 0$ *is as large as we want, provided we choose* $T(n) \ge n^{\gamma}$.

Proof *of Proposition 2.3.3.* We have the following

$$u^{T(n),n}(0,y) = \int_{D_n} \mathbb{1}_{\{\tau_n \wedge T(n) = T(n)\}} g(z)\, \mathrm{d}\mathbb{P}_{Y_y(T(n))}(z), \quad \text{from (2.3.45)}$$

$$\overline{g} = \mu(g) = \int_{\mathbb{R}^m} g(z)\, \mathrm{d}\mu(z) = \int_{D_n} g(z)\, \mathrm{d}\mu(z) + \int_{D_n^c} g(z)\, \mathrm{d}\mu(z).$$

Hence

$$\begin{aligned}\left| u^{T(n),n}(0,y) - \overline{g} \right| &\le \left| \int_{D_n} \mathbb{1}_{\{\tau_n \wedge T(n) = T(n)\}} g(z)\, \mathrm{d}\left(\mathbb{P}_{Y_y(T(n))} - \mu\right)(z) \right| \\ &\qquad + \left| \int_{D_n^c} g(z)\, \mathrm{d}\mu(z) \right| \\ &\le C(1+n^2) \|\mathbb{P}_{Y_y(T(n))}(\cdot) - \mu(\cdot)\|_{TV} + \sqrt{\mu(g^2)}\sqrt{1-\mu(D_n)}\end{aligned}$$

where, for the second integral term in the first inequality, we used Hölder inequality together with the fact that μ has finite forth moment guaranteed by Lemma 2.3.1 and is a probability measure i.e. positive and such that $\mu(\mathbb{R}^m) = \int_{\mathbb{R}^m} 1 \mathrm{d}\mu = 1$, while for the first integral term, we upper-bounded $\mathbb{1}_{\{\tau_n \wedge T(n) = T(n)\}}$ by 1, then we used the quadratic growth of g from (2.2.8), and extended the integral domain D_n to the whole space $\mathbb{R}^m$ on which we considered the absolute variation of the signed measure $\mathbb{P}_{Y_y(T(n))}(\cdot) - \mu(\cdot)$ on Borel sets of $\mathbb{R}^m$. Now using Lemma 2.3.1, there exist $C, d, k > 0$ such that

$$\|\mathbb{P}_{Y_y(T(n))}(\cdot) - \mu(\cdot)\|_{TV} \le C(1+|y|^d)(1+T(n))^{-(1+k)}$$

Therefore, we can choose a sequence of Cauchy problems (2.3.43) where the final time $T(n)$ is sufficiently large such that $T(n) \geq n^2$ in order to have

$$\left| u^{T(n),n}(0,y) \; - \; \mu(g) \right| \leq C(1+n^2)(1+|y|^d)\frac{1}{(1+n^2)^{1+k}} \; + \; \sqrt{\mu(g^2)(1-\mu(D_n))} \; \longrightarrow \; 0.$$

Finally, the regularity of $\overline{g}$ and the growth condition can be obtained, using the definition of $\overline{g}$, from condition (2.2.8) and the regularity of the invariant measure stated in Proposition 2.3.1. □

2.4 The convergence theorem for the value function

We will prove that the value function $V^\varepsilon(t,x,y)$, solution to (2.2.9)-(2.2.10), converges locally uniformly, as $\varepsilon \to 0$, to a function $V(t,x)$ which will be characterised as the unique solution of the Cauchy problem

$$\begin{cases} -V_t + \overline{H}(t,x,D_xV,D^2_{xx}V) + \lambda V(x) = 0, & \text{in } \; (0,T)\times\mathbb{R}^n \\ \qquad\qquad\qquad\qquad V(T,x) = \overline{g}(x), & \text{in } \; \mathbb{R}^n \end{cases} \tag{2.4.1}$$

where $\overline{H}$ is the effective Hamiltonian (2.3.40) and $\overline{g}(x) = \int_{\mathbb{R}^m} g(x,y)\,\mathrm{d}\mu_x(y)$ the effective initial data as in Proposition 2.3.3. To this end, we will need the following Liouville type result. Recall the *fast subsystem* (2.3.1) associated to (2.2.1) for a fixed $x \in \mathbb{R}^n$

$$\mathrm{d}Y_t = b(x,Y_t)\,\mathrm{d}t + \sqrt{2}\varrho(x,Y_t)\,\mathrm{d}W_t, \quad Y_0 = y \in \mathbb{R}^m.$$

under the standing assumptions.

Lemma 2.4.1. *([132, Proposition 3.1]) Fix $x \in \mathbb{R}^n$ and consider the problem*

$$-\mathcal{L}V(y) = -b(x,y)\cdot\nabla V(y) - \mathit{trace}(\varrho(x,y)\varrho(x,y)^\top D^2V(y)) = 0, \quad \mathit{in}\ \mathbb{R}^m. \tag{2.4.2}$$

Assume that there exist a function $\omega \in C^\infty(\mathbb{R}^m)$ and $R_0 > 0$ such that

$$-\mathcal{L}\omega \geq 0 \quad \mathit{in}\ \overline{B(0,R_0)}^C, \quad \omega(y) \to +\infty \;\; \mathit{as} \;\; |y| \to +\infty. \tag{2.4.3}$$

Then:

1. *every viscosity subsolution $V \in USC(\mathbb{R}^m)$ to (2.4.2) such that $\limsup\limits_{|y|\to\infty} \frac{V}{\omega} \leq 0$ is constant;*

2. *every viscosity supersolution* $V \in LSC(\mathbb{R}^m)$ *to* (2.4.2) *such that* $\liminf_{|y|\to\infty} \frac{V}{\omega} \geq 0$ *is constant;*

Proof *of Lemma 2.4.1.* The Lemma is proven in [132] assuming that the diffusion ϱ satisfies the Hörmander condition. It suffices to notice that the non degeneracy condition (2.2.6) implies Hörmander condition. A more general result can be found in [22]. □

Remark 2.4.1. *The Liouville property replaces the standard strong maximum principle and is the key ingredient for extending some results of [4] to the non periodic setting. Roughly speaking, it says that when a harmonic function is bounded, then it can only be constant. This property is also reminiscent of other similar conditions about ergodicity of diffusion processes in the whole space, see for example [29, 47, 48, 109, 126] and [21, Remark 2.1].*

We are now ready to state and prove our first main result (Theorem 2.4.1) in this chapter on the convergence of the value function of the stochastic optimal control problem with singular perturbations to the unique viscosity solution of the effective Cauchy problem.

Theorem 2.4.1. *We assume the standing assumptions in Sections §2.2.1 and §2.2.2, and we assume the assumptions in Proposition 2.3.1 and in Lemma 2.4.1. Then the solution* V^ε *to* (2.2.9) *converges uniformly on compact subsets of* $[0,T) \times \mathbb{R}^n \times \mathbb{R}^m$ *to the unique continuous viscosity solution to the limit problem* (2.4.1) *satisfying a quadratic growth condition in* x*, i.e.*

$$\exists\, K > 0 \text{ such that } |V(t,x)| \leq K(1+|x|^2), \quad \forall\, (t,x) \in [0,T] \times \mathbb{R}^n \tag{2.4.4}$$

Proof *of Theorem 2.4.1.* Using our previous results on the effective data of the problem, we can conduct the proof in the line of the one of [23, Theorem 5.1] (see also [21, Theorem 3.2]), but still with some difference (in particular in Step 3). It is divided into several steps:

- *Step 1.* we define the relaxed semilimits that will be used all along the proof,
- *Step 2.* we show that the relaxed semilimits do not depend on y using the Liouville type result in Lemma 2.4.1,
- *Step 3.* we show that the relaxed semilimits are sub- and supersolutions to the limit PDE,
- *Step 4.* we show that the relaxed semilimits are continuous at the final time T,

- *Step 5.* we conclude by using a Comparison Principle for Bellman equations under quadratic growth.

Step 1. (Relaxed semilimits)
Recall that by (2.2.15) the functions V^ε are locally equibounded in $[0,T]\times\mathbb{R}^n\times\mathbb{R}^m$, uniformly in ε. We define the half-relaxed semilimits (see [19, Chap. V]) in $[0,T]\times\mathbb{R}^n\times\mathbb{R}^m$:

$$\underline{V}(t,x,y)=\liminf_{\substack{\varepsilon\to 0\\ t'\to t, x'\to x, y'\to y}} V^\varepsilon(t',x',y'),\quad \overline{V}(t,x,y)=\limsup_{\substack{\varepsilon\to 0\\ t'\to t, x'\to x, y'\to y}} V^\varepsilon(t',x',y')$$

for $t<T, x\in\mathbb{R}^n, y\in\mathbb{R}^m$, and

$$\underline{V}(T,x,y)=\liminf_{\substack{\varepsilon\to 0\\ t'\to T^-, x'\to x, y'\to y}} V^\varepsilon(t',x',y'),\quad \overline{V}(T,x,y)=\limsup_{\substack{\varepsilon\to 0\\ t'\to T^-, x'\to x, y'\to y}} V^\varepsilon(t',x',y').$$

It is immediate to get by definitions that the estimate (2.2.15) holds also for $\underline{V}$ and $\overline{V}$, that is

$$|\underline{V}(t,x,y)|,\ |\overline{V}(t,x,y)|\ \le K(1+|x|^2+|y|^2),\quad \forall\, t\in[0,T],\ x\in\mathbb{R}^n,\ y\in\mathbb{R}^m \tag{2.4.5}$$

Step 2. ($\underline{V},\overline{V}$ do not depend on y)
We check that $\underline{V}(t,x,y),\overline{V}(t,x,y)$ do not depend on y for every $t\in[0,T)$ and $x\in\mathbb{R}^n$. Arguing as in Step 2 of the proof of [23, Theorem 5.1], we get that $\overline{V}(t,x,y)$ (resp., $\underline{V}(t,x,y)$) is, for every $t\in(0,T)$ and $x\in\mathbb{R}^n$, a viscosity subsolution (resp., supersolution) to

$$-\mathcal{L}(x,y,D_yV,D^2_{yy}V)=0\quad \text{in } \mathbb{R}^m \tag{2.4.6}$$

where $\mathcal{L}$ is the differential operator defined in (2.2.13). Consider now the function ω defined on $\mathbb{R}^m\setminus\{0\}$ such that

$$\omega(y)=\frac{1}{2}|y|^2\log|y|. \tag{2.4.7}$$

and such that $\omega(0)=0$. It is easy to check that

$$\nabla\omega(y)=\left(\frac{1}{2}+\log(|y|)\right)y$$
$$D^2\omega(y)=\left(\frac{1}{2}+\log(|y|)\right)\mathbb{I}_m+\frac{y\otimes y}{|y|^2}$$

where we recall $y \in \mathbb{R}^m$ and $\mathbb{I}_m$ is the identity matrix of dimension m. Therefore, recalling $a = \varrho\varrho^\top$, one has

$$\begin{aligned} -\mathcal{L}\omega &= -\left(\frac{1}{2} + \log(|y|)\right)\langle b(y), y\rangle - \left(\frac{1}{2} + \log(|y|)\right)\mathrm{trace}(a(y)) - \frac{1}{|y|^2}\mathrm{trace}((y \otimes y)a(y)) \\ &\geq -\left(\frac{1}{2} + \log(|y|)\right)\left(\langle b(y), y\rangle + m\overline{\Lambda}\right) - \overline{\Lambda} \xrightarrow[|y|\to\infty]{} +\infty \end{aligned} \tag{2.4.8}$$

thanks to assumption (2.2.5) and (2.2.6). This means that one can find $R > 0$ such that

$$-\mathcal{L}\omega(y) \geq 0 \quad \text{in } \overline{B(0,R)}^C, \qquad \text{and } \omega(y) \xrightarrow[|y|\to\infty]{} +\infty \tag{2.4.9}$$

We can now use Lemma 2.4.1 with such a Lyapunov function ω, since $\overline{V}, \underline{V}$ have at most a quadratic growth in y, to conclude that the functions $y \mapsto \overline{V}(t,x,y)$, $y \mapsto \underline{V}(t,x,y)$ are constants for every $(t,x) \in (0,T) \times \mathbb{R}^n$. Finally, using the definition it is immediate to see that this implies that also $\overline{V}(T,x,y)$ and $\underline{V}(T,x,y)$ do not depend on y.

Step 3. ($\underline{V}, \overline{V}$ are super- and subsolutions of the limit PDE)
First we show that $\overline{V}$ and $\underline{V}$ are sub and supersolutions to the PDE in (2.4.1) in $(0,T) \times \mathbb{R}^n$. We prove it only for $\overline{V}$ since the other case is completely analogous. The proof adapts the perturbed test function method [78]. We fix $(\bar{t}, \bar{x}) \in (0,T) \times \mathbb{R}^n$ and we show that $\overline{V}$ is a viscosity subsolution at $(\bar{t}, \bar{x})$ of the limit problem. This means that if ψ is a smooth function such that $\psi(\bar{t}, \bar{x}) = \overline{V}(\bar{t}, \bar{x})$ and $\overline{V} - \psi$ has a strict maximum at $(\bar{t}, \bar{x})$ then

$$-\psi_t(\bar{t}, \bar{x}) + \overline{H}(\bar{t}, \bar{x}, D_x\psi(\bar{t}, \bar{x}), D^2_{xx}\psi(\bar{t}, \bar{x})) + \lambda\overline{V}(\bar{t}, \bar{x}) \leq 0 \tag{2.4.10}$$

Without loss of generality, we assume that the maximum is strict in $B((\bar{t}, \bar{x}), r)$ and that $0 < \bar{t} - r < \bar{t} + r < T$.

Set $\overline{p} = D_x\psi(\bar{t}, \bar{x})$ and $\overline{P} = D^2_{xx}\psi(\bar{t}, \bar{x})$ and consider the following $\delta(n)$-cell problem

$$\begin{cases} \delta\chi_\delta(y) - \mathcal{L}(\overline{x}, y, D\chi_\delta, D^2\chi_\delta) + H^\varepsilon(\bar{t}, \overline{x}, y, \overline{p}, \overline{P}, 0) = 0, & \text{in } D_n \\ \chi_\delta(y) = 0, & \text{in } \mathbb{R}^m \setminus D_n \end{cases} \tag{2.4.11}$$

where $\delta := \delta(n) = O\left(\frac{1}{n^{4+\alpha}}\right)$ and D_n is the euclidean open ball centered in 0 with radius n large enough. For simplicity of notations, we drop in what follows the dependency of δ on n. The solution χ_δ of the above δ-cell problem is C^2 in a neighborhood of $(\bar{t}, \overline{x}, \overline{p}, \overline{P})$.

Thanks to the convergence result for the effective Hamiltonian, we have for every $\kappa > 0$ there exists $\delta_o > 0$ (or equivalently $n_o > 0$) such that for every $\delta \leq \delta_o$ (or equivalently for every $n \geq n_o$) one has

$$|\delta\chi_\delta(y) + \overline{H}(\bar{t}, \overline{x}, \overline{p}, \overline{P})| \leq \kappa, \quad \forall y \in B(\overline{y}, \overline{R}) \tag{2.4.12}$$

where $\overline{y}, \overline{R}$ will be soon after made precise, and n is chosen large enough to insure $B(\overline{y}, \overline{R}) \subset D_n$.

We consider now the Lyapunov function ω in (2.4.7), and let $\overline{y}$ be such that $\omega(\overline{y}) = \min_{y\in\mathbb{R}^m} \omega(y)$. It is easy to see that such $\overline{y}$ satisfies $|\overline{y}| = e^{-1/2}$. However, by evaluating the equality in (2.4.8) at $\overline{y}$, one has $-\mathcal{L}(\omega)(\overline{y}) \leq -\underline{\Lambda} < 0$. So we will construct a perturbed test function by compensating the negative gap of $-\mathcal{L}\omega$ when evaluated in a neighborhood of $\overline{y}$, and which will contain a new global minimizer. We define the perturbed test function as

$$\psi^\varepsilon(t, x, y) := \psi(t, x) + \varepsilon\chi_\delta(y) + \omega(y) - \eta\frac{|y - \overline{y}|^2}{2}\zeta(y) \tag{2.4.13}$$

Here ε is independent of δ and n. Moreover, $\eta > 0$ is to be made precise, and $\zeta \in C^2(\mathbb{R}^m)$ is a cut-off function with a compact support subset of $B(\overline{y}, \overline{R})$ where $\overline{R} > 0$ is to be made precise and such that $\zeta(y) = 1$ for all $y \in B(\overline{y}, \overline{R}-\theta)$ for some $\theta > 0$ small enough, and $\zeta(y) = 0$ for all $y \in \overline{B(\overline{y}, \overline{R})}^C$.

<u>*Claim:*</u> There exist $\eta, \overline{R} > 0$ such that

$$\hat{y} := \operatorname*{argmin}_{y\in\mathbb{R}^m} \left\{\omega(y) - \eta\frac{|y - \overline{y}|^2}{2}\zeta(y)\right\}$$

is a global strict minimum. Moreover, $\hat{y} \in B(\overline{y}, \overline{R} - \theta)$ and

$$-\mathcal{L}\left(\omega(y) - \eta\frac{|y - \overline{y}|^2}{2}\zeta(y)\right) \geq 0, \quad \forall y \in B(\overline{y}, \overline{R} - \theta) \tag{2.4.14}$$

Assume the *claim* holds true. Observe that

$$\begin{aligned} \limsup_{\substack{\varepsilon\to 0 \\ t'\to t, x'\to x, y'\to y}} \quad & V^\varepsilon(t', x', y') - \psi^\varepsilon(t', x', y') \\ &= \overline{V}(t, x) - \psi(t, x) - \left(\omega(y) - \eta\frac{|y - \overline{y}|^2}{2}\zeta(y)\right) \end{aligned} \tag{2.4.15}$$

and $(\bar{t}, \overline{x}, \hat{y})$ is a strict local maximum of $(t,x,y) \mapsto \overline{V}(t,x) - \psi(t,x) - \Big(\omega(y) - \eta\frac{|y-\overline{y}|^2}{2}\zeta(y)\Big)$.
Arguing as in [19, Lemma V.1.6] we get sequences

$$\varepsilon_k \to 0, \quad \text{and} \quad (t_k, x_k, y_k) \in B := B((\bar{t}, \overline{x}), r) \times B(\overline{y}, \overline{R} - \theta)$$

such that $(t_k, x_k, y_k) \to (\bar{t}, \overline{x}, \hat{y})$ where $\hat{y}$ is as in the *claim*, and, when $k \to +\infty$,

$$V^{\varepsilon_k}(t_k, x_k, y_k) - \psi^{\varepsilon_k}(t_k, x_k, y_k) \to \overline{V}(\bar{t}, \overline{x}) - \psi(\bar{t}, \overline{x}) - \left(\omega(\hat{y}) - \eta\frac{|\hat{y} - \overline{y}|^2}{2}\zeta(\hat{y})\right)$$

and (t_k, x_k, y_k) is a maximum of $V^{\varepsilon_k} - \psi^{\varepsilon_k}$ in B.
Then using the fact that V^ε is a subsolution to (2.2.9), we get[4]

$$\begin{aligned} -\psi_t + H^{\varepsilon_k}(t_k, x_k, y_k, D_x\psi, D^2_{xx}\psi, 0) + \lambda V^{\varepsilon_k} - \mathcal{L}(x_k, y_k, D_y\chi_\delta, D^2_{yy}\chi_\delta) \\ -\frac{1}{\varepsilon_k}\mathcal{L}\left(\omega(y_k) - \eta\frac{|y_k - \overline{y}|^2}{2}\zeta(y_k)\right) \leq 0 \end{aligned} \tag{2.4.16}$$

where $V^{\varepsilon_k}, \psi, \chi_\delta$ and ω (and their derivatives) are computed respectively in (t_k, x_k, y_k), (t_k, x_k) and in y_k. Using (2.4.14) we get from the previous inequality that

$$-\psi_t + H^{\varepsilon_k}(t_k, x_k, y_k, D_x\psi, D^2_{xx}\psi, 0) + \lambda V^{\varepsilon_k} - \mathcal{L}(x_k, y_k, D_y\chi_\delta, D^2_{yy}\chi_\delta) \leq 0 \tag{2.4.17}$$

We now recall that χ_δ solves the δ-cell problem (2.4.11), thus

$$\begin{aligned} &-\psi_t + H^{\varepsilon_k}(t_k, x_k, y_k, D_x\psi(t_k, x_k), D^2_{xx}\psi(t_k, x_k), 0) \\ &- H^{\varepsilon_k}(\bar{t}, \overline{x}, y_k, \overline{p}, \overline{P}, 0) - \delta\chi_\delta(y_k) + \lambda V^{\varepsilon_k}(t_k, x_k, y_k) \leq 0. \end{aligned} \tag{2.4.18}$$

By taking the limit as $k \to \infty$ the second and the third term of this inequality cancel out. Next we use (2.4.12) to replace $-\delta\chi_\delta$ with $\overline{H} - \kappa$ and get that

$$-\psi_t(\bar{t}, \overline{x}) + \overline{H}(\bar{t}, \overline{x}, \overline{p}, \overline{P}) + \lambda\overline{V}(\bar{t}, \overline{x}) \leq \kappa. \tag{2.4.19}$$

Finally, since the latter holds for all $\kappa > 0$, we conclude.
To prove that $\underline{V}$ is a supersolution to (2.4.1) we proceed exactly in the same way, provided we choose as a perturbed test function

$$\psi^\varepsilon(t, x, y) := \psi(t, x) + \varepsilon\chi_\delta(y) - \omega(y) + \eta\frac{|y - \overline{y}|^2}{2}\zeta(y).$$

[4]We use both notations: $\mathcal{L}\Phi(y) = \mathcal{L}(x, y, D\Phi(y), D^2\Phi(y))$ when Φ is a function of y.

Step 4. (Behavior of $\overline{V}$ *and* $\underline{V}$ *at time T)*
In this step, we adapt the *Step 4* in the proof of [23, Theorem 5.1] or in [21, Theorem 3.2] using our result in Proposition 2.3.3. The main difference relies in the use of the sequence of Cauchy problems with bounded domains (2.3.43) instead of the Cauchy problem (2.3.41) that was used in [21, 23]. We repeat the proof for the sake of consistency and clarity.
We prove only the statement for subsolution, since the proof for the supersolution is completely analogous.

We fix $\overline{x} \in \mathbb{R}^n$ and $t_0 > 0$, and we consider, for some $n > 0$ to be later made precise, the unique bounded solution $\omega^{r,n}$ to the Cauchy problem in $[0, T(n)] \times D_n$ where $T(n) \coloneqq n^2 t_0$ and D_n is the ball of radius n in $\mathbb{R}^m$

$$\begin{cases} \omega_t - \mathcal{L}(y, D\omega, D^2\omega) = 0, & \text{in } (0, T(n)] \times D_n, \\ \omega(0, y) = \sup\limits_{\{|x-\overline{x}| \le r\}} g(x, y), & \text{in } D_n, \\ \omega(t, y) = 0, & \text{in } [0, T(n)] \times \partial D_n. \end{cases} \tag{2.4.20}$$

Using stability properties of viscosity solutions it is not hard to see that $\omega^{r,n}$ converges, as $r \to 0$, to ω^n solution to (2.3.43) set in $[0, T(n)] \times D_n$.

Denote by

$$\overline{g}(\overline{x}) \coloneqq \mu_{\overline{x}}(g(\overline{x}, \cdot)) = \int_{\mathbb{R}^m} g(\overline{x}, y)\, \mathrm{d}\mu_{\overline{x}}(y).$$

Using the convergence result in Proposition 2.3.3 and the uniform convergence of $\omega^{r,n}$ to ω^n, it is easy to see that for every $\eta > 0$ there exist r_0 and $n_0 > 0$ such that

$$\forall\, n \ge n_0 : \quad |\omega^{r,n}(T(n), y) - \overline{g}(\overline{x})| \le \eta, \quad \forall\, r < r_0,\, y \in D_n \supseteq \overline{D}_{n_0}. \tag{2.4.21}$$

We now fix $r < r_0$ and a constant M_r such that $V^\varepsilon(t, x, y) \le M_r$ and $|g(x, y)| \le M_r/2$ for every $\varepsilon > 0$, $x \in \overline{B} \coloneqq \overline{B(\overline{x}, r)}$ and $y \in \overline{D} \coloneqq \overline{D}_{n_0}$. This is possible by Proposition 2.2.1 and assumption (2.2.8). Moreover we fix a smooth non-negative function ψ such that $\psi(\overline{x}) = 0$ and $\psi(x) + \inf\limits_{y \in \overline{D}} g(x, y) \ge 2M_r$ for every $x \in \partial B$. Let C_r be a positive constant such that

$$|H^\varepsilon(t, x, y, D\psi(x), D^2\psi(x), 0)| \le C_r \quad \text{for } x \in \overline{B},\ y \in \overline{D} \text{ and } \varepsilon > 0$$

where H^ε is defined in (2.2.12). Note that such constant exists thanks to assumptions (2.2.2) and (2.2.8). We define the function

$$\psi_r^\varepsilon(t,x,y) = \omega^{r,n}\left(\frac{T-t}{\varepsilon}, y\right) + \psi(x) + C_r(T-t),$$

for some fixed $n > n_0$, and we claim that it is a supersolution to the parabolic problem

$$\begin{cases} -V_t + F^\varepsilon\left(t,x,y,V,D_xV,\dfrac{D_yV}{\varepsilon},D^2_{xx}V^\varepsilon,\dfrac{D^2_{yy}V}{\varepsilon},\dfrac{D^2_{xy}V}{\sqrt{\varepsilon}}\right) = 0, & \text{in } (0,T)\times B\times\overline{D} \\ V(t,x,y) = M_r, & \text{in } (0,T)\times\partial B\times\overline{D} \\ V(T,x,y) = g(x,y), & \text{in } \overline{B}\times\overline{D} \end{cases} \tag{2.4.22}$$

where F^ε is defined in (2.2.11). Indeed

$$\begin{aligned} &-(\psi_r^\varepsilon)_t + F^\varepsilon\left(t,x,y,D_x\psi_r^\varepsilon,\frac{D_y\psi_r^\varepsilon}{\varepsilon},D^2_{xx}\psi_r^\varepsilon,\frac{D^2_{yy}\psi_r^\varepsilon}{\varepsilon},\frac{D^2_{xy}\psi_r^\varepsilon}{\sqrt{\varepsilon}}\right) \\ &\quad = \frac{1}{\varepsilon}\left[(\omega^{r,n})_t - \mathcal{L}(y,D\omega^{r,n},D^2\omega^{r,n})\right] + C_r + H^\varepsilon(t,x,y,D\psi(x),D^2\psi(x),0) \geq 0. \end{aligned}$$

Moreover $\psi_r^\varepsilon(T,x,y) = \sup_{\{|x-\overline{x}|\leq r\}} g(x,y) + \psi(x) \geq g(x,y)$. Finally, observe that the constant function $\min\{\,0\,;\,\inf_{y\in\overline{D}}\sup_{\{|x-\overline{x}|\leq r\}} g(x,y)\,\}$ is always a subsolution to (2.4.20) and then by a standard comparison principle we obtain $\omega^{r,n}(t,y) \geq \min\{\,0\,;\,\inf_{y\in\overline{D}}\sup_{\{|x-\overline{x}|\leq r\}} g(x,y)\,\}$. This implies

$$\begin{aligned} \psi_r^\varepsilon(t,x,y) \geq\ & \min\{\,0\,;\,\inf_{y\in\overline{D}}\sup_{\{|x-\overline{x}|\leq r\}} g(x,y)\,\} + 2M_r - \inf_{y\in\overline{D}} g(x,y) + C_r(T-t), \quad \forall\, x\in\partial B \\ \geq\ & M_r \end{aligned}$$

where we have used either the fact that $|g(x,y)| \leq M_r/2$, and hence $-\inf_{y\in\overline{D}} g(x,y) \geq -M_r/2$, when we have $\min\{\,0\,;\,\inf_{y\in\overline{D}}\sup_{\{|x-\overline{x}|\leq r\}} g(x,y)\,\} = 0$, or otherwise, we have used the fact that $\inf_{y\in\overline{D}}\sup_{\{|x-\overline{x}|\leq r\}} g(x,y) - \inf_{y\in\overline{D}} g(x,y) \geq 0$. In the first case, we get $\psi_r^\varepsilon(t,x,y) \geq 3M_r/2$ and in the second case we have $\psi_r^\varepsilon(t,x,y) \geq 2M_r$. Then ψ_r^ε is a supersolution to (2.4.22). For our choice of M_r we get that V^ε is a subsolution to (2.4.22). Moreover both V^ε and ψ_r^ε are bounded in $[0,T]\times\overline{B}\times\overline{D}$, because of the estimate (2.2.15), of the boundedness of $\omega^{r,n}$ and of the regularity of ψ. So, a standard comparison principle for viscosity solutions gives

$$\begin{aligned}V^\varepsilon(t,x,y) \leq\ & \psi_r^\varepsilon(t,x,y)\\ &= \omega^{r,n}\left(\frac{T-t}{\varepsilon}, y\right) + \psi(x) + C_r(T-t)\end{aligned}$$

for every $0 < r < r_0$, $n > n_0$ $\varepsilon > 0$, $(t,x,y) \in [0,T] \times \overline{B} \times \overline{D}$. We compute the upper limit of both sides of the previous inequality as $(\varepsilon, t, x, y) \to (0, t', x', y')$ for $t' \in (0,T)$, $x' \in B$, $y' \in D$ and $\varepsilon := \varepsilon(n) = \frac{T-t}{T(n)}$ (recalling $T(n) = n^2 t_0$) and get, using (2.4.21),

$$\overline{V}(t',x') \leq \overline{g}(\overline{x}) + \eta + \psi(x') + C_r(T-t').$$

Then taking the upper limit for $(t',x') \to (T,\overline{x})$, we obtain obtain $\overline{V}(T,\overline{x}) \leq \overline{g}(\overline{x}) + \eta$ which permits to conclude recalling that η is arbitrary.

The proof for $\underline{V}$ is completely analogous, once we replace the Cauchy problem (2.4.20) with

$$\begin{cases}\omega_t - \mathcal{L}(y, D\omega, D^2\omega) = 0, & \text{in } (0,T(n)] \times D_n,\\ \omega(0,y) = \inf\limits_{\{|x-\overline{x}|\leq r\}} g(x,y), & \text{in } D_n,\\ \omega(t,y) = 0, \quad \text{in } [0,T(n)] \times \partial D_n. \end{cases}$$

Step 5. (Uniform convergence)
We observe that by definition $\overline{V} \geq \underline{V}$ and that both $\overline{V}$ and $\underline{V}$ satisfy the same quadratic growth condition (2.4.4). Moreover the Hamiltonian $\overline{H}$ defined in(2.3.40) inherit all the regularity properties of H in (2.2.12), as easily seen by their definitions. Therefore we can use the comparison result between sub- and supersolutions to parabolic problems satisfying a quadratic growth condition, given in [71, Theorem 2.1], to deduce $\underline{V} \geq \overline{V}$. Therefore $\underline{V} = \overline{V} =: V$. In particular V is continuous, and by definition of half-relaxed semilimits, this implies that V^ε converges locally uniformly to V (see [19, Lemma V.1.9]).

Proof of the claim (2.4.14)*:*
In what follows, and for the sake of clarity, we will omit the dependency on x for the fast subsystem (2.3.1). Its infinitesimal generator writes

$$-\mathcal{L}V(y) = -\langle b(y), \nabla V(y)\rangle - \operatorname{trace}(a(y)D^2V(y)) \tag{2.4.23}$$

It is easy to see that any $\overline{y}$ such that $|\overline{y}| = e^{-1/2}$ is a global minimizer of ω. Fix such $\overline{y}$. We look for $(\eta, \overline{R})$ such that $\hat{y} := \operatorname*{argmin}_{y\in\mathbb{R}^m} \left\{\omega(y) - \eta\frac{|y-\overline{y}|^2}{2}\zeta(y)\right\} \in B(\overline{y}, \overline{R} - \theta)$ and

$$-\mathcal{L}\left(\omega(y)-\eta\frac{|y-\overline{y}|^2}{2}\zeta(y)\right)\geq 0,\quad \forall y\in B(\overline{y},\overline{R}-\theta) \tag{2.4.24}$$

We have, when $\zeta(y)=1$,

$$\begin{aligned}
-\mathcal{L}&\left(\omega(y)-\eta\frac{|y-\overline{y}|^2}{2}\right)\\
&=-\left(\frac{1}{2}+\log(|y|)\right)\langle b(y),y\rangle-\left(\frac{1}{2}+\log(|y|)\right)\mathrm{trace}(a(y))\\
&\qquad-\frac{1}{|y|^2}\mathrm{trace}((y\otimes y)a(y))+\eta\,\langle b(y),y-\overline{y}\rangle+\eta\,\mathrm{trace}(a(y)) \qquad (2.4.25)\\
&\geq-\left(\frac{1}{2}+\log(|y|)\right)\left(\langle b(y),y\rangle+m\overline{\Lambda}\right)-\overline{\Lambda}\\
&\qquad+\eta\Big(\langle b(y),y-\overline{y}\rangle+m\underline{\Lambda}\Big)
\end{aligned}$$

So we need to find a pair (η,R) such that $\forall\, y\in B(\overline{y},R)$, and hence also in $B(\overline{y},R-\theta)$,

$$\eta\Big(\langle b(y),y-\overline{y}\rangle+m\underline{\Lambda}\Big)\geq\left(\frac{1}{2}+\log(|y|)\right)\left(\langle b(y),y\rangle+m\overline{\Lambda}\right)+\overline{\Lambda} \tag{2.4.26}$$

We start from the left-hand side of the above inequality.
Thanks to assumption (2.2.3), for $y\in B(\overline{y},R)$, i.e. $|y-\overline{y}|\leq R$ and recalling $|\overline{y}|=e^{-1/2}$, we have

$$|\langle b(y),y-\overline{y}\rangle|\quad\leq\quad C(1+|y|)|y-\overline{y}|\quad\leq\quad C(1+e^{-1/2}+R)R$$

$$\Rightarrow\quad -C(1+e^{-1/2}+R)R+m\underline{\Lambda}\quad\leq\quad\langle b(y),y-\overline{y}\rangle+m\underline{\Lambda}\quad\leq\quad C(1+e^{-1/2}+R)R+m\underline{\Lambda}$$

We look for $R>0$ such that the left hand side of the above (double) inequality is positive and hence $0<\langle b(y),y-\overline{y}\rangle+m\underline{\Lambda}$. Denote by $R_\pm$ the roots of the polynomial

$$P(Z):=-C(1+e^{-1/2}+Z)Z+m\underline{\Lambda}=-CZ^2-C(1+e^{-1/2})Z+m\underline{\Lambda}$$

It is easy to see that there exist two real roots defined by

$$R_\pm:=\frac{C(1+e^{-1/2})\pm\sqrt{C^2(1+e^{-1/2})^2+4Cm\underline{\Lambda}}}{-2C}$$

Denote by $R_c=|R_+|\wedge|R_-|$. Hence, for any $R\in]0,R_c[$, $P(R)>0$ and we have $\forall\, y\in B(\overline{y},R)$

$$0<P(R)\quad<\quad\langle b(y),y-\overline{y}\rangle+m\underline{\Lambda}\quad\leq\quad C(1+e^{-1/2}+R)R+m\underline{\Lambda} \tag{2.4.27}$$

Choose $\overline{R}$ and fix $\theta > 0$ small enough such that $\overline{R} \pm \theta \in]0, R_c[$. Hence, to get (2.4.26), we choose

$$\eta \geq \frac{\left(\frac{1}{2} + \log(|y|)\right)\left(\langle b(y), y\rangle + m\overline{\Lambda}\right) + \overline{\Lambda}}{\langle b(y), y - \overline{y}\rangle + m\underline{\Lambda}}, \quad \forall\, y \in B(\overline{y}, \overline{R}) \tag{2.4.28}$$

Finally, it is easy to notice that both the numerator and denominator are bounded for all $y \in B(\overline{y}, R)$: the denominator is bounded thanks to (2.4.27), and the numerator is a continuous function on $B(\overline{y}, \overline{R})$. Note in addition that the right hand side when evaluated in $y = \overline{y}$ is equal to $\frac{\overline{\Lambda}}{m\underline{\Lambda}}$ and hence is a lower-bound of the expression we are maximizing. Therefore, by choosing such $\overline{R} \in]0, R_c[$ and η such that

$$\eta \geq \max_{|y-\overline{y}|\leq \overline{R}} \frac{\left(\frac{1}{2} + \log(|y|)\right)\left(\langle b(y), y\rangle + m\overline{\Lambda}\right) + \overline{\Lambda}}{\langle b(y), y - \overline{y}\rangle + m\underline{\Lambda}} \;\geq\; \frac{\overline{\Lambda}}{m\underline{\Lambda}} \;>\; 0$$

we have (2.4.26), which in turn implies (2.4.24).

We need now to check that $\hat{y} := \operatorname*{argmin}_{y\in\mathbb{R}^m} \left\{\omega(y) - \eta\frac{|y-\overline{y}|^2}{2}\zeta(y)\right\} \in B(\overline{y}, \overline{R} - \theta)$. This is easy to see since we know that $\overline{y}$ is a global minimizer (not unique) of ω. And we are subtracting a positive quantity in a neighborhood of $\overline{y}$ thanks to the cut-off function ζ. Therefore we recover a new global minimizer (unique) which belongs to the domain where $\zeta = 1$, that is $B(\overline{y}, \overline{R} - \theta)$. □

2.5 Deep relaxation of controlled Stochastic Gradient Descent

2.5.1 Introduction and motivation

A gradient descent in its continuous version is given as

$$\mathrm{d}X_t = -\nabla F(X_t)\,\mathrm{d}t$$

It is well known that such process converges, under some mild assumptions, to a local minimum of $F(x)$ and a huge literature exists on this topic. However, in most of the applications nowadays, the function to be minimized lacks many properties: it is usually non convex and also not smooth. Moreover, the dimension of the unknown variable is tremendously large, and this issue is known as *the curse of dimensionality*. This is the case in deep neural networks and more precisely in supervised learning when one wishes to find the optimal parameters of the neural network in order to fit the data set on which it performs the "training". A particular (and well known) example of such algorithms is the image classification. To overcome this issues (non smoothness of the function to be minimized/maximized, large dimension, non convexity, ...), one performs a *stochastic gradient descent* (SGD) where the stochasticity is added artificially to recover some richness in the exploration performed by a "non-exact" gradient descent. In fact, one only computes a sample of the gradient (called *mini-batch*), that is a gradient with respect to only some variables, then adds a noise to the latter. The stochastic gradient descent in the continuous version writes

$$\mathrm{d}X_t = -\nabla_{\mathrm{mb}} F(X_t)\mathrm{d}t + \sigma \mathrm{d}W_t$$

where $W_\cdot$ is a Wiener process, σ some constant, and ∇_{mb} is the gradient performed over a mini-batch, that is a subset of variables. Again, a huge literature exists on this topic, and many algorithms are performed and improved for the convergence of such process towards a minimum of the function F, called *loss function*. It is worth noticing that such process is of Smoluchowski type, provided F is a smooth confining potential (see Definition 2.6.1). We have seen however that smoothness is also an issue when dealing with such optimization problems. One way to handle this problem is the well known use of convolution together with a mollifier, see e.g. [151, §7.2]: if η_γ is a smooth (say C^∞) function such that $\int_{\mathbb{R}^n} \eta_\gamma = 1$, then $F_\gamma := F * \eta_\gamma$ may be considered as a *convex weighted average* of F which enjoys more smoothness properties. Moreover, as $\gamma \to 0$, F_γ is known to be a good approximation of F [151, Lemma 7.1]. In this way one recovers a more regular version of the loss function to be minimized, and hence expects a better performance of the SGD since the gradients will be computed in a more accurate way. Still, the high dimensionality of the optimization problem will be an obstacle towards the computation of the whole gradient. And this is where singular perturbations will be used, together with the regularization obtained with the mollifiers.

This modification of the loss function has been introduced in [18, 63] where the authors used a regularized version of the loss function called *local entropy*, whereas the additional use of homogenization appeared in [64]. In the sequel, we shall apply our convergence result of trajectories to the system of SDEs introduced in [64] and whose limit is a gradient descent (not stochastic) of a regularized version of the loss function; the local entropy. Moreover, we will add a control parameter in the same spirit as in [123, §4.1]. We therefore recover a convergence result for a singularly perturbed system towards a *Controlled Stochastic Gradient Descent* of a local entropy function that plays the role of the regularized loss function.

2.5.2 The model with singular perturbations

In what follows, we consider a generic non-convex optimization problem

$$\min_{x\in\mathbb{R}^n} f(x) \tag{2.5.1}$$

where f is a scalar function, a priori not smooth. We will give more precise assumption on f later on. Such an optimization problem arises in training a deep neural network. We don't give further features on the loss function f here and refer to the large existing literature.

The authors in [64] introduced f_γ, a regularization of the loss function f, such that

$$f_\gamma := -\frac{1}{\beta}\log\left(G_{\beta^{-1}\gamma} * \exp(-\beta f(x))\right) \tag{2.5.2}$$

where

$$G_{\beta^{-1}\gamma}(x) := (2\pi\gamma)^{-n/2}\exp\left(-\frac{\beta}{2\gamma}|x|^2\right)$$

is the heat kernel, and $\beta, \gamma > 0$ are fixed parameter. The function f_γ plays the role of a local entropy and, as $\gamma \to 0$, it is a smooth approximation of f. The parameter β corresponds in physics to the inverse of the temperature (see [144, Chapter 6]) and as $\beta \to \infty$, the heat kernel tends to Dirac measure supported on 0 (see [144, Chapter 7, p.236]).

It is now easy to see that the gradient of (2.5.2) has the following nice structure.

Lemma 2.5.1. *([64, Lemma 1]) The gradient of the regularized loss function (2.5.2) is given by*

$$\nabla f_\gamma(x) = \int_{\mathbb{R}^n} \frac{x-y}{\gamma}\rho_\beta^\infty(dy;x) \tag{2.5.3}$$

where $\rho_\beta^\infty(y;x) := Z^{-1}\exp\left(-\beta\left(f(y) + \frac{1}{2\gamma}|x-y|^2\right)\right)$ *and* Z *is a normalizing constant.*

Now the fact that ∇f_γ in (2.5.3) is defined as an average of $y \mapsto \frac{1}{\gamma}(x-y)$ over a Gibbs measure ρ_β^∞ is reminiscent to what one usually expects to obtain in homogenization and in singular perturbations.

Indeed, with this observation, we would like to build a system of singularly perturbed SDEs whose limiting behavior yields (2.5.3). This motivates the following system of *singularly perturbed SDEs* as introduced in [64] where we first define

$$V(y,x) := f(y) + \frac{1}{2\gamma}|x-y|^2 \tag{2.5.4}$$

and then we set

$$\begin{aligned} \mathrm{d}X_s &= -\nabla_x V(Y_s, X_s)\,\mathrm{d}s, \quad X_0 = x \in \mathbb{R}^n \\ \mathrm{d}Y_s &= -\frac{1}{\varepsilon}\nabla_y V(Y_s, X_s)\,\mathrm{d}s + \sqrt{\frac{2}{\varepsilon}}\beta^{-1/2}\,\mathrm{d}W_s, \quad Y_0 = y \in \mathbb{R}^n. \end{aligned} \tag{2.5.5}$$

Therefore, we expect the limit as $\varepsilon \to 0$ in the above system of SDEs to be

$$\mathrm{d}\hat{X}_s = \int_{\mathbb{R}^n} -\frac{1}{\gamma}(X_s - y)\rho^\infty_\beta(\mathrm{d}y; X_s)\,\mathrm{d}s, \quad \hat{X}_0 = x \in \mathbb{R}^n$$

which writes (using Lemma 2.5.1) as

$$\mathrm{d}\hat{X}_s = -\nabla f_\gamma(\hat{X}_s)\mathrm{d}s, \quad \hat{X}_0 = x \in \mathbb{R}^n \tag{2.5.6}$$

that is the gradient descent (not stochastic) of the regularized loss function.

Several preliminary questions arise before the study of the limit $\varepsilon \to 0$:

- Well-posedness of the SDES in (2.5.5) and in (2.5.6): a sufficient condition for existence and uniqueness of strong solutions is to have f Lipschitz continuous and with at most a linear growth.

- Existence and uniqueness of ρ^∞_β: it is well defined when V is a *confining potential* (see Definition 2.6.1 in the appendix), and it is the unique invariant probability measure corresponding to the stochastic process $Y_\cdot$ in (2.5.5) by classical results that we recall in the appendix (in particular Proposition 2.6.1 and Theorem 2.6.1).

The limit as $\varepsilon \to 0$ in (2.5.5) has been studied in various ways under different assumptions, and several references are mentioned in section 2.1. Our goal in the sequel is to show this limit in the context of stochastic optimal control theory with an application to controlled stochastic gradient descent as we shall describe in the next section.

2.5.3 Controlled Stochastic Gradient Descent

Following the model in [123, §4], we can introduce in (2.5.5) a control parameter u which plays the role of a Learning Rate. Its optimization will allow us to control how far the process $X_\cdot$ (and equivalently $\hat{X}_\cdot$) should follow the gradient descent, in other words, how trustful is the gradient descent direction. Usually the control u takes values in $[0,1]$. In the sequel we consider U as a general compact set of values that the control u would take, and we write the new system of *singularly perturbed controlled SDEs* as

$$\begin{aligned} \mathrm{d}X_s &= -u_s\,\nabla_x V(Y_s,X_s)\,\mathrm{d}s + \sqrt{2}\sigma(X_s,Y_s,u_s)\,\mathrm{d}W_s, \quad X_0 = x \in \mathbb{R}^n \\ \mathrm{d}Y_s &= -\frac{1}{\varepsilon}\nabla_y V(Y_s,X_s)\,\mathrm{d}s + \sqrt{\frac{2}{\varepsilon}}\beta^{-1/2}\,\mathrm{d}W_s, \quad Y_0 = y \in \mathbb{R}^n \end{aligned} \tag{2.5.7}$$

where V is defined in (2.5.4), and σ is a diffusion term that also depends on the learning rate which is the control, and is allowed to be zero. The optimal learning rate should provide a balance between *exploration* (how fast at each step should we follow the drift) and *exploitation* (how much at each step should we diffuse and look around). Given an appropriate cost function for the problem of tuning the learning rate, we can write an optimal control problem of the form

$$\min_u \mathbb{E}\left[g(X_T,Y_T)e^{\lambda(t-T)} + \int_t^T \ell(s,X_s,Y_s,u_s)e^{\lambda(s-T)}\,\mathrm{d}s \;\middle|\; X_t = x,\ Y_t = y \right]$$

subject to (2.5.7), where λ is a non negative constant, and g,ℓ satisfy some growth assumptions that we will later made precise in section 2.2.2, and can be chosen according to the performance we seek (sparsity, momentum, covariance, $\mathbb{E}[f(X_T)]$, ...).

Our main result (Theorem 2.5.2) insures that at the limit $\varepsilon \to 0$, the stochastic control problem with singular perturbations (2.5.7) is again a control problem that is subject to dynamics of the form

$$\mathrm{d}\hat{X}_s = -\nu_s \nabla f_\gamma(\hat{X}_s)\,\mathrm{d}s + \sqrt{2}\overline{\sigma}(\hat{X}_s,\nu_s)\,\mathrm{d}W_s \tag{2.5.8}$$

where

$$\overline{\sigma}(\hat{x},\nu) := \sqrt{\int_{\mathbb{R}^n} \sigma(\hat{x},y,\nu)\sigma^\top(\hat{x},y,\nu)\rho^\infty(\mathrm{d}y;x)}$$

Note that by taking $\sigma \equiv 0$ and $U = \{1\}$, we recover the particular case of (2.5.5) and (2.5.6).

The benefit of this approximation is that we can allow the use of *mini-batches* in the fast variables of (2.5.5) and recover at the limit as $\varepsilon \to 0$, a *full* gradient of the regularized loss function in (2.5.6) while controlling the learning rate (or any other parameter in the dynamics). We show in what follows an example of such application and how it can benefit in optimization problems.

A practical example.

Let U be a compact subset of $\mathbb{R}$ and $\mathcal{U}$ be the set progressively measurable functions from $[0,T]$ to U and $T>0$. Recalling the definition of the regularized loss function f_γ as in (2.5.2), we are interested in the following optimal control problem (where $\sigma=0$ is allowed, in which case we would have a deterministic control problem)

$$\begin{aligned} \mathfrak{U}(x) := \min_{u_\cdot \in \mathcal{U}} \ & \mathbb{E}[f(X_T)] \\ \text{s.t.} \ \ & \mathrm{d}X_t = -u_t \nabla f_\gamma(x_t)\,\mathrm{d}t + \sigma\,\mathrm{d}W_t \\ & X_0 = x \in \mathbb{R}^n, \quad t \in [0,T] \end{aligned} \tag{2.5.9}$$

We introduce the *singularly perturbed* optimal control problem

$$\begin{aligned} \mathfrak{U}^\varepsilon(x,y) := \min_{u_\cdot \in \mathcal{U}} \ & \mathbb{E}[f(X_T^\varepsilon)] \\ \text{s.t.} \ \ & \mathrm{d}X_t^\varepsilon = -u_t\gamma^{-1}(X_t^\varepsilon - Y_t^\varepsilon)\,\mathrm{d}t + \sigma\,\mathrm{d}W_t \\ & \mathrm{d}Y_t^\varepsilon = -\frac{1}{\varepsilon}\left(\nabla f(Y_t^\varepsilon) - \frac{1}{\gamma}(X_t^\varepsilon - Y_t^\varepsilon)\right)\mathrm{d}t + \sqrt{\frac{2}{\varepsilon}}\beta^{-1/2}\,\mathrm{d}W_t \\ & X_0^\varepsilon = x \in \mathbb{R}^n,\ Y_0^\varepsilon = y \in \mathbb{R}^n, \quad t \in [0,T] \end{aligned} \tag{2.5.10}$$

Theorem 2.5.1. *Let f be Lipschitz continuous. Then we have*

$$\lim_{\varepsilon\to 0} \mathfrak{U}^\varepsilon(x,y) \le \mathfrak{U}(x)$$

locally uniformly in $x, y \in \mathbb{R}^n$, i.e. the dynamics with singular perturbations yields a lower value than the one with a controlled full gradient descent.

In other words, the latter theorem insures that with a controlled and singularly perturbed system of SDEs, one reaches a value of the function f to be minimized (in expectation) lower than any other choice of the learning rate (which is here represented by the control).

The proof is postponed to the end of section 2.5.4.

Reminder 2.5.1. *We believe that allowing mini-batches in the dynamics of $Y_\cdot^\varepsilon$ does not alter the result, since at the limit $\varepsilon \to 0$ we recover the invariant measure of the process, which is independent of which subgradients we choose or not to implement, i.e., those chosen in the mini-batches. And the invariant measure $\mu_x(\cdot)$, where x is fixed, only depends on which potential $V(\cdot, x)$ we are considering; in our case it is $y \mapsto V(y,x) := f(y) + \frac{1}{2}\gamma^{-1}|x-y|^2$ and is explicitly given by its density ρ^∞ as in Lemma 2.5.1 or as in Proposition 2.6.1.*

More generally, we conjecture that there exists a natural number n^* strictly less than the space dimension n, such that the invariant measure associated to a stochastic process (or to the corresponding stationary Fokker-Planck equation), with constant diffusion and a drift given by $-\nabla V$ where V is a confining potential, remains unchanged when we consider the same process but with a drift given by $-\nabla_{n^*} V$ where $\nabla_{n^*} = (\alpha_1 \frac{\partial}{\partial x_1}, \ldots, \alpha_n \frac{\partial}{\partial x_n})^\top$ and $\alpha_i \in \{0, 1\}$ satisfying $\sum_{i=1}^{n} \alpha_i = n^* < n$. Such an operator ∇_{n^*} corresponds to the gradient with mini-batches ∇_{mb} as presented earlier in this section.

2.5.4 A control interpretation of the limit PDE

The following result allows to represent the effective Hamiltonian (2.3.40) as a Bellman Hamiltonian associated to an *effective* optimal control problem that we will construct by a relaxation procedure.

Proposition 2.5.1. *Under the standing assumptions, the effective Hamiltonian* (2.3.40) *writes*

$$\overline{H}(t, x, p, P) = \min_{\nu \in \mathcal{U}^{ex}(x)} \int_{\mathbb{R}^m} \left[-trace(\sigma\sigma^\top P) - f \cdot p - \ell \right] \, d\mu_x(y) \tag{2.5.11}$$

where σ, f are computed in (x, y, ν) and ℓ in (t, x, y, ν) , and $\mathcal{U}^{ex}(x)$ is the set of progressively measurable processes taking values in the extended control set $U^{ex}(x) := L^2((\mathbb{R}^m, \mu_x), U)$.

Reminder 2.5.2.
- *The set $\mathcal{U}^{ex}(x)$ contains a copy of U given by constant functions since μ_x is a probability measure.*
- *The choice of $U^{ex}(x) := L^2((\mathbb{R}^m, \mu_x), U)$ is justified by the quadratic term of the Hamiltonian, that is $trace(\sigma\sigma^\top M)$. The drift f is supposed to have a linear growth in the control, and since μ_x is finite, we have $L^2((\mathbb{R}^m, \mu_x), U) \subset L^1((\mathbb{R}^m, \mu_x), U)$.*
- *Since any process $\nu(\cdot) \in U^{ex}(x)$ takes values in a compact set U, it is in particular bounded and hence in $L^\infty(\mathbb{R}^m, U)$. The latter being a subset of $U^{ex}(x)$, one can write* (2.5.11) *with $\mathcal{U}^{ex}(x)$ being the set of progressively measurable processes taking values in $L^\infty(\mathbb{R}^m, U)$ which is independent of x.*

Proof. (Proposition 2.5.1)
Fix $(t, x, p, M) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{S}^n$, and define

$$F(y, u) = -\text{trace}(\sigma(x, y, u)\sigma^\top(x, y, u)M) - f(x, y, u) \cdot p - \ell(t, x, y, u), \quad \text{in } \mathbb{R}^m \times U.$$

Now define LHS (resp., RHS) the left hand side (resp., right hand side) of (2.5.11) by

$$LHS := \int_{\mathbb{R}^m} \min_{u\in U} F(y,u)\ \mathrm{d}\mu_x(y), \quad \text{and} \quad RHS(\nu) := \int_{\mathbb{R}^m} F(y,\nu(y))\,\mathrm{d}\mu_x(y)$$

then (2.5.11) writes

$$LHS = \min_{\nu\in\mathcal{U}^{ex}(x)} RHS(\nu) \tag{2.5.12}$$

Step 1. $(LHS \leq \min_{\nu\in\mathcal{U}^{ex}(x)} RHS(\nu))$.
This is the easy inequality. It suffices to notice that, for all $\nu(\cdot)\in\mathcal{U}^{ex}(x)$

$$RHS(\nu) \geq \int_{\mathbb{R}^m} \min_{u\in U} F(y,u)\ \mathrm{d}\mu_x(y) = LHS$$

and in particular, taking the minimum in $\mathcal{U}^{ex}(x)$ yields the desired inequality.

Step 2. $(LHS \geq \min_{\nu\in\mathcal{U}^{ex}(x)} RHS(\nu))$.
We first start by choosing a sequence $(I_i)_{i\in\mathbb{Z}}$ of open intervals in $\mathbb{R}^m$ such that $I_i\cap I_j=\emptyset$ and $\mathbb{R}^m = \cup_{i\in\mathbb{Z}}\overline{I}_i$, where $\overline{I}_i$ is the closure of I_i. We denote by $RHS(i)$ the integral defining RHS but where the integration is done only on I_i. Therefore, one has for an arbitrary chosen $\nu\in\mathcal{U}^{ex}(x)$

$$\mathrm{RHS}(i) = \int_{I_i} F(y,\nu(y))\,\mathrm{d}\mu_x(y)$$

Recall that μ_x is a probability measure and hence is positive with a total variation equal to 1. Denote by $A_k = \cup_{i=-k}^{k}\overline{I}_i$ and let $y\mapsto F_k(y)$ be the sequence of functions defined as

$$F_k(y) = \mathbb{1}_{\{y\in A_k\}}(y)F(y,\nu(y)), \quad \forall\, y\in\mathbb{R}^m,\ \forall\, k\in\mathbb{N}$$

where ν is an arbitrary fixed element of $\mathcal{U}^{ex}$ that we omit in F_k for the sake of clarity, and $\mathbb{1}_{\{y\in A_k\}}$ is the characteristic function of A_k. Notice that $\lim_{k\to+\infty} A_k = \mathbb{R}^m$.
It is clear that the family $\{F_k\}_k$ is uniformly integrable over $\mathbb{R}^m$, that is:
For every $\varepsilon>0$, one can find $\delta>0$ such that

$$\text{if } D\subseteq\mathbb{R}^m \text{ with } \mu_x(D)<\delta, \text{ then } \textstyle\int_D |F_k(y)|\,\mathrm{d}\mu_x(y)<\varepsilon,\ \forall\, k.$$

This is true since $F_k\leq|F|$ which in turn is integrable with respect to the measure μ_x that has finite moments. Therefore, Vitali's convergence theorem applies:

Since $F_k(y) \xrightarrow[k\to+\infty]{} F(u, \nu(y))$ for μ-almost every y, one has

$$\sum_{i=-k}^{k} \int_{I_i} F(u, \nu(y))\,\mathrm{d}\mu_x(y) = \int_{\mathbb{R}^m} F_k(y)\,\mathrm{d}\mu_x(y) \xrightarrow[k\to+\infty]{} \int_{\mathbb{R}^m} F(y, \nu(y))\,\mathrm{d}\mu_x(y).$$

We shall now consider the minimization problem

$$\min_{u\in U} F(y,u) =: H_i(y) \in F(y,U)$$

where $y \in I_i$. Since U is compact, $H_i(y)$ and $F(y,U)$ are continuous, and μ_x is finite, a classical selection theorem (e.g., [94, Theorem 7.1, p. 66]) implies the existence of a measurable selector $\overline{\nu}_i$ such that

$$H_i(y) = F(y, \overline{\nu}_i(y)).$$

Therefore the minimization is obtained with $\overline{\nu}_i(\cdot)$ and one has

$$\forall i \in \mathbb{Z},\ \exists\, \overline{\nu}_i \in \mathcal{U}^{ex}(x),\ \text{s.t.}\ \ \forall y \in I_i,\ H_i(y) = \min_{u\in U} F(y,u) = F(y, \overline{\nu}_i(y)).$$

Now consider $\overline{\nu} \in \mathcal{U}^{ex}(x)$ defined as $y \mapsto \overline{\nu}(y) = \{\overline{\nu}_i(y),\ \text{if } y \in I_i,\ \forall\, i \in \mathbb{Z}\}$. Therefore, one has

$$\begin{aligned}
LHS = \int_{\mathbb{R}^m} \min_{u\in U} F(y,u)\,\mathrm{d}\mu_x(y) &= \sum_{i\in\mathbb{Z}} \int_{I_i} \min_{u\in U} F(y,u)\,\mathrm{d}\mu_x(y) \\
&= \sum_{i\in\mathbb{Z}} \int_{I_i} F(y, \overline{\nu}_i(y))\,\mathrm{d}\mu_x(y) \\
&= \int_{\mathbb{R}^m} F(y, \overline{\nu}(y))\,\mathrm{d}\mu_x(y) \\
&\geq \min_{\nu\in\mathcal{U}^{ex}(x)} \int_{\mathbb{R}^m} F(y, \nu(y))\,\mathrm{d}\mu_x(y) = \min_{\nu\in\mathcal{U}^{ex}(x)} RHS(\nu).
\end{aligned}$$

This yields the desired second inequality, and proves (2.5.12) and equivalently (2.5.11). □

Armed with this result, together with the Lipschitz regularity of the invariant measure in Proposition 2.3.1, we can construct the effective dynamics as

$$\begin{cases} d\hat{X}_t = \displaystyle\int_{\mathbb{R}^m} f(\hat{X}_t, y, \nu_t(y))\mathrm{d}\mu_{\hat{X}_t}(y)\mathrm{d}t + \sqrt{2}\sqrt{\int_{\mathbb{R}^m} \sigma\sigma^\top(\hat{X}_t, y, \nu_t(y))\mathrm{d}\mu_{\hat{X}_t}(y)}\, \mathrm{d}W_t, \\ \nu_t(\cdot) \in \mathcal{U}^{ex}(\hat{X}_t), \quad \text{and} \quad \hat{X}_0 = x \in \mathbb{R}^n. \end{cases} \tag{2.5.13}$$

We recall that $\mathcal{U}^{ex}(\hat{X}_t)$ is the set of progressively measurable processes taking values in the extended control set $U^{ex}(\hat{X}_t) := L^2((\mathbb{R}^m, \mu_{\hat{X}_t}), U)$, that is

$$\nu_\cdot(\cdot) : t \mapsto \nu_t(\cdot) \in L^2((\mathbb{R}^m, \mu_{\hat{X}_t}), U) = \left\{ \phi(\cdot) : y \mapsto \phi(y) \in U \;\middle|\; \int_{\mathbb{R}^m} |\phi(y)|^2 \,\mathrm{d}\mu_{\hat{X}_t}(y) \right\}$$

and the measure $\mu_{\hat{X}_t}$ for a fixed $t \geq 0$ is the unique invariant probability measure associated to the fast subsystem

$$\mathrm{d}Y_s = b(\hat{X}_t, Y_s)\,\mathrm{d}s + \sqrt{2}\varrho(\hat{X}_t, Y_s)\,\mathrm{d}W_s, \quad Y_0 = y \in \mathbb{R}^m.$$

Reminder 2.5.3. *It is immediate to see that both the drift and the diffusion*

$$F(x, \nu) := \int_{\mathbb{R}^m} f(x, y, \nu(y))\, d\mu_x(y) \quad \textit{and} \quad G(x, \nu) := \sqrt{\int_{\mathbb{R}^m} \sigma\sigma^\top(x, y, \nu(y))\, d\mu_x(y)} \tag{2.5.14}$$

are Lipschitz continuous in x with at most a linear growth, and uniformly bounded in ν, that is they satisfy assumption (2.2.2) *with now no dependency on the y variable. Hence the SDE in* (2.5.13) *has a strong solution.*

The dynamics (2.5.13) can be written in fact as a stochastic differential inclusion (SDI) (see [110, 111]). Let us introduce the following set-valued functions $\overline{F}$ from $\mathbb{R}^n$ to $\mathbb{R}^n$ and $\overline{G}$ from $\mathbb{R}^n$ to $\mathbb{R}^{n,r}$ defined using F, G in (2.5.14) as

$$\begin{aligned} \overline{F}(x) &:= F(x, \mathcal{U}^{ex}(x)) = \left\{ \int_{\mathbb{R}^m} f(x, y, \nu(y))\,\mathrm{d}\mu_x(y), \text{ s.t. } \nu(\cdot) \in U^{ex}(x) \right\} \\ \overline{G}(x) &:= G(x, \mathcal{U}^{ex}(x)) = \left\{ \sqrt{\int_{\mathbb{R}^m} \sigma(x, y, \nu(y))\sigma^\top(x, y, \nu(y))\,\mathrm{d}\mu_x(y)}, \text{ s.t. } \nu(\cdot) \in U^{ex}(x) \right\} \end{aligned} \tag{2.5.15}$$

We define the SDI, using the notation $(\overline{F} \circ X)_s(\omega) = \overline{F}(X_s(\omega))$, $(\overline{G} \circ X)_s(\omega) = \overline{G}(X_s(\omega))$ for $t \geq 0$ and $\omega \in \Omega$, where now the *extended* control ν is seen as an element of $\mathcal{U}^{ex}(\cdot)$, and write (2.5.13) as

$$\hat{X}_{t_2} - \hat{X}_{t_1} \in \int_{t_1}^{t_2} (\overline{F} \circ \hat{X})_s\,\mathrm{d}s + \sqrt{2}\int_{t_1}^{t_2} (\overline{G} \circ \hat{X})_s\,\mathrm{d}W_s \tag{2.5.16}$$

We can now write the effective optimal control problem as follows

$$V(t,x) = \sup \hat{J}(t,x,\nu.(\cdot)), \quad \text{subject to } (2.5.13) \tag{$\overline{OCP}$}$$

where the effective pay off is

$$\hat{J}(t,x,\nu.(\cdot)) = \mathbb{E}\left[e^{\lambda(t-T)}\overline{g}(\hat{X}_s) + \int_t^T \overline{\ell}(s,\hat{X}_s,\nu_s)e^{\lambda(s-T)}\,\mathrm{d}s \,\middle|\, \hat{X}_t = x \right] \tag{2.5.17}$$

with

$$\overline{g}(x) := \int_{\mathbb{R}^m} g(x,y)\,\mathrm{d}\mu_x(y) \quad \text{and} \quad \overline{\ell}(s,x,u) := \int_{\mathbb{R}^m} \ell(s,x,y,u)\,\mathrm{d}\mu_x(y)$$

Theorem 2.5.2. *The value function* ($\overline{OCP}$) *is the unique viscosity solution to the Cauchy problem* (2.4.1)*. In particular, it is the limit of* V^ε *defined in* ($OCP(\varepsilon)$)*.*

Proof. (Theorem 2.5.2)
We are in the framework of Proposition 2.2.1, since the dynamics (2.5.13) and the cost function (2.5.17) satisfy the conditions in §2.2.1 following Remark 2.5.3 and thanks to Lipschitz regularity of the invariant measure in Proposition 2.3.1. This insures that the value function as defined by ($\overline{OCP}$) is a viscosity solution to the Cauchy problem (2.4.1) satisfying moreover the quadratic growth condition.
But we know from Theorem 2.4.1 that the limit problem has a unique viscosity solution which is the one given by the limit of V^ε solution to (2.2.9). Therefore V as defined in ($\overline{OCP}$) is the limit of V^ε defined by ($OCP(\varepsilon)$). □

Reminder 2.5.4. *Combining Theorem 2.4.1 and Theorem 2.5.2 yields the following (for example with* $\lambda = 0$ *and* $g \equiv 0$*):*

$$\mathbb{E}\left[\int_t^T \ell(s,X_s^\varepsilon,Y_s^\varepsilon,u_s)\,ds\right] \xrightarrow[\varepsilon\to 0]{} \mathbb{E}\left[\int_t^T \overline{\ell}(s,\hat{X}_s,\nu_s)\,ds\right], \quad \forall\,\ell \text{ continuous, as in } (2.2.8)$$

where we denoted by $(X^\varepsilon_\cdot, Y^\varepsilon_s, u.)$ *an optimal solution to* ($OCP(\varepsilon)$) *and by* $(\hat{X}., \nu.)$ *an optimal solution to* ($\overline{OCP}$)*. In particular, when* ℓ *is independent of* y*, we have* $\ell = \overline{\ell}$*, and choosing it for simplicity also independent of* u*, yields*

$$\mathbb{E}\left[\int_t^T \ell(s,X_s^\varepsilon)\,ds\right] \xrightarrow[\varepsilon\to 0]{} \mathbb{E}\left[\int_t^T \ell(s,\hat{X}_s)\,ds\right], \quad \forall\,\ell \text{ continuous, as in } (2.2.8)$$

This will be used in the next subsection.

We are now ready to prove Theorem 2.5.1 in the practical example at the end of section 2.5.3 as a direct consequence of the previous results.

Proof. (Theorem 2.5.1)
From Theorem 2.4.1 and Theorem 2.5.2, we know that $\mathfrak{U}^\varepsilon(x,y)$ converges locally uniformly to the value function $\overline{\mathfrak{U}}(x)$ defined by the optimal control problem

$$\begin{aligned}\overline{\mathfrak{U}}(x) := \min_{\nu_\cdot(\cdot)\in\mathcal{U}^{\mathrm{ex}}(\cdot)} \quad & \mathbb{E}[f(\overline{X}_T)] \\ \text{s.t.} \quad & \mathrm{d}\overline{X}_t = -\int_{\mathbb{R}^n} \nu_t(y)\gamma^{-1}(\overline{X}_t - y)\,\mathrm{d}\mu_{\overline{X}_t}(y)\,\mathrm{d}t \\ & \overline{X}_0 = x \in \mathbb{R}^n, \quad t\in[0,T]\end{aligned}$$

It suffices then to notice that the set of admissible controls $\mathcal{U}$ is a subset of the extended control set $\mathcal{U}^{\mathrm{ex}}$ since the latter contains controls which are constant with respect to y and μ_x is a probability measure (see Remark 2.5.2).
Hence we have $\lim_{\varepsilon\to 0}\mathfrak{U}^\varepsilon(x,y) = \overline{\mathfrak{U}}(x) \le \mathfrak{U}(x)$. □

2.5.5 Convergence of trajectories in multiscale optimal control

We have shown so far that the value function V^ε in ($OCP(\varepsilon)$) converges locally uniformly to the value function V in ($\overline{OCP}$) as $\varepsilon \to 0$. In this section, we are interested in the link between the singularly perturbed dynamics (2.2.1) and the corresponding effective one (2.5.13) (equivalently (2.5.16)). Mainly we will show that, under the standing assumptions and if $\sigma = 0$ in (2.2.4), then as $\varepsilon \to 0$, every solution to (2.5.13) is approximated by a sequence of processes of the form (2.2.1), in a sense that we will soon after make precise, and conversely, any converging sequence of trajectories (2.2.1), can be represented with a solution to (2.5.13).

In this subsection, we will assume, besides the standing assumptions of §2.2, that the limit in (2.2.4) is null, that is,

$$\lim_{\varepsilon\to 0}\sigma^\varepsilon(x,y,u) = 0 \quad \text{locally uniformly.} \tag{2.5.18}$$

In this case, the effective dynamics (2.5.13) becomes

$$\begin{cases}\dfrac{\mathrm{d}\hat{x}_t}{\mathrm{d}t} = \displaystyle\int_{\mathbb{R}^m} f(\hat{x}_t, y, \nu_t(y))\mathrm{d}\mu_{\hat{x}_t}(y) \\ \nu_t(\cdot) \in \mathcal{U}^{ex}(\hat{x}_t), \quad \text{and} \quad \hat{x}_0 = x \in \mathbb{R}^n.\end{cases} \tag{2.5.19}$$

Note that, since there is no randomness, $\mathcal{U}^{ex}(\hat{x}_t) \equiv U^{ex}(\hat{x}_t) := L^2((\mathbb{R}^m, \mu_{\hat{x}_t}), U)$. And following the last point in Remark 2.5.2, one can take instead of $\mathcal{U}^{ex}(\hat{x}_t)$ the set $L^\infty(\mathbb{R}^m, U) =: \overline{U}^{ex}$ that is independent of x. Hence, the effective dynamics (2.5.19) equivalently writes as

$$\begin{cases} \dfrac{\mathrm{d}\hat{x}_t}{\mathrm{d}t} = \displaystyle\int_{\mathbb{R}^m} f(\hat{x}_t, y, \nu_t(y))\mathrm{d}\mu_{\hat{x}_t}(y) \\ \nu_t(\cdot) \in \overline{U}^{ex}, \quad \text{and } \hat{x}_0 = x \in \mathbb{R}^n. \end{cases} \Leftrightarrow \quad \hat{x}_{t_2} - \hat{x}_{t_1} \in \int_{t_1}^{t_2} \overline{F}(\hat{x}_s)\,\mathrm{d}s. \tag{2.5.20}$$

where $\overline{F}$ is as defined in (2.5.15) and (2.5.16).

Theorem 2.5.3. *Under the standing assumptions of §2.2 and assuming* (2.5.18) *holds, every solution* $\hat{x}_\cdot$ *to the effective dynamics* (2.5.20) *is an accumulation point to a sequence* $\{X^\varepsilon_\cdot\}_{\varepsilon>0}$ *of trajectories* (2.2.1) *in the sense*

$$\lim_{\varepsilon\to 0} \mathbb{E}\left[\,\left|X^\varepsilon_s - \hat{x}_s\right|^p\,\right] = 0, \quad a.e.\ s \in [t, T]$$

for some $p \in (0, 2]$.

Proof. (Theorem 2.5.3)
The proof is in the same spirit as for Theorem 1.3.2: we will construct an optimal control problem of the form ($OCP(\varepsilon)$), then exploit the convergence result of the value function to finally deduce the desired result using a particular choice of pay off function.

We start by fixing an initial condition $y \in \mathbb{R}^m$ for the fast process $Y^\varepsilon_\cdot$, and we consider a fixed pair $(\hat{x}_\cdot, \nu_\cdot) : [0, T] \to \mathbb{R}^n \times \overline{U}^{ex}$ satisfying (2.5.20) with $\hat{x}_0 = x$ fixed in $\mathbb{R}^n$. We choose $X^\varepsilon_\cdot$ with the same initial condition as for $\hat{x}_\cdot$ and a diffusion σ^ε satisfying (2.5.18), which together with $Y^\varepsilon_\cdot$ solves (2.2.1). We then choose a pay off functional of the form (2.2.7) with $g \equiv 0$ and a running cost

$$\ell(s, x) = -\left|x - \hat{x}_s\right|^p, \quad p \in (0, 2] \tag{2.5.21}$$

that is, ℓ has at most a quadratic growth. We choose, for simplicity, the discount factor $\lambda = 0$. Therefore, our optimal control problem ($OCP(\varepsilon)$) writes

$$\begin{aligned} V^\varepsilon(t, x, y) := \quad & \sup_{u_\cdot \in \mathcal{U}} \mathbb{E}\left[\int_t^T -\left|X^\varepsilon_s - \hat{x}_s\right|^p \mathrm{d}s\right] \\ \text{s.t.} \quad & \mathrm{d}X^\varepsilon_s = f(X^\varepsilon_s, Y^\varepsilon_s, u_s)\,\mathrm{d}s + \sqrt{2}\,\sigma^\varepsilon(X^\varepsilon_s, Y^\varepsilon_s, u_s)\,\mathrm{d}W_s, \ X^\varepsilon_t = x \in \mathbb{R}^n \\ & \mathrm{d}Y^\varepsilon_s = \frac{1}{\varepsilon} b(X^\varepsilon_s, Y^\varepsilon_s)\,\mathrm{d}s + \sqrt{\frac{2}{\varepsilon}}\,\varrho(X^\varepsilon_s, Y^\varepsilon_s)\,\mathrm{d}W_s, \quad Y^\varepsilon_t = y \in \mathbb{R}^m. \end{aligned}$$

The minus sign in the running cost is due to fact that we have a maximization problem. Thanks to the convergence result of the value function in Theorem 2.4.1 we deduce that $V^\varepsilon(t, x, y)$ converges locally uniformly to $V(t, x)$ which solves an effective optimal control problem of the form ($\overline{OCP}$). Notice that here, the pay off functional is unchanged, since it is independent of the variable y. Now, because we have $V(t,x) \leq 0$ together with the fact that $\hat{x}.$ is an admissible solution (by definition), then when the cost functional is evaluated in the latter trajectory, it yields $V(t,x) = 0$ and hence $\hat{x}.$ is indeed an optimal solution. This means that $V^\varepsilon(t,x,y)$ converges locally uniformly to 0 as $\varepsilon \to 0$, i.e.

$$\forall\, \delta > 0,\ \exists\, E > 0 \ \text{ s.t.: } \ \forall\, \varepsilon \leq E, \quad |V^\varepsilon(t,x,y)| \leq \frac{\delta}{2}$$

Fix $\delta > 0$. Let us denote again by $(X^\varepsilon, Y^\varepsilon)$ the suboptimal ($\frac{\varepsilon}{2}$-optimal) solution associated to $V^\varepsilon(t,x,y)$, so we get

$$-\frac{\delta}{2} \leq V^\varepsilon(t,x,y) \leq \int_t^T -\mathbb{E}\left[\,\left|X_s^\varepsilon - \hat{x}_s\right|^p\,\right]\, \mathrm{d}s + \frac{\varepsilon}{2}$$

and since ε can be chosen as small as we want, we can choose it such that $0 < \varepsilon \leq \delta$. Hence, one gets

$$-\delta \leq -\frac{\delta}{2} - \frac{\varepsilon}{2} \leq V^\varepsilon(t,x,y) \leq \int_t^T -\mathbb{E}\left[\,\left|X_s^\varepsilon - \hat{x}_s\right|^p\,\right]\, \mathrm{d}s \leq 0$$

which finally yields

$$\forall\, \delta > 0,\ \exists\, E > 0 \ \text{ s.t.: } \ \forall\, \varepsilon \leq E, \quad 0 < \int_t^T \mathbb{E}\left[\,\left|X_s^\varepsilon - \hat{x}_s\right|^p\,\right]\, \mathrm{d}s \leq \delta$$

in particular

$$\lim_{\varepsilon \to 0} \mathbb{E}\left[\,\left|X_s^\varepsilon - \hat{x}_s\right|^p\,\right] = 0, \quad \text{a.e. } \ s \in [t,T].$$

□

The next result shows that every limit (in a sense that we will make precise) of a sequence of controlled and singularly perturbed dynamics can be approximated by a sequence of effective dynamics (2.5.20) and is, moreover, a solution to the convexified effective dynamics.

Theorem 2.5.4. *Under the standing assumptions of §2.2 and assuming* (2.5.18) *holds, if a given sequence of controlled processes* $X^\varepsilon_\cdot$ *of* (2.2.1) *converges to some (deterministic) process* $\overline{x}.$ *in the sense*

$$\lim_{\varepsilon \to 0} \mathbb{E}\left[\,\left|X_s^\varepsilon - \overline{x}_s\right|^p\,\right] = 0, \quad \textit{a.e.} \ \ s \in [t,T], \tag{2.5.22}$$

for some $p \in [1,2]$, *then* $\overline{x}_\cdot$ *satisfies*

$$\dot{\overline{x}}_s \in \overline{co}\,\overline{F}(\overline{x}_s), \quad a.e.\ s \in [t,T],$$

where $\overline{F}$ *is as in* (2.5.20) *and* $\overline{co}$ *denotes the closed convex hull.*

Proof. (Theorem 2.5.4)
In the same spirit as in the proof of Theorem 2.5.3, we will construct an optimal control problem of the form ($OCP(\varepsilon)$) then study its limit and deduce the desired convergence.

We start by choosing a sequence $X^\varepsilon_\cdot$ solution to (2.2.1) which converges to a deterministic process $\overline{x}_\cdot$ in the sense (2.5.22) for some $p \in (0,2]$ fixed. We need to show that the limit process $\overline{x}_\cdot$ can be approximated by a sequence of dynamics solving (2.5.20). We consider an optimal control problem of the form ($OCP(\varepsilon)$) where the final cost $g \equiv 0$ and the running cost is (2.5.21). Since $X^\varepsilon_\cdot$ is an admissible solution to ($OCP(\varepsilon)$), we have

$$\int_t^T -\mathbb{E}\left[\left|X^\varepsilon_s - \overline{x}_s\right|^p\right] \mathrm{d}s \le V^\varepsilon(t,x,y) \le 0.$$

Using the fact that X^ε converges to $\overline{x}$ in the sense (2.5.22), we deduce that $V^\varepsilon(t,x,y)$ converges to 0 as $\varepsilon \to 0$. This means that the limit value function $V(t,x)$ of the effective optimal control problem ($\overline{OCP}$) also equals 0. And hence, one can consider a minimizing sequence $\{x^k_\cdot\}_k$ of the effective problem ($\overline{OCP}$) such that

$$\int_t^T \left|x^k_s - \overline{x}_s\right|^p \mathrm{d}s \xrightarrow[k\to+\infty]{} 0$$

which yields

$$\lim_{k\to+\infty} \left|x^k_s - \overline{x}_s\right|^p = 0, \quad \text{a.e. } s \in [t,T].$$

We need now to apply [67, Theorem 4.1.11, p.186] which provides a subsequence (again denoted by) $x^k_\cdot$ that converges uniformly to $z_\cdot$ and whose derivatives converge weakly to $\dot{z}_\cdot$ where

$$\dot{z}_s \in \overline{\text{co}}\,\overline{F}(z_s), \quad \text{a.e. } s \in [t,T] \tag{2.5.23}$$

using the notation in (2.5.20). The latter theorem holds true, since for every x, $\overline{\text{co}}\,\overline{F}(x)$ is a nonempty compact convex set, moreover $\overline{F}(x)$ is upper semicontinuous[5] as a direct consequence of [16, Proposition 1.4.14, p.47] hence also $\overline{\text{co}}\,\overline{F}(x)$, and finally every element of $\overline{\text{co}}\,\overline{F}(x)$ is upper bounded by an affine function of $\|x\|$ which follows from

[5] F is upper semicontinuous in x if $\forall\,\varepsilon > 0$, $\exists\,\delta > 0$ s.t. $|x - x'| \le \delta \Rightarrow F(x') \subset F(x) + \varepsilon\, B$ where B is the unit ball (see [15, Def. 1.1.5, p. 45] and the discussion afterwards, or [16, p. 39]).

(2.2.2). Therefore, and when $p \geq 1$, one has

$$|z_s - \overline{x}_s|^p \leq \|x^k_\cdot - z_\cdot\|^p_\infty + |x^k_s - \overline{x}_s|^p \xrightarrow[k\to+\infty]{} 0$$

and $z_s = \overline{x}_s$ for almost every $s \in [t,T]$ and $\overline{x}$ satisfies (2.5.23). $\square$

2.6 Conclusion

We managed to provide a SGD version which combines the results in [64] concerning the (uncontrolled) singularly perturbed system, and those in [123] concerning the control of the learning rate. So we presented a convergence result which allows to justify the approximation of a controlled SGD by a system of controlled and singularly perturbed SDEs. And of course this holds for any SDE of Smoluchowski type and also to more general dynamics satisfying the assumptions presented earlier.

Advantages. Using singular perturbations as an approximation procedure allows us ultimately to gain more reliability in the gradient descent, since we get the "full gradient". Moreover, since we have a regularized version of the loss function, then we gain in smoothness and hence the gradient descent will be more effective and trustful. We refer to [63] and the references therein, where the entropy-guided SGD is introduced in the framework of deep neural networks and is well studied.

Drawbacks. To implement such controlled system of SDEs, we need an explicit optimal control (which is in our application the learning rate parameter). But as it is usually the case for control systems, the computation of an optimal control can be costly from the numerical point of view, especially when it is given in a feedback form. However, it turns out that for some particular choices of cost functions, one can get the explicit controls and then plug them directly in the dynamics. This is the case for instance in the linear-quadratic cost functions. We refer to [123, §4.1.2 & §4.1.3] where the use of such controlled SGD has been presented and explicit computation for optimal controls have been performed and tested.

2.6.1 Known results on Smoluchowski equation

The results in this section are well known and are borrowed from [144, §4.5]. We recall them for completeness.

Consider the following stochastic process

$$\mathrm{d}X_t = -\nabla V(X_t)\,\mathrm{d}t + \sqrt{2\beta^{-1}}\,\mathrm{d}W_t, \quad X_0 = x \tag{2.6.1}$$

The corresponding infinitesimal generator writes

$$\mathcal{L}\bullet = -\nabla V(x)\cdot\nabla\bullet + \beta^{-1}\Delta\ \bullet\,. \tag{2.6.2}$$

Assume the initial condition for X_t is a random variable with probability density function $\rho_0(x)$. The probability density function $\rho(x,t)$ of X_t is the solution of the initial value problem:

$$\begin{aligned} \frac{\partial\rho}{\partial t} &= \nabla\cdot(\nabla V\rho) + \beta^{-1}\Delta\rho \\ \rho(x,0) &= \rho_0(x) \end{aligned} \tag{2.6.3}$$

It is not possible to calculate the time-dependent solution for arbitrary potentials. We can however calculate the stationary solution when it exists.

Definition 2.6.1. *[144, Definition 4.2] A potential V will be called confining if*

$$\lim_{|x|\to\infty} V(x) = +\infty \quad \textit{and} \quad \exp(-\eta V(x)) \in L^1(\mathbb{R}^d),\ \forall\eta\in\mathbb{R}^+ \tag{2.6.4}$$

With such potential, one expects nice ergodic properties.

Proposition 2.6.1. *[144, Proposition 4.2] Let $V(x)$ be a smooth confining potential and $\beta > 0$ a constant. Then the Markov process with generator (2.6.2) is ergodic. The unique invariant distribution is the Gibbs distribution*

$$\rho_{\beta}^{\infty}(x) = \frac{1}{Z}\exp(-\beta V(x)) \tag{2.6.5}$$

where the normalization factor Z is

$$Z = \int_{\mathbb{R}^d} \exp(-\beta V(x))\,dx \tag{2.6.6}$$

In general, for $\mathrm{d}X_t = b(X_t)\mathrm{d}t + \Sigma(X_t)\mathrm{d}W_t$, we can write the Fokker-Planck equation in the form of a continuity equation

$$\frac{\partial\rho}{\partial t} + \nabla\cdot J = 0 \tag{2.6.7}$$

where the probability flux (current) is

$$J := b(x)\rho - \frac{1}{2}\nabla \cdot (\Sigma(x)\rho) \tag{2.6.8}$$

Therefore, in the case of (2.6.1), direct computations yield

$$\nabla \cdot J(\rho_\beta^\infty) = 0, \tag{2.6.9}$$

that is, ρ_β^∞ is an invariant distribution.

For uniqueness, we need to show that the infinitesimal generator (2.6.2) has a spectral gap (the monograph [129] contains more details on this topic) or equivalently that ρ_{stat} satisfies a Poincaré inequality. This is the object of th next result.

Theorem 2.6.1. *[144, Theorem 4.3] Let $V \in C^2(\mathbb{R}^d)$, $\beta = 1$ and define $\mu(dx) = \frac{1}{Z}\exp(-V)dx$. If*

$$\lim_{|x|\to\infty} \left(\frac{1}{2}|\nabla V(x)|^2 - \Delta V(x)\right) = +\infty \tag{2.6.10}$$

then $\mu(dx)$ satisfies the Poincaré inequality with constant $\lambda > 0$:

$$\exists\, \lambda > 0 \ s.t. \ \ \forall f \in C^1(\mathbb{R}^d) \cap L^2(\mu), \ with \int f\mu(dx) = 0 \ : \ \lambda\|f\|^2_{L^2(\mu)} \le \|\nabla f\|^2_{L^2(\mu)} \tag{2.6.11}$$

A condition that ensures that the probability measure $\mu(dx) = \frac{1}{Z}\exp(-V(x))dx$ satisfies the Poincaré inequality with constant λ is the uniform convexity condition (or Bakry-Emery criterion)

$$D^2 V \ge \lambda I, \tag{2.6.12}$$

Proposition 2.6.2. *[144, Proposition 4.3] Assume that $V(x)$ is a smooth confining potential. Then the operator*

$$\mathcal{L}\, \bullet = -\nabla V(x) \cdot \nabla \bullet + \beta^{-1}\Delta\, \bullet$$

is self-adjoint in $L^2(\rho_\beta^\infty)$. Furthermore, it is non-positive, and its kernel consists of constants.

Remark 2.6.1. *The generator $\mathcal{L}$ is self-adjoint in the space of square integrable functions weighted by the invariant density of X_t:*

$$L^2(\rho_\beta^\infty) := \left\{ f, \ \ such\ that \ \int_{\mathbb{R}^d} |f|^2 \rho_\beta^\infty(dx) < \infty \right\} \tag{2.6.13}$$

which is a Hilbert space with inner product $\langle f, h\rangle := \int_{\mathbb{R}^d} f(x)h(x)\rho_\beta^\infty(dx)$.

The Poincaré inequality yields exponentially fast convergence to equilibrium in the right function space.

Theorem 2.6.2. *[144, Theorem 4.4] Let $\rho(x,t)$ denote the solution of the FP equation (2.6.3) with $\rho_0(x) \in L^2(\mathbb{R}^d; (\rho_\beta^\infty)^{-1})$, and assume that the potential V satisfies a Poincaré inequality with constant λ. the $\rho(x,t)$ converges to the Gibbs distribution ρ_β^∞ defined in (2.6.5) exponentially fast*

$$\|\rho(\cdot,t) - \rho_\beta^\infty\|_{L^2((\rho_\beta^\infty)^{-1})} \le e^{-\lambda\beta^{-1}t}\|\rho_0(\cdot) - \rho_\beta^\infty\|_{L^2((\rho_\beta^\infty)^{-1})}. \tag{2.6.14}$$

Thanks to these results, we can directly link a potential V with its invariant measure. In fact, this is one of the few cases where we can have an explicit formula for the invariant measure. Other cases where one has explicitly the invariant measure of a diffusion can be found for example in [124].

2.7 Future perspective

An application to Energy production

We briefly describe a possible future application of our results on the asymptotics of such system of controlled and singularly perturbed SDEs

$$\begin{cases} \mathrm{d}X_t = f(X_t, Y_t, u_t)\,\mathrm{d}t + \sqrt{2}\,\sigma^\varepsilon(X_t, Y_t, u_t)\,\mathrm{d}W_t, \quad X_0 = x \in \mathbb{R}^n \\ \mathrm{d}Y_t = \dfrac{1}{\varepsilon}\, b(X_t, Y_t)\,\mathrm{d}t + \sqrt{\dfrac{2}{\varepsilon}}\,\varrho(X_t, Y_t)\,\mathrm{d}W_t, \quad Y_0 = y \in \mathbb{R}^m \end{cases}$$

In the context of energy production, the slow dynamics $X_\cdot$ represents the production of energy which we can control (e.g., start and stop): fossil thermal production, nuclear production or hydropower production, whereas the fast dynamics $Y_\cdot$ represents the renewable energies which we do not control. The latter are highly volatile since they depend on weather conditions, and moreover the energy produced must be disposed of at any cost, i.e. they benefit from priority over other means of production. Therefore, one expects the dynamics $Y_\cdot$ to be ruled by $b = b(y)$ and $\varrho = \varrho(y)$, and the randomness in X shall come from the one of Y, i.e. $\sigma^\varepsilon = \sigma^\varepsilon(y)$, in particular, it is reasonable to assume $\lim\limits_{\varepsilon\to 0} \sigma^\varepsilon(y) = 0$ since the *effective* control problem is expected to be deterministic (although one can still include a diffusion for X coming from the randomness of the demand of energy). The optimal control is then of the form ($OCP(\varepsilon)$). We refer to [58] for some explicit models of energy production.

The limit procedure as studied in the present chapter would allow us to construct an *effective* optimal control problem of the form ($\overline{OCP}$), where the new *effective* dynamics and cost only concern the means of energy production which we do control but that now incorporate the "effect" of the renewable energies. This shall then be a model reduction technique for the initial problem of control, since we would have reduced the dimension of the system from $n+m$ to n. Moreover, we have an established link between the dynamics of the singularly perturbed control problem and the *effective* (limit) one as in §2.5.5.

Let us finally mention that one can consider additional constraints of the form $X_{\cdot} \geq 0$ which could be meaningful in this application. Our results can still be adapted to this setting following the techniques in [23].

Chapter 3

Global optimization: an optimal control approach

3.1 A parameterized control problem

Let $\nu = (\Omega, \{\mathcal{F}_s\}, \mathbb{P}, W.)$ be some reference probability system, where Ω is a sample space, $\{\mathcal{F}_s\}$ a filtration, $\mathbb{P}$ a probability measure, and $W.$ a $\mathbb{P}$-Brownian motion adapted to $\{\mathcal{F}_s\}$. Given $\varepsilon > 0$, we introduce a controlled stochastic process X_s solution to

$$\begin{aligned} \mathrm{d}X_s &= \alpha_s \,\mathrm{d}s + \sqrt{2\varepsilon}\,\mathrm{d}W_s, \\ X_0 &= x \in \mathbb{R}^n \end{aligned} \tag{3.1.1}$$

where the control α_s is $\mathbb{R}^n$-valued $\mathcal{F}_s$-progressively measurable process satisfying $|\alpha_s| \leq M$ for all $s \geq 0$ and for some constant $M > 0$. We denote by $\mathcal{A}^\nu_M$ the set of all such control processes $\alpha.$. Then

$$\mathcal{A}^\nu = \bigcup_{M>0} \mathcal{A}^\nu_M \tag{3.1.2}$$

is the set of all admissible control processes. In the sequel, we will omit the explicit dependency on ν when there is no confusion and simply write $\mathcal{A}$ or $\mathcal{A}_M$ for some given fixed ν. And the goal is to choose $\alpha. \in \mathcal{A}^\nu$ for some reference probability system ν, such that it minimizes a given criterion J that we will made precise in the next sections. Note that in the case where (3.1.1) has pathwise unique solution, then the reference probability system ν can be arbitrary.

As pointed out in [85, Example 8.2, p.137], the dynamics (3.1.1) represents the position of some particle at time s, in the setting of Nelson's theory of stochastic mechanics [138]. For a particle with such dynamics, the velocity is undefined since brownian paths are nowhere differentiable with probability 1, but one can still represent its local "average velocity" and which is represented by α_s. The classical *action* associated to such particle (of mass 1) takes then the form $\frac{1}{2}|\alpha|^2 + f(x)$ where α plays the role of the velocity and $-f(x)$ is the potential energy at position x.

In this section we prove Theorem 3.1.1 which allows us to use classical results (e.g. [85] and [117] for the discounted problem) for proving existence of an optimal control and characterizing it via the solution of a HJB equation, before we study its asymptotic behavior.
The main issue is in the control set: while the classical results concern controls with values in a compact set, if we want to make explicit the maximization in the HJB representation then this requires the control set to be unbounded (in fact, equal to the whole space) a priori. To remedy this, we show instead that the gradient of the value function (which captures the optimal control) is indeed bounded and hence there is no loss in generality when considering a bounded control set (provided it is large enough); see Lemma 3.1.1.
A similar result is in [84, §4]. The main difference between our setting and the one in [84] is in the assumptions on the dynamics (3.1.1). In [84] it is assumed that the dynamics has an inward-pointing drift (independent from the control) which guarantees strong ergodicity of the process X. (see [84, (3.1)-c]). This assumption plays a crucial role in their estimates. In our case, we do not consider any drift (besides the control), in particular we do not have ergodicity, but we will take advantage of the semiconcavity of f in the running cost (which is not present in [84]).

We consider a finite-horizon and discounted control problem. We fix a small discount factor $\lambda > 0$ and a time horizon $t > 0$, and consider the cost functional defined by

$$J_\lambda(t, x, \alpha.) = \mathbb{E}\left[\int_0^t \left(\frac{1}{2}|\alpha_s|^2 + f(X_s)\right) e^{-\lambda s}\,\mathrm{d}s \;\middle|\; X_0 = x\right] \tag{3.1.3}$$

where X_s is the solution to (3.1.1) for a fixed $\varepsilon > 0$ and f is a bounded continuous function, s.t.

$$\exists\, \underline{f}, \overline{f} \text{ s.t. } \underline{f} \leq f(x) \leq \overline{f}, \quad \forall\, x \in \mathbb{R}^n. \tag{3.1.4}$$

Let us fix a bound $M > 0$ for the admissible controls and consider the problem of minimizing the cost function (3.1.3) over the set $\mathcal{A}_M$ as defined earlier in this section. We will later prove that the choice of M can be made arbitrary when it is large enough;

see Remark 3.1.1. We define the value function of the latter control problem as

$$u_\lambda^\varepsilon(x,t) = \inf_{\alpha.\in\mathcal{A}_M} \; J_\lambda(t,x,\alpha.) \quad \text{s.t.} \quad (3.1.1). \tag{3.1.5}$$

In the light of [85, Theorem IV.4.2 and Remark IV.4.1] (see also [84, §4]) and references therein, results about parabolic PDEs and a verification theorem insure that $u_\lambda^\varepsilon(x,t)$ is a classical solution to the dynamic programming equation

$$\begin{cases} \partial_t u_\lambda^\varepsilon - \varepsilon\Delta u_\lambda^\varepsilon + \max\limits_{|\alpha|\le M}\left\{-\alpha\cdot\nabla u_\lambda^\varepsilon - \dfrac{1}{2}|\alpha|^2\right\} + \lambda\, u_\lambda^\varepsilon = f(x) \\ u_\lambda^\varepsilon(x,0) = 0 \end{cases} \tag{3.1.6}$$

that is, $u_\lambda^\varepsilon(x,t)$ and the partial derivatives $\partial_t u_\lambda^\varepsilon$, $\partial_{x_i} u_\lambda^\varepsilon$ and $\partial^2_{x_i x_j} u_\lambda^\varepsilon$, $i,j = 1,\dots,n$, are continuous. We will now prove some estimates that will be later needed.

Lemma 3.1.1. *Assume* (3.1.4) *holds and let* $C_1 = \sup|\nabla f|$. *Then the following hold for every* $x\in\mathbb{R}^n$, $\lambda > 0$, $t > 0$

(i) Bounds on the value function $u_\lambda^\varepsilon(x,t)$*:*

$$\lambda^{-1}\underline{f}(1-e^{-\lambda t}) \le u_\lambda^\varepsilon(x,t) \le \min(\lambda^{-1},t)\|f\|_\infty \tag{3.1.7}$$

in particular we have

$$\underline{f}(1-e^{-\lambda t}) \le \lambda u_\lambda^\varepsilon(x,t) \le \|f\|_\infty, \text{ and } \underline{f}\frac{1-e^{-\lambda t}}{\lambda t} \le \frac{1}{t}u_\lambda^\varepsilon(x,t) \le \|f\|_\infty. \tag{3.1.8}$$

(ii) Bounds on its time derivative: for every $t_1, t_2 > 0$

$$|u_\lambda^\varepsilon(x,t_1) - u_\lambda^\varepsilon(x,t_2)| \le (M^2 + \|f\|_\infty)\min\left(|t_1-t_2|, \; \frac{|e^{-\lambda t_1} - e^{-\lambda t_2}|}{\lambda}\right) \tag{3.1.9}$$

in particular we have

$$|\partial_t u_\lambda^\varepsilon(x,t)| \le (M^2 + \|f\|_\infty). \tag{3.1.10}$$

(iii) bounds on its spatial derivative:

$$|\nabla u_\lambda^\varepsilon(x,t)| \le \min(\lambda^{-1},t)C_1 \tag{3.1.11}$$

(iv) If moreover f *is* C_2*-semiconcave, then there exists* $C > 0$ *such that* u_λ^ε *is* C*-semiconcave and* C *is independent of* λ, t *and of the diffusion parameter* $\varepsilon \le 1$.

Remark 3.1.1. *The latter results state that for fixed $\lambda, T > 0$, we have for any $(x,t) \in \mathbb{R}^n \times [0,T]$, $|\nabla u_\lambda^\varepsilon(x,t)| \leq \min(\lambda^{-1}, T)C_1 =: M_\lambda^T$. Therefore if we choose the constant M large enough in the admissible control set $\mathcal{A}_M$ and such that $M > M_\lambda^T$, then the maximum in the HJB equation* (3.1.6) *is an interior maximum, achieved at $\alpha_s^* = -\nabla u_\lambda^\varepsilon(x,s)$. Thus the HJB equation* (3.1.6) *writes as*

$$\begin{cases} \partial_t u_\lambda^\varepsilon - \varepsilon \Delta u_\lambda^\varepsilon + \dfrac{1}{2}|\nabla u_\lambda^\varepsilon|^2 + \lambda\, u_\lambda^\varepsilon = f(x), \quad (x,t) \in \mathbb{R}^n \times (0,T] \\ u_\lambda^\varepsilon(x,0) = 0, \quad x \in \mathbb{R}^n \end{cases} \tag{3.1.12}$$

which is the one satisfied, in the classical sense, by the value function of the same optimal control problem but where we replace the minimization over bounded controls $|\alpha| \leq M$ with the minimization over controls taking values in $\mathbb{R}^n$.

Proof *of Lemma 3.1.1.* The inequalities in the statement *(i)* can be immediately obtained by direct estimates using (3.1.3). The statement in *(ii)* relies on the well-known inequality $\inf_{\alpha.\in\mathcal{A}_M} J_\lambda(t_1,x,\alpha.) - \inf_{\alpha.\in\mathcal{A}_M} J_\lambda(t_2,x,\alpha.) \leq \sup_{\alpha.\in\mathcal{A}_M} \{J_\lambda(t_1,x,\alpha.) - J_\lambda(t_2,x,\alpha.)\}$, together with direct estimates.

Proof of (iii).

For $\delta > 0$ take a δ-optimal control for the problem with initial position $x+h$ and denote with $X_\cdot^{x+h}$ the corresponding trajectory. Then use the same control for the initial position x and denote with $X_\cdot^x$ the trajectory. Then using $C_1 = \sup|\nabla f|$, we have

$$\begin{aligned} u_\lambda^\varepsilon(x,t) - u_\lambda^\varepsilon(x+h,t) &\leq \mathbb{E}\left[\int_0^t (f(X_s^x) - f(X_s^{x+h}))e^{-\lambda s}\,\mathrm{d}s\right] + \delta \\ &\leq C_1 \mathbb{E}\left[\int_0^t |X_s^x - X_s^{x+h}| e^{-\lambda s}\,\mathrm{d}s\right] + \delta \end{aligned}$$

Since $X_\cdot^{x+h}$ and $X_\cdot^x$ have the same control (hence the same drift) and the same constant diffusion, then

$$u_\lambda^\varepsilon(x,t) - u_\lambda^\varepsilon(x+h,t) \leq C_1|h|(1-e^{-\lambda t})\lambda^{-1} + \delta \leq C_1|h|\lambda^{-1} + \delta$$

or if we use $e^{-\lambda s} \leq 1$, then

$$u_\lambda^\varepsilon(x,t) - u_\lambda^\varepsilon(x+h,t) \leq C_1|h|t + \delta$$

By reversing the roles of x and $x+h$ and then letting $\delta \to 0$, we obtain the desired inequality.

Proof of (iv).

Let us denote by ξ a vector of $\mathbb{R}^n$ such that $|\xi| = 1$, and let $\omega_\lambda(x,t) := D^2_{\xi\xi}u^\varepsilon_\lambda(x,t)$ be the second order derivative in the direction ξ. The proof of the statement *(iv)* is conducted in two steps.

Step 1. (*We show that* $\omega_\lambda(x,t) \le \min(\lambda^{-1},t)C_2$)

This is equivalent to showing that the value function $u^\varepsilon_\lambda(x,t)$ is $\min(\lambda^{-1},t)C_2$-semiconcave in the spatial variable x. Let $\delta > 0$ and take a $\frac{\delta}{2}$-optimal control for the initial point x. Then use the same control for the initial points $x+h$ and $x-h$ where $h \in \mathbb{R}^n$. Consider the following inequality

$$u^\varepsilon_\lambda(x+h,t) - 2u^\varepsilon_\lambda(x,t) + u^\varepsilon_\lambda(x-h,t) - \delta \le \mathbb{E}\left[\int_0^t (f(X_s^{x+h}) - 2f(X_s^x) + f(X_s^{x-h}))e^{-\lambda s}\,\mathrm{d}s\right] \tag{3.1.13}$$

From (3.1.1), we have $X_s^x = \frac{1}{2}\left(X_s^{x+h} + X_s^{x-h}\right)$. And since f is C_2-semiconcave, we have

$$\begin{aligned}\mathbb{E}&\left[\int_0^t (f(X_s^{x+h}) - 2f(X_s^x) + f(X_s^{x-h}))e^{-\lambda s}\,\mathrm{d}s\right]\\ &\le\; C_2\mathbb{E}\left[\int_0^t \frac{1}{4}\left|X_s^{x+h} - X_s^{x-h}\right|^2 e^{-\lambda s}\,\mathrm{d}s\right]\\ &\le\; \min(\lambda^{-1},t)C_2\,|h|^2\end{aligned} \tag{3.1.14}$$

This holds for any $\delta > 0$, we therefore have $u^\varepsilon_\lambda(\cdot,t)$ is $(\min(\lambda^{-1},t)C_2)$-semiconcave, which then implies that $w(x,t) \le \min(\lambda^{-1},t)C_2$ for all $(x,t) \in \mathbb{R}^n \times (0,+\infty)$ and $\lambda > 0$.

Step 2. (*We show that* $\omega_\lambda(x,t) \le C$ *for some* $C > 0$ *independent of* x, t, λ)

Let $T > 0$ that we will later made precise. From (3.1.12), we have ω satisfies

$$\partial_t\omega_\lambda - \varepsilon\Delta\omega_\lambda + Du^\varepsilon_\lambda \cdot D\omega_\lambda + |D_\xi Du^\varepsilon_\lambda|^2 + \lambda\omega_\lambda = D_{\xi\xi}f, \quad \text{in } \mathbb{R}^n \times (0,T]. \tag{3.1.15}$$

And since $\omega_\lambda^2 \le |D_\xi Du^\varepsilon_\lambda|^2$ and using the semiconcavity assumption $D^2_{\xi\xi}f \le C_2$, then ω_λ satisfies

$$\partial_t\omega_\lambda - \varepsilon\Delta\omega_\lambda + Du^\varepsilon_\lambda \cdot D\omega_\lambda + \omega_\lambda^2 + \lambda\omega_\lambda \le C_2, \quad \text{in } \mathbb{R}^n \times (0,+\infty). \tag{3.1.16}$$

Now set $g(x) := \log(1+|x|^2)$ and $\Phi_\lambda(x,t) := \omega_\lambda(x,t) - \beta g(x)$, in $\mathbb{R}^n \times (0,+\infty)$ for some $\beta > 0$ to be made precise. From Step 1, ω_λ is bounded from above uniformly in x and for every $t \le T$, therefore $\Phi_\lambda(x,t) \to -\infty$ as $|x| \to +\infty$ and hence Φ_λ admits a global maximum in $\mathbb{R}^n \times [0,T]$. Set $(\overline{x},\overline{t}) \in \mathbb{R}^n \times [0,T]$ such that $\Phi_\lambda(\overline{x},\overline{t}) := \max\limits_{\mathbb{R}^n\times[0,T]} \Phi_\lambda(x,t)$ (clearly $(\overline{x},\overline{t})$ depends on λ and T) . Hence, evaluating (3.1.16) in $(\overline{x},\overline{t})$ and supposing

$\bar{t} \in (0,T)$ yields[1]

$$\omega_\lambda^2(\overline{x},\bar{t}) + \lambda\omega_\lambda(\overline{x},\bar{t}) \leq C_2 + 2\varepsilon\beta\frac{n+(n-2)|\overline{x}|^2}{(1+|\overline{x}|^2)^2} - 2\beta Du_\lambda^\varepsilon(\overline{x},\bar{t})\cdot\frac{\overline{x}}{1+|\overline{x}|^2} \tag{3.1.17}$$

Note that $x \in \mathbb{R}^n \mapsto \frac{n+(n-2)|x|^2}{(1+|x|^2)^2}$ has a global maximum in $x = 0$ when $n \geq 1$, also $\frac{x}{1+|x|^2}$ is bounded from above by 1, and together with (3.1.11) and the fact that $\bar{t} < T$, the bound in (3.1.17) writes

$$\omega_\lambda^2(\overline{x},\bar{t}) + \lambda\omega_\lambda(\overline{x},\bar{t}) \leq C_2 + 2\varepsilon\beta n + 2\beta\min(\lambda^{-1},T)C_1$$

We choose $\beta > 0$ small enough, such that $2\beta\min(\lambda^{-1},T) \leq 1$, and since we are interested in $\lambda \to 0$ or $T \to +\infty$, then we can choose λ small enough or T large enough, such that we have in particular $\beta < 1$, then

$$\omega_\lambda^2(\overline{x},\bar{t}) + \lambda\omega_\lambda(\overline{x},\bar{t}) \leq C_2 + \max\{C_1; n\varepsilon\} \leq C_2 + C_1 + n\varepsilon \tag{3.1.18}$$

where the right hand side is now independent of T and x. Now if $\bar{t} = 0$, then since $u_\lambda^\varepsilon(x,0) = 0$ for all x, then $\omega(\overline{x},0) = 0$ and (3.1.18) still holds. And if $\bar{t} = T$ then either we choose $T' > T$ and we maximize Φ over $\mathbb{R}^n \times [0,T']$ or if we have $(\overline{x},T) = \underset{\mathbb{R}^n\times[0,T]}{\operatorname{argmax}}\,\Phi(x,t)$ for all $T > 0$, then $\partial_t\Phi(\overline{x},T) \geq 0$ i.e. $\partial_t\omega(\overline{x},T) \geq 0$ and (3.1.18) again still holds. Moreover, note that $\frac{1}{2}(z^2-\lambda^2) \leq z^2 + \lambda z$ holds[2] for any $z \in \mathbb{R}$. Therefore, from (3.1.18) and for $\lambda < 1$ we have

$$\omega_\lambda(\overline{x},\bar{t})^2 \leq 2(C_1 + C_2 + \varepsilon n) + 1 \tag{3.1.19}$$

where the right hand side is now independent of λ, T and x.
Let us set $C_3 := \sqrt{2(C_1 + C_2 + \varepsilon n) + 1}$, and suppose by contradiction that

$$\exists\ (y,s) \in \mathbb{R}^n \times (0,+\infty) \text{ s.t. } \ \omega_\lambda(y,s) > C_3. \tag{3.1.20}$$

Denote by $\delta = \omega_\lambda(y,s) - C_3 > 0$. Without loss of generality, we can choose $T > 0$ large enough such that $s < T$, and denote again by $(\overline{x},\bar{t})$ the maximizer of Φ_λ over $\mathbb{R}^n \times [0,T]$. And let us choose $\beta > 0$ small enough such that it satisfies $2\beta\min(\lambda^{-1},T) \leq 1$ and also

[1]
$\partial_t\Phi_\lambda(\overline{x},\bar{t}) = 0 \Rightarrow \partial_t\omega_\lambda(\overline{x},\bar{t}) = 0$,
$D_x\Phi_\lambda(\overline{x},\bar{t}) = 0 \Rightarrow D_x\omega_\lambda(\overline{x},\bar{t}) = 2\beta\frac{\overline{x}}{1+|\overline{x}|^2}$, and $\Delta\Phi_\lambda(\overline{x},\bar{t}) \leq 0 \Rightarrow \Delta\omega_\lambda(\overline{x},\bar{t}) \leq 2\beta\frac{n+(n-2)|\overline{x}|^2}{(1+|\overline{x}|^2)^2}$.

[2] $0 \leq \frac{1}{2}(z+\lambda)^2 = z^2 + \lambda z - \frac{1}{2}\left(z^2-\lambda^2\right)$

$\beta g(y) \leq \frac{\delta}{2}$ (it suffices to take T large and $\lambda < 1$ small). Then, one has

$$0 < \frac{\delta}{2} \leq \delta - \beta g(y) = \omega_\lambda(y,s) - \beta g(y) - C_3 = \Phi_\lambda(y,s) - C_3$$

and hence from the definition of $(\overline{x}, \overline{t})$

$$0 < \frac{\delta}{2} \leq \Phi_\lambda(\overline{x}, \overline{t}) - C_3. \tag{3.1.21}$$

But from (3.1.19) we have

$$\omega_\lambda(\overline{x}, \overline{t}) \leq C_3$$

which yields

$$\Phi_\lambda(\overline{x}, \overline{t}) - C_3 \leq -\beta g(\overline{x}) \leq 0$$

and contradicts (3.1.21). Therefore, the statement in (3.1.20) cannot be true and hence

$$\omega_\lambda(x,t) \leq C_3, \quad \text{for all } (x,t) \in \mathbb{R}^n \times (0, +\infty).$$

This proves the semiconcavity of u_λ^ε uniformly in λ, t and x, and for every ε less than some constant, say $\varepsilon \leq 1$. □

We are now ready to pass to the limit either for $\lambda > 0$ fixed and $t \to +\infty$, or for $t > 0$ fixed and $\lambda \to 0$.

Theorem 3.1.1. *Under the standing assumptions as in the previous lemma, the following holds*

(i) the limit $u^\varepsilon(x,t) := \lim_{\lambda \to 0} u_\lambda^\varepsilon(x,t)$ exists uniformly in x, locally uniformly in t. Moreover, $u^\varepsilon(x,t)$ is the value function of the corresponding finite-horizon optimal control problem (3.1.5) *with $\lambda = 0$ in* (3.1.3)*, and for any $T > 0$, it satisfies in the classical sense*

$$\begin{cases} \partial_t u^\varepsilon - \varepsilon \Delta u^\varepsilon + \frac{1}{2}|\nabla u^\varepsilon|^2 = f(x), & (x,t) \in \mathbb{R}^n \times (0,T] \\ u^\varepsilon(x,0) = 0, & x \in \mathbb{R}^n \end{cases} \tag{3.1.22}$$

(ii) the limit $u_\lambda^\varepsilon(x) := \lim_{t \to +\infty} u_\lambda^\varepsilon(x,t)$ exists, uniformly in x. Moreover, $u_\lambda^\varepsilon(x)$ is the value function of the corresponding infinite-horizon discounted optimal control problem (3.1.5) *with $t = +\infty$ in* (3.1.3)*, and satisfies in the classical sense*

$$\lambda\, u_\lambda^\varepsilon - \varepsilon \Delta u_\lambda^\varepsilon + \frac{1}{2}|\nabla u_\lambda^\varepsilon|^2 = f(x), \quad x \in \mathbb{R}^n. \tag{3.1.23}$$

Proof. The results are consequences of the previous estimates in Lemma 3.1.1. Indeed, as previously mentioned in Remark 3.1.1, the value function $u^\varepsilon_\lambda(x,t)$ satisfies in the classical sense the PDE (3.1.12).

Proof of (i).

Let us first consider $(x,t) \in \mathbb{R}^n \times [0,T]$ for a fixed $T > 0$. From (3.1.7), we have

$$|u^\varepsilon_\lambda(x,t)| \le \|f\|_\infty T, \quad \forall\, (x,t) \in \mathbb{R}^n \times [0,T] \tag{3.1.24}$$

This is a direct consequence of (3.1.7), noticing that $\frac{z}{1+z} < 1 - e^{-z} < z$ for any $z > -1$. We also get from (3.1.11) the following bound

$$|\nabla u^\varepsilon_\lambda(x,t)| \le T\, C_1, \quad \forall\, (x,t) \in \mathbb{R}^n \times [0,T] \tag{3.1.25}$$

Therefore, $\{u^\varepsilon_\lambda(x,t)\}_{\lambda>0}$ is a bounded and equicontinuous family of functions. We can then apply Ascoli-Arzelà theorem and extract a subsequence $0 \le \lambda_m \le 1$ satisfying $\lambda_m \to 0$ as $m \to +\infty$ and such that $u^\varepsilon_{\lambda_m}$ converges uniformly in $\mathbb{R}^n \times [0,T]$ to a function u^ε. Using again (3.1.24), we have for any (x,t), $\lambda_m u_{\lambda_m} \to 0$ uniformly in x, and then u^ε solves (3.1.22) for any $(x,t) \in \mathbb{R}^n \times [0,T]$. And using (3.1.25), we have moreover $|\nabla u^\varepsilon(x,t)| \le TC_1$ for any $x \in \mathbb{R}^n$. Since $\lambda_m u^\varepsilon_{\lambda_m}$, $\nabla u^\varepsilon_{\lambda_m}$ and also (using (3.1.10)) $\partial_t u^\varepsilon_{\lambda_m}$ are bounded independently of λ_m, then $\Delta u^\varepsilon_{\lambda_m}$ is also bounded independently of λ_m uniformly in x. Therefore, standard arguments for quasilinear parabolic PDEs (see e.g. [118, Chapter VI]) insure that u^ε is a classical solution to (3.1.22). On the other hand, the value function (3.1.5) associated to the optimal control problem with $\lambda = 0$ in (3.1.3) and with an admissible control set $\mathcal{A}_M$ such that $M = TC_1$ (or any larger constant as in Remark 3.1.1), is the unique classical solution to the HJB equation (3.1.6) where we set $\lambda = 0$ (see [85, Theorem IV.4.2]). And with our choice of M, the maximum is an interior one, and the value function is again the unique classical solution to (3.1.22) which coincides with the limit function u^ε.

Proof of (ii).

Fix $\lambda > 0$ and set $M := \lambda^{-1} C_1$ (or any larger constant as we previously discussed in Remark 3.1.1). The proof is in the line of *(i)*, using the following two estimates that we easily deduce from (3.1.7)

$$|u^\varepsilon_\lambda(x,t)| \le \lambda^{-1} \|f\|_\infty, \quad \forall\, (x,t) \in \mathbb{R}^n \times (0,+\infty) \tag{3.1.26}$$

and

$$|\nabla u^\varepsilon_\lambda(x,t)| \le \lambda^{-1} C_1, \quad \forall\, (x,t) \in \mathbb{R}^n \times (0,+\infty). \tag{3.1.27}$$

We also have from (3.1.9) where we set $t := t_1$ and $t_2 = t + h$

$$|u_\lambda^\varepsilon(x,t) - u_\lambda^\varepsilon(x,t+h)| \leq \lambda^{-1}\left(M^2 + \|f\|_\infty\right)|e^{-\lambda t} - e^{-\lambda(t+h)}|.$$

Dividing by $|h|$ and letting $h \to 0$, we get

$$|\partial_t u_\lambda^\varepsilon(x,t)| \leq (M^2 + \|f\|_\infty)e^{-\lambda t}$$

which therefore yields $\lim\limits_{t\to+\infty} |\partial_t u_\lambda^\varepsilon(x,t)| = 0$ uniformly in x. Since $u_\lambda^\varepsilon(x,t)$ satisfies (3.1.12) in the classical sense, and together with (3.1.26) and (3.1.27), we have $\Delta u_\lambda^\varepsilon$ is also bounded uniformly in x, t. This insures that the limit $\lim\limits_{t\to+\infty} u_\lambda^\varepsilon(x,t) =: u_\lambda^\varepsilon(x)$, uniform in x, exists, and by standard estimates for semilinear parabolic PDEs (see e.g. [117, Theorem 1]) , we have $u_\lambda^\varepsilon \in C^2(\mathbb{R}^n)$ and solves (3.1.23). In addition, $u_\lambda^\varepsilon(x)$ satisfies the dynamic programming equation (3.1.6) and hence is the value function of the corresponding infinite-horizon discounted optimal control problem (3.1.5) with $t = +\infty$ in (3.1.3). $\square$

Before we end this section, let us comment the latter theorem.
First, we have the problem (3.1.6) which does admit a classical solution $u_\lambda^\varepsilon(x,t)$. Then using the estimates in Lemma 3.1.1, the latter PDE writes as (3.1.12). This was the object of Remark 3.1.1. We next consider two problems

- Fix the time horizon $t > 0$ and let the discount factor $\lambda \to 0$: this is the statement (i) in Theorem 3.1.1. We showed that this limit PDE admits a classical solution $u(x,t)$ and moreover the gradient $\nabla u(x,t)$ admits a bound uniform in x, but which depends on T such that $t \in [0,T]$, that is of the form (3.1.25).

- Fix $\lambda > 0$ and let $t \to +\infty$: this is the statement (ii) in Theorem 3.1.1. The limit PDE admits a classical solution and its gradient $\nabla u_\lambda^\varepsilon$ satisfies a bound uniform in x but which depends on λ as in (3.1.27).

In both cases, the admissible control set is of the form $\mathcal{A}_M$ and the optimal strategies are feedback controls given by the gradient of the value function when M is chosen large enough. However, the constant M in both situations depends on the parameter t in the first case, and on λ in the second, which prevents in the limit ($t \to +\infty$ or $\lambda \to 0$) for the ergodic case. In the next sections, we shall need an estimate on the gradient that is independent of the parameters t, λ.

3.2 Degenerate Eikonal equation

3.2.1 Introduction

Let $f \in C(\mathbb{R}^n)$ be a bounded function attaining the global minimum. Global optimization is concerned with the search of the minimum points, i.e., finding the set $\mathfrak{M} = \operatorname{argmin} f$. For convex smooth functions this is achieved by the gradient flow, i.e., by following the trajectories of $\dot{y}(s) = -\nabla f(y(s))$ from any initial point $x = y(0)$. However, if the function f is not convex the trajectory $y(\cdot)$ may converge to a local minimum or a saddle point. Several alternative algorithms have been designed to handle non-convex optimization, such as the stochastic gradient descent, simulated annealing, or consensus-based methods. In particular the case of non-smooth f in high dimensions is important for the applications to machine learning, see, e.g., the recent paper [59] and the references therein.

In this section we construct and study a Lipschitz function $v : \mathbb{R}^n \to \mathbb{R}$ such that the following normalized non-smooth gradient descent differential inclusion

$$\dot{y}(s) \in \left\{ -\frac{p}{|p|} \,,\ p \in D^- v(y(s)) \right\}, \quad \text{for a.e. } \ s > 0, \tag{3.2.1}$$

has a solution for any initial condition $x = y(0)$ and all solutions converge to $\mathfrak{M}$ as $t \to +\infty$. Here $D^- v$ is the sub-differential of the theory of viscosity solutions (see, e.g., [19]). The construction of such a generating function v is based on a classical problem for Hamilton-Jacobi equations: find a constant c such that the stationary equation

$$H(x, Dv) = c \quad \text{in } \mathbb{R}^n \tag{3.2.2}$$

has a solution v. The minimal c with this property is the critical value of the Hamiltonian H and, if $H(x, \cdot)$ is convex, it is also the value of an optimal control problem with ergodic cost having H as its Bellman Hamiltonian. If the critical solution v is interpreted in the viscosity sense, the problem fits in the weak KAM theory, and it is well-known that, for $H = |p|^2 - f(x)$ with f periodic, $c = -\min f$ [81, 127]; moreover the same holds for any bounded $f \in C^2(R^n)$ by a result of Fathi and Maderna [83]. In Section 3.2.2 we extend such result to non-smooth f, provided it is Lipschitz and semiconcave. We also prove that $\min f$ and v solving the critical equation

$$\min f + \frac{1}{2}|\nabla v(x)|^2 = f(x) \quad \text{in } \mathbb{R}^n$$

can be approximated in two ways: by the solution of the stationary equation

$$\lambda u_\lambda + \frac{1}{2}|Du_\lambda|^2 = f(x), \quad x \in \mathbb{R}^n, \tag{3.2.3}$$

as $\lambda \to 0+$, the so-called small discount limit, as well as by the long-time limit of the solution of the evolution equation

$$\partial_t u + \frac{1}{2}|Du|^2 = f(x), \text{ in } \mathbb{R}^n \times (0, +\infty), \quad u(x,0) = 0. \tag{3.2.4}$$

More precisely, for the evolutive equation (3.2.4) we prove

$$\lim_{t\to+\infty} (u(x,t) - t \min f) = v(x) \quad \text{locally uniformly in } \mathbb{R}^n. \tag{3.2.5}$$

Note that the two problems (3.2.3) and (3.2.4) do not require the a-priori knowledge of $\min f$ and $\operatorname{argmin} f$. Moreover we show that Du_λ and $D_x u(\cdot, t)$ both converge (a.e.) to Dv, therefore giving an approximation of the gradient descent equation (3.2.1).

The main result of this section is the convergence of the gradient descent trajectories (3.2.1) to the set $\mathfrak{M}$ of minima of f. This is done in Section 3.2.3.1 after observing that v solves also the Dirichlet problem for the eikonal equation

$$\begin{cases} |\nabla v(x)| = \ell(x), & x \in \mathbb{R}^n \setminus \mathfrak{M} \\ v(x) = 0, & x \in \mathfrak{M} \end{cases} \tag{3.2.6}$$

with $\ell(x) := \sqrt{2(f(x) - \min f)}$. (In fact, our analysis of this problem requires only that $\ell \in C(\mathbb{R}^n)$ is bounded, non-negative, and $\mathfrak{M} = \{x : \ell(x) = 0\}$). We exploit that the unique solution of (3.2.6) is the value function

$$v(x) = \inf_{\alpha(\cdot)} \int_0^{t_x(\alpha)} \ell(y_x^\alpha(s))\,\mathrm{d}s, \quad \dot{y}_x^\alpha(s) = \alpha(s), \text{ for } s > 0, \quad y_x^\alpha(0) = x,$$

where α is measurable, $|\alpha(s)| \leq 1$, and $t_x(\alpha)$ is the first time the trajectory y_x^α hits $\mathfrak{M}$. We show that optimal trajectories exist, satisfy the gradient descent inclusion (3.2.1), and tend to $\mathfrak{M}$ as $t \to +\infty$ under a slightly strengthened positivity condition on ℓ. A crucial new tool for the proof are the occupational measures associated to these functions. Finally, we give a sufficient condition for such trajectories to reach $\mathfrak{M}$ in finite time.

In the third part of this chapter, §3.3, we also study the approximation of v and $\mathfrak{M}$ by vanishing viscosity. We add to (3.2.3) a term $-\varepsilon\Delta u_\lambda$ and let $\lambda \to 0+$ to get the viscous critical equation

$$U^\varepsilon - \varepsilon\Delta v^\varepsilon(x) + \frac{1}{2}|\nabla v^\varepsilon(x)|^2 = f(x) \quad \text{in } \mathbb{R}^n,$$

where U^ε is a constant. We prove that $0 \leq U^\varepsilon - \min f \leq C\varepsilon^\beta$ for some $\beta > 0$. Then we define the approximate stochastic gradient descent

$$\mathrm{d}X_s = -\nabla u_\lambda(X_s)\,\mathrm{d}s + \sqrt{2\varepsilon}\,\mathrm{d}W_s,$$

and show that the trajectories converge to $\mathfrak{M}$ in a suitable sense, for small λ and ε.

Note that (3.2.4) is the classical Hamilton-Jacobi equation with the mechanical Hamiltonian $H(x,p) = |p|^2 - f(x)$, where $-f$ is the potential energy. Then our results of Section 3.2.2 have an interpretation in analytical mechanics. For instance, the long-time behavior (3.2.5) describes a thermodynamical trend to equilibrium in a non-turbulent gas or fluid: see [56, 57].

We do not attempt to review all the literature related to the topics mentioned above. For weak KAM theory on compact manifolds we refer to [80, 81, 82], and for the PDE approach to ergodic control, mostly under periodicity assumptions, the reader can consult [4, 8] and the references therein. When the state space is not bounded one must add conditions to get some compactness. In addition to [83] already quoted, such problems were studied in all $\mathbb{R}^n$ by [12, 52, 139] assuming that f is large at infinity, and by [87, 105] for equations involving a linear first order term that satisfies a recurrence condition. Here, instead, we get compactness from the semiconcavity of f. Several of the results just quoted were used for homogenisation and singular perturbation problems, e.g., [4, 12, 127, 139], so we believe that also our results will have such applications.

The Dirichlet problem (3.2.6) with ℓ vanishing at the boundary was studied, e.g., in [131, 153]. The synthesis of an optimal feedback from the value function v leading to (3.2.1) uses method from [19] based on the earlier papers [34, 86].

We do not try here to design algorithms for global optimization based on the previous results. Let us mention, however, that some efficient numerical method for computing at the same time c and v in the critical/ergodic PDE (3.2.2) were proposed in [49].

The second part of this chapter, §3.2, is organized as follows. **Section 3.2.2** concerns a weak KAM theorem and approximation of the critical solution: in **subsection 3.2.2.1** we prove the weak KAM theorem by the small discount approximation (3.2.3) and in **subsection 3.2.2.2** we study the long-time asymptotics of solutions to (3.2.4). Then, in **section 3.2.3** we address the problem of reaching the minima via optimal control:

subsection 3.2.3.1 is devoted to the optimal control problem with target $\mathfrak{M}$ associated to (3.2.6) and **subsection 3.2.3.2** to deriving the gradient descent inclusion (3.2.1) for the optimal trajectories, then in **subsection 3.2.3.3** we prove that such trajectories converge to $\mathfrak{M}$. Finally, in **section 3.2.4** we show a case where the hitting time is finite. And before we move to the third and last part of this chapter, §3.3, we provide in the appendix in **section 3.2.5** a counterexample to uniqueness for (3.2.2).

3.2.2 A weak KAM theorem and approximation of the critical solution

Throughout this section we assume the following.

A1. $f : \mathbb{R}^n \to \mathbb{R}$ is continuous and

$$\exists\, \underline{f},\, \overline{f} \text{ s.t. } \; \underline{f} \le f(x) \le \overline{f}, \quad \forall\, x \in \mathbb{R}^n, \tag{3.2.7}$$

A2. f attains the minimum, i.e.,

$$\mathfrak{M} := \{x \in \mathbb{R}^n \;:\; f(x) = \underline{f} := \min_{z\in\mathbb{R}^n} f(z)\} \neq \emptyset \tag{3.2.8}$$

A3. f is C_1-Lipschitz continuous, i.e. $C_1 = \|\nabla f\|_\infty$,

A4. f is C_2-semiconcave, i.e., $D^2_{\xi\xi} f \le C_2$ a.e. for all $\xi \in \mathbb{R}^n$ s.t. $|\xi| = 1$, where $D^2_{\xi\xi} f$ is the second order derivative of f in the direction ξ.

A weak KAM theorem for the Hamiltonian $H(x,p) = |p|^2 - f(x)$ should give conditions under which there exists a constant $U \in \mathbb{R}$, the (Mané) critical value, such that the equation

$$U + |\nabla v(x)|^2 = f(x), \quad \text{ in } \mathbb{R}^n. \tag{3.2.9}$$

has a viscosity solution v. Clearly any critical value must satisfy $U \le \underline{f}$. In this section we prove under the current assumptions that $\underline{f}$ is a critical value and construct the solution v by two different approximation procedures, both classical and with an interpretation in terms of ergodic problems in optimal control.

The fact that $\underline{f}$ is the maximal critical value was proved in [83] for $f \in C^2$ and with $\mathbb{R}^n$ replaced by any complete Riemannian manifold, by methods of weak KAM theory different form ours.

3.2.2.1 The small discount limit

We consider the stationary approximation of (3.2.9)

$$\lambda u_\lambda + \frac{1}{2}|Du_\lambda|^2 = f(x), \quad x \in \mathbb{R}^n, \tag{3.2.10}$$

where $\lambda > 0$ will be sent to 0. The viscosity solution u_λ is known to be the value function of the following infinite horizon discounted optimal control problem

$$\begin{aligned} u_\lambda(x) = \inf_{\alpha.} J(x,\alpha.), \quad J(x,\alpha.) &:= \int_0^{+\infty} \left(\frac{1}{2}|\alpha_t|^2 + f(x(t))\right) e^{-\lambda t}\, \mathrm{d}t, \\ \text{s.t.} \;\; \dot{x}(s) &= \alpha_s, \quad x(0) = x \in \mathbb{R}^n, \quad s \geq 0 \end{aligned} \tag{3.2.11}$$

where the controls $\alpha. : [0,+\infty) \to \mathbb{R}^n$ are measurable function (see, e.g.,[19, Chapter III]). The main result of this section is the following.

Theorem 3.2.1. *Under the standing assumptions (A1–A4), as $\lambda \to 0$,*

$$\lambda u_\lambda(x) \to \underline{f} \quad \textit{and} \quad u_\lambda(x) - \underline{f}\lambda^{-1} \to v(x) \quad \textit{locally uniformly in } \mathbb{R}^n,$$

$$Du_{\lambda_k}(x) \to Dv(x) \quad \textit{a.e.},$$

where $v(\cdot)$ is a Lipschitz continuous viscosity solution to

$$\underline{f} + \frac{1}{2}|Dv(x)|^2 = f(x), \quad x \in \mathbb{R}^n. \tag{3.2.12}$$

Moreover $v \geq 0$ in $\mathbb{R}^n$ and null on $\mathfrak{M}$, and it is the unique viscosity solution of (3.2.12) *in $\mathbb{R}^n \setminus \mathfrak{M}$ vanishing on $\partial\mathfrak{M}$ and bounded from below.*

For the proof we need some estimates uniform in λ. The following lemmata are direct consequences of Lemma 3.1.1 in the previous section.

Lemma 3.1. *Under the assumption (A1,A3,A4), for all $x \in \mathbb{R}^n$ and $\lambda > 0$,*

$$\underline{f} \;\leq\; \lambda u_\lambda(x) \;\leq\; \overline{f} \tag{3.2.13}$$

$$|Du_\lambda(x)| \leq \sqrt{4\|f\|_\infty} \quad \textit{a.e.} \tag{3.2.14}$$

Lemma 3.2. *Let (A1,A3,A4) be satisfied. Then u is $\widetilde{C}_3$−semiconcave, where $\widetilde{C}_3$ is a positive constant independent of $\lambda \geq 0$.*

Proof *of Theorem 3.2.1.* First we claim that $\lambda u_\lambda(\bar{x}) = \underline{f}$ if $\bar{x} \in \mathfrak{M}$ (i.e., $f(\bar{x}) = \underline{f} = \min f$), for all $\lambda > 0$. In fact, for such $\bar{x}$,

$$u_\lambda(\overline{x}) = \inf_{\alpha.} \int_0^{+\infty} \left(\frac{1}{2}|\alpha_t|^2 + f(x(t))\right) e^{-\lambda t}\, \mathrm{d}t \le \int_0^{+\infty} f(\overline{x}) e^{-\lambda t}\, \mathrm{d}t = \underline{f}\lambda^{-1},$$

where the inequality follows from the choice $\alpha. \equiv 0$. The other inequality $\ge$ is true for all $x \in \mathbb{R}^n$ by Lemma 3.1, so the claim is proved.

Now we denote $R := \sqrt{4\|f\|_\infty}$ and use the gradient bound (3.2.14) to get

$$|\lambda u_\lambda(x) - \underline{f}| \le \lambda R \,\mathrm{dist}(x, \mathfrak{M}) \quad \forall\, x \in \mathbb{R}^n.$$

Then $\lambda u_\lambda(x) \to \underline{f}$ locally uniformly.

Define $\varphi_\lambda(\cdot) := u_\lambda(\cdot) - \underline{f}\lambda^{-1} \ge 0$ and use (3.2.14) to get, for all $x, y \in \mathbb{R}^n$,

$$|\varphi_\lambda(x)| \le R \,\mathrm{dist}(x, \mathfrak{M}), \qquad |\varphi_\lambda(x) - \varphi_\lambda(y)| \le R\,|x - y|. \tag{3.2.15}$$

Hence, $\{\varphi_\lambda(\cdot)\}_{\lambda \in (0,1)}$ is a uniformly bounded and equi-continuous family on any ball of $\mathbb{R}^n$. So we can choose a sequence $\lambda_k \to 0$ as $k \to +\infty$, such that $\varphi_{\lambda_k}(\cdot) \to v(\cdot) \in C(\mathbb{R}^n)$ locally uniformly. Plugging φ_λ in (3.2.10) we get

$$\lambda\varphi_\lambda + \underline{f} + \frac{1}{2}|D\varphi_\lambda(x)|^2 = f(x), \quad x \in \mathbb{R}^n.$$

We let $\lambda_k \to 0$ and use the stability of viscosity solutions to find that v satisfies (3.2.12).

Now we note that (3.2.12) is an eikonal equation with right hand side $f(x) - \underline{f} > 0$ in $\mathbb{R}^n \setminus \mathfrak{M}$, $v \ge 0$ and $v = 0$ on $\partial\mathfrak{M}$. This Dirichlet boundary value problem is known to have a unique viscosity solution bounded from below. Therefore the convergence of φ_λ is for $\lambda \to 0$ and not only on subsequences.

The convergence of the gradient $Du_\lambda(\cdot)$ to $Dv(\cdot)$ is a direct consequence of [53, Theorem 3.3.3], recalling that $|\varphi_\lambda(x)| \le R\,|x|$ and using the uniform semiconcavity estimate in Lemma 3.2. □

3.2.2.2 Long time asymptotics

Here we consider the evolutive Hamilton-Jacobi equation

$$\begin{cases} \partial_t u(x,t) + \dfrac{1}{2}|Du(x,t)|^2 = f(x), & (x,t) \in \mathbb{R}^n \times (0, +\infty) \\ u(x,0) = 0, & x \in \mathbb{R}^n. \end{cases} \tag{3.2.16}$$

and we will study the limit as $t \to +\infty$. The viscosity solution $u(x,t)$ is known to be the value function of the following finite-horizon optimal control problem

$$\begin{aligned} u(x,t) = \inf_{\alpha.} J(x,t,\alpha.) := \int_0^t \frac{1}{2}|\alpha_s|^2 + f(x(s))\,\mathrm{d}s, \\ \text{s.t.}\ \ \dot{x}(s) = \alpha_s, \quad x(0) = x \in \mathbb{R}^n \end{aligned} \tag{3.2.17}$$

where $\alpha. : [0,+\infty) \to \mathbb{R}^n$ are measurable functions (see e.g. [85, Chapter II] or [19, Chapter III]). The main result of this section is the following.

Theorem 3.2.2. *Under the standing assumptions (A1–A4), as $t \to +\infty$,*

$$\frac{u(x,t)}{t} \to \underline{f} \quad \textit{and} \quad u(x,t) - \underline{f}t \to v(x) \quad \textit{locally uniformly in } \mathbb{R}^n,$$

$$D_x u(x,t) \to Dv(x) \quad \textit{a.e.},$$

where $v(\cdot)$ is the viscosity solution of (3.2.12) *found in Theorem 3.2.1.*

To proceed with its proof we need some estimates uniform in t.

Lemma 3.3. *Under the assumption (A1,A3,A4), for all $(x,t) \in \mathbb{R}^n \times (0,+\infty)$,*

$$\underline{f} \leq \frac{u(x,t)}{t} \leq \overline{f}, \tag{3.2.18}$$

$$|\partial_t u(x,t)| \leq \|f\|_\infty \quad \textit{a.e.}, \tag{3.2.19}$$

$$|Du(x,t)| \leq \sqrt{4\|f\|_\infty} \quad \textit{a.e.} \tag{3.2.20}$$

Proof. The arguments are standard and follow from Lemma 3.1.1.

We only show (3.2.19).

Fix $h \in \mathbb{R}$ and $x \in \mathbb{R}^n$. Note first that $|u(x,h)| \leq |h|\|f\|_\infty$. Let us now denote $\overline{v}(x,t) := u(x,t+h)+|h|\|f\|_\infty$. Both u and $\overline{v}$ solve the same PDE in (3.2.16) with initial conditions $u(x,0) = 0$ and $\overline{v}(x,0) = u(x,h) + |h|\|f\|_\infty \geq 0$, hence by the comparison principle in [72, Theorem 2.1] we get $u(x,t) \leq \overline{v}(x,t)$.

Conversely, $\underline{v}(x,t) := u(x,t+h)-|h|\|f\|_\infty$ solves the same PDE in (3.2.16) with initial condition $\underline{v}(x,0) = u(x,h)-|h|\|f\|_\infty \leq u(x,0) = 0$. The same comparison principle now implies that $\underline{v}(x,t) \leq u(x,t)$. Therefore, one gets $|u(x,t+h) - u(x,t)| \leq |h|\|f\|_\infty$. □

Lemma 3.4. *Let (A1,A3,A4) be satisfied. Then u is C_3-semiconcave, where C_3 is a positive constant independent of $t \geq 0$.*

Proof. It is a consequence of *(iv)* in Lemma 3.1.1. □

Proof *of Theorem 3.2.2.* First we observe that $\frac{1}{t}u(x,t) = \underline{f}$ if $\bar{x} \in \mathfrak{M}$. In fact, for such $\bar{x}$,

$$u(\overline{x}, t) = \inf_{\alpha.} \int_0^t \frac{1}{2}|\alpha_s|^2 + f(x(s))\,\mathrm{d}s \le \int_0^t f(\overline{x})\,\mathrm{d}t = t\underline{f},$$

where the inequality follows from the choice $\alpha. \equiv 0$. The other inequality $\geq$ is true for all $x \in \mathbb{R}^n$ by Lemma 3.3.

Denote $R := \sqrt{4\|f\|_\infty}$ and use the gradient bound (3.2.20) to get

$$\left|\frac{1}{t}u(x,t) - \underline{f}\right| \le \frac{1}{t}R\,\mathrm{dist}(x, \mathfrak{M}) \quad \forall\, x \in \mathbb{R}^n,\ t > 0.$$

Then $u(x,t) \to \underline{f}$ locally uniformly as $t \to \infty$.

Define now $\varphi_t(\cdot) := u(\cdot, t) - \underline{f}t$. We observe that, in view of (3.2.20), $|\varphi_t(x)| \le R\,\mathrm{dist}(x, \mathfrak{M})$ and $|\varphi_t(x) - \varphi_t(y)| \le R|x - y|$. Hence, $\{\varphi_t(\cdot)\}_{t\ge 0}$ is a locally uniformly bounded and equi-continuous family.

We claim that $\varphi_t(\cdot) \to \psi(\cdot) \in C(\mathbb{R}^n)$ locally uniformly as $t \to +\infty$ and $\psi(\cdot)$ is a viscosity solution of

$$\underline{f} + \frac{1}{2}|D\psi(x)|^2 = f(x), \qquad \text{in } \mathbb{R}^n. \tag{3.2.21}$$

To prove the claim define $u_\eta(x,t) := \varphi_{t/\eta}(x) = u\left(x, \frac{t}{\eta}\right) - \frac{t}{\eta}\underline{f}$. Then we have

$$\eta\partial_t u_\eta + \underline{f} + \frac{1}{2}|Du_\eta|^2 = f(x), \qquad \text{in } \mathbb{R}^n \times (0, \infty).$$

Now consider the upper and lower relaxed semilimits (see [19, Definition V.I.4, p. 288])

$$\theta(x,t) := \limsup_{\eta\to 0,\, s\to t,\, y\to x} u_\eta(y,s), \quad \zeta(x,t) := \liminf_{\eta\to 0,\, s\to t,\, y\to x} u_\eta(y,s),$$

and note that they are finite by the local equiboundedness of φ_t. It is well-known from the stability properties of viscosity solutions (see, e.g., [19]) that they are, respectively, a sub- and supersolution of (3.2.21) for any $t > 0$. Moreover, for all $t > 0$,

$$\theta(x,t) = \limsup_{s\to+\infty,\, y\to x} \varphi_s(y) = \limsup_{s\to+\infty} \varphi_s(x),$$

where the last equality comes from the equicontinuity of φ_t. Similarly,

$$\zeta(x,t) = \liminf_{s\to+\infty} \varphi_s(x)$$

and so both θ and ζ do not depend on t. Next note that $\varphi_s(x) = 0$ for all $x \in \mathfrak{M}$ and it is non-negative everywhere. Then $\theta(x) = \zeta(x) = 0$ on $\partial\mathfrak{M}$, and they are a sub- and a supersolution bounded from below of (3.2.21) in $\mathbb{R}^n \setminus \mathfrak{M}$, where $f(x) - \underline{f} > 0$. Then a standard comparison principle for the Dirichlet problem associated to eikonal equations gives $\theta(x) = \zeta(x)$. This proves that φ_t converges pointwise to $\psi := \theta = \zeta \geq 0$, and the convergence is locally uniform by the Ascoli-Arzela theorem, which gives the claim. Moreover ψ coincides with the function v found in Theorem 3.2.1.

Finally, the convergence of the gradient $D_x u(\cdot, t) = D\varphi_t$ to $D\psi$ is a direct consequence of [53, Theorem 3.3.3], recalling that $|\varphi_t(x)| \leq R\,\mathrm{dist}(x, \mathfrak{M})$ and using the uniform semiconcavity estimate in Lemma 3.4. □

3.2.3 Reaching the minima via deterministic optimal control

3.2.3.1 The optimal control problem with target

In this section we consider the Dirichlet problem

$$\begin{cases} |\nabla v(x)| = \ell(x), & x \in \mathbb{R}^n \setminus \mathfrak{M}, \\ v(x) = 0, & x \in \mathfrak{M}, \end{cases} \tag{3.2.22}$$

motivated by the ergodic equation (3.2.12) of the previous section if $\ell(x) = \sqrt{2(f(x) - \underline{f})}$. Here, however, the standing assumptions are only

B1. $\mathfrak{M} \subseteq \mathbb{R}^n$ is a closed nonempty set, possibly unbounded,

B2. $\ell \in C(\mathbb{R}^n)$ satisfies

$$\ell \text{ is bounded }, \; \ell(x) > 0 \text{ if } x \in \mathbb{R}^n \setminus \mathfrak{M}, \;\; \ell \equiv 0 \text{ on } \mathfrak{M}. \tag{3.2.23}$$

The Lipschitz and semiconcavity conditions used in the previous section are not needed here.

We recall that the continuous viscosity solution of (3.2.22) is the value function of the control problem

$$v(x) = \inf_{\alpha} \int_0^{t_x(\alpha)} \ell(y_x^\alpha(s))\,\mathrm{d}s, \tag{3.2.24}$$

where α (an admissible control) is a measurable function $[0, +\infty) \to B(0,1)$, the unit ball in $\mathbb{R}^n$, $t_x(\alpha) := \inf\{s \geq 0 \,:\, y_x^\alpha(s) \in \mathfrak{M}\}$, and

$$\dot{y}_x^\alpha(s) = \alpha(s), \, \forall\, s \geq 0, \quad y_x^\alpha(0) = x. \tag{3.2.25}$$

Theorem 3.2.3. *Under the standing assumptions (B1,B2), there exists an optimal control α^* for the problem* (3.2.24).

Proof. Note first that, since $\ell(x) = 0$ for all $x \in \mathfrak{M}$ and otherwise $\ell > 0$, we can write without loss of generality

$$v(x) = \inf_{\alpha \in B(0,1)} \int_0^{+\infty} \ell(y_x^\alpha(s))\,\mathrm{d}s.$$

Fix $x \in \mathbb{R}^n$ and consider now a minimizing sequence $(y_k, \alpha_k)_k$ i.e. satisfying

$$\lim_{k\to+\infty} \int_0^{+\infty} \ell(y_k(t))\,\mathrm{d}t = v(x) \tag{3.2.26}$$

and

$$y_k(t) = x + \int_0^t \alpha_k(s)\,\mathrm{d}s, \text{ for all } t \geq 0.$$

Fix $N \in \mathbb{N}$. Then using Alaoglu's theorem, we can extract a subsequence that we denote by $(y_{k(N)}, \alpha_{k(N)})$ where $k(N) \to +\infty$ and such that

$$\begin{aligned}
\alpha_{k(N)} &\overset{*}{\rightharpoonup} \alpha_N^*, \quad \text{a.e. in } [0,N] \\
y_{k(N)} &\to y_N^*, \text{ loc. unif. on } [0,N] \\
\text{and } \; y_N^*(t) &= x + \int_0^t \alpha_N^*(s)\,\mathrm{d}s, \text{ for all } t \in [0,N].
\end{aligned}$$

We repeat this procedure for $N+1$, that is from the subsequence indexed by $k(N)$ we extract again by Alaoglu's theorem another subsequence $(y_{k(N+1)}, \alpha_{k(N+1)})$ where $k(N+1) \to +\infty$ and such that

$$\begin{aligned}
\alpha_{k(N+1)} &\overset{*}{\rightharpoonup} \alpha_{N+1}^*, \quad \text{a.e. in } [0,N+1], \\
y_{k(N+1)} &\to y_{N+1}^*, \text{ loc. unif. on } [0,N+1], \\
\text{and } \; y_{N+1}^*(t) &= x + \int_0^t \alpha_{N+1}^*(s)\,\mathrm{d}s, \text{ for all } t \in [0,N+1].
\end{aligned}$$

Moreover we have

$$\begin{aligned}
\alpha_{N+1}^* &= \alpha_N^*, \quad \text{a.e. in } [0,N] \\
y_{N+1}^* &= y_N^*, \quad \text{in } [0,N]
\end{aligned}$$

This suggests the definition of the candidate optimal pair (y^*, α^*) as

$$(y^*, \alpha^*) \coloneqq (y_N^*, \alpha_N^*) \qquad \text{in } [0,N].$$

Before we prove its optimality, we first check that the subsequences $(y_{k(N)}, \alpha_{k(N)})$, for $N \in \mathbb{N}$, converge along their diagonal elements $(y_{N(N)}, \alpha_{N(N)})$. Indeed, let us fix $T > 0$ and choose some $\overline{N} \geq T$. From the previous construction, we have

$$\begin{aligned} \alpha_{k(\overline{N})} &\stackrel{*}{\rightharpoonup} \alpha^*, \ \text{ a.e. in } \ [0,T], \\ y_{k(\overline{N})} &\to y^*, \ \text{loc. unif. on } \ [0,T], \\ \text{and } \ y^*(t) &= x + \int_0^t \alpha^*(s)\,\mathrm{d}s, \ \text{for all } \ t \in [0,T]. \end{aligned} \tag{3.2.27}$$

It suffices then to note that $N(N)$ is a subsequence of $k(\overline{N})$ for all $N \geq \overline{N}$. Therefore $(y_{N(N)}, \alpha_{N(N)})$ satisfies the convergence properties as in (3.2.27) where $k(\overline{N})$ is now replace by $N(N)$. We are now left with the proof of optimality of (y^*, α^*). This is a consequence of Fatou's lemma which here writes as

$$\int_0^{\infty} \liminf_{N\to+\infty} \ell(y_{N(N)}(t))\,\mathrm{d}t \leq \liminf_{N\to\infty} \int_0^{+\infty} \ell(y_{N(N)}(t))\,\mathrm{d}t.$$

But, recalling (3.2.26), the right-hand side is nothing but $v(x)$ since $y_{N(N)}$ is a subsequence of y_k. And ℓ being continuous, we conclude the proof taking the limits in N with the latter convergence result of $y_{N(N)}$ and get

$$\int_0^{\infty} \liminf_{N\to+\infty} \ell(y_{N(N)}(t))\,\mathrm{d}t = \int_0^{+\infty} \ell(y^*(t))\,\mathrm{d}t \leq v(x),$$

that is (y^*, α^*) is an optimal pair solution to (3.2.24). □

Next we show that the fraction of time spent by an optimal trajectory away from the minimizers of ℓ tends to zero as $t \to +\infty$. For a given fixed $\delta > 0$ we define the set of quasi-minimizers

$$K_\delta := \{x \in \mathbb{R}^n \ : \ \ell(x) \leq \delta\}$$

and the fraction of time ρ_t^δ spent by an optimal trajectory starting from x away from K_δ

$$\rho_t^\delta := \frac{1}{t}\big|\{s \in [0,t] \ : \ y_x^{\alpha^*}(s) \notin K_\delta\}\big|,$$

where $\big|I\big|$ denotes the Lebesgue measure of $I \subseteq \mathbb{R}$.

Theorem 3.2.4. *Under the standing assumptions (B1,B2), for any $x \in \mathbb{R}^n$ and $\delta > 0$, an optimal trajectory $y_x^{\alpha^*}(\cdot)$ for the problem* (3.2.24) *satisfies*

$$\rho_t^\delta \leq \frac{\overline{\ell}}{t\,\delta}\, dist(x, \mathfrak{M}). \tag{3.2.28}$$

In particular, $\lim_{t\to+\infty} \rho_t^\delta = 0.$

Proof. Since $\ell \geq 0$, using the characteristic function $\mathbb{1}_Q(y) = 1$ if $y \in Q$ and 0 otherwise,

$$\int_0^t \ell(y_x^{\alpha^*}(s))\mathrm{d}s \geq \int_0^t \mathbb{1}_{K_\delta^c}(y_x^{\alpha^*}(s))\,\ell(y_x^{\alpha^*}(s))\,\mathrm{d}s \;\geq \delta \int_0^t \mathbb{1}_{K_\delta^c}(y_x^{\alpha^*}(s))\,\mathrm{d}s$$

and hence

$$\frac{1}{t}\int_0^t \ell(y_x^{\alpha^*}(s))\mathrm{d}s \;\geq\; \delta\,\rho_t^\delta.$$

Now, since $\ell(y_x^{\alpha^*}(s)) = 0$ for all $s \geq t_x(\alpha^*)$, we have

$$\begin{aligned} \forall\, t \geq 0 \qquad \int_0^t \ell(y_x^{\alpha^*}(s))\,\mathrm{d}s \;&\leq\; \int_0^{t_x(\alpha^*)} \ell(y_x^{\alpha^*}(s))\,\mathrm{d}s, \\ &=\; v(x) \;\leq\; \bar{\ell}\,\inf\{t_x(\alpha) : (3.2.25) \text{ holds with } |\alpha(s)| \leq 1\}\,. \end{aligned}$$

The second factor on the right-hand side is the minimal time function whose optimal trajectories are the straight lines from the initial position x to its orthogonal projection on the set $\mathfrak{M}$, with maximal speed 1. Therefore the right-hand side in the last inequality is less or equal $\bar{\ell}|z-x|$ for any $z \in \mathfrak{M}$, and then

$$v(x) \leq \bar{\ell}\,\mathrm{dist}(x, \mathfrak{M}).$$

Combining the inequalities we get

$$0 \leq\; \delta\,\rho_t^\delta \;\leq\; \frac{1}{t}\int_0^t \ell(y_x^{\alpha^*}(s))\,\mathrm{d}s \leq\; \frac{v(x)}{t} \;\leq\; \frac{\bar{\ell}}{t}\,\mathrm{dist}(x, \mathfrak{M}),$$

which concludes the proof. □

3.2.3.2 A gradient descent inclusion for the optimal trajectories

So far, we showed that an optimal control exists and the corresponding optimal trajectory does not leave the set of minimizers in average as time goes to infinity, i.e. in the sense of (3.2.28). We now synthesize optimal feedback controls that give the gradient descent differential inclusion anticipated in the Introduction.

Theorem 3.2.5. *A control α with corresponding trajectory $y(\cdot) := y_x^\alpha(\cdot)$ is optimal if and only if*

$$\dot{y}(s) \in \left\{-\frac{p}{|p|}\,,\; p \in D^- v(y(s))\right\}, \quad \textit{for a.e. } s \in\; (0, t_x(\alpha)). \tag{3.2.29}$$

Proof. By the dynamic programming principle, the function

$$h(t) := v(y_x^\alpha(t)) + \int_0^t \ell(y_x^\alpha(s))\mathrm{d}s, \quad 0 \le t \le t_x(\alpha) \tag{3.2.30}$$

is non decreasing for all α, and nonincreasing (hence constant) if and only if α is optimal. And since h is locally Lipschitz, we get

$$\alpha \text{ is optimal} \quad \text{if and only if} \quad h'(t) \le 0 \text{ a.e. } t.$$

Proof of Necessity. Assume α is optimal, and so $h' \le 0$. Let $y(\cdot) := y_x^\alpha(\cdot)$.
Claim 1. $p \cdot \dot{y}(t) + \ell(y(t)) \le 0$ for all $p \in D^- v(y(t))$ a.e. t.

Let $\partial^- v(x;q)$ be the lower Dini derivative at x in the direction q (see equation (2.47) in [19, page 125]). Then by [19, Lemma 2.50, p. 135], one has

$$\partial^-(v \circ y)(s;1) = \partial^- v(y(s;,\dot{y}(s))$$

and for almost every t, $h'(t) = \partial^- v(y(t), \dot{y}(t)) + \ell(y(t))$. Next, using [19, Lemma 2.37, p. 126], one has, for any $z \in \mathbb{R}^n$

$$D^- v(z) = \{\, p \,:\, p \cdot q \le \partial^- v(z;q),\ \forall\, q \in \mathbb{R}^n\},$$

and hence, for almost every t and for all $p \in D^- v(y(t))$,

$$p \cdot \dot{y}(t) + \ell(y(t)) \le \partial^- v(y(t); \dot{y}(t)) + \ell(y(t)) = h'(t) \le 0.$$

Claim 2. $\dot{y}(t) = -\frac{p}{|p|}$ for all $p \in D^- v(y(t))$, a.e. t.

By [19, Proposition 5.3, p. 344], v is a bilateral supersolution of $|Dv(x)| - \ell(x) = 0$ in $\mathbb{R}^n \setminus \mathfrak{M}$, i.e. $|p| - \ell(x) = 0$ for all $p \in D^- v(x)$. This implies in particular that $p \neq 0$ if $x \notin \mathfrak{M}$. Hence, and using claim 1 together with $\dot{y} \in B(0,1)$, one gets

$$|p| = \ell(y(t)) \le -p \cdot \dot{y}(t) \le |p|$$

that is $\dot{y}(t) = -\frac{p}{|p|}$.

Proof of sufficiency. Assume (3.2.29) holds. Then for almost every t, one has

$$h'(t) = -\partial^- v(y(t); -\dot{y}(t)) + \ell(y(t)) \le -p \cdot (-\dot{y}(t)) + \ell(y(t)), \quad \forall\, p \in D^- v(y(t)).$$

Hence, if $y(\cdot)$ solves (3.2.29), then

$$h'(t) = -p \cdot \frac{p}{|p|} + \ell(y(t)) = -|p| + \ell(y(t)) \leq 0$$

since $p \in D^- v(y(t))$, by definition of v being a supersolution of $|Dv| - \ell = 0$. This concludes the proof. □

Remark 3.2.1. *Combining Theorem 3.2.3 and Theorem 3.2.5, the differential inclusion* (3.2.29) *has at least a solution and all such solutions are optimal.*

We recall the definition of limiting gradient of a Lipschitz function

$$D^* v(z) \coloneqq \{ p \,:\, p = \lim_{n \to +\infty} Dv(x_n) \ \text{ for some } \ x_n \to z \}.$$

Proposition 3.2.1. *(i) A necessary condition for the optimality of $y(\cdot)$ is*

$$\dot{y}(t) = -\frac{p}{|p|}, \quad \forall\, p \in D^+ v(y(t)),\, p \neq 0, \ \ a.e.\ t.$$

In particular, $D^+ v(y(t))$ is a singleton for a.e. t.

(ii) A sufficient condition for the optimality of $y(\cdot)$ is

$$\dot{y}(t) \in -\left\{ \frac{p}{|p|} \ :\ p \in D^* v(y(t)) \cap D^+ v(y(t)),\, p \neq 0 \right\}, \ \ a.e.\ t. \tag{3.2.31}$$

Proof. To prove *(i)* we take h defined by (3.2.30) and let $\partial^+ v(x; q)$ be the upper Dini derivative of v in direction q, with $|q| = 1$.

Claim 1. $p \cdot \dot{y}(t) + \ell(y(t)) \leq 0$, for all $p \in D^* v(y(t))$, a.e. t.

Using [19, Lemma 2.37, p. 126], one has, for any $z \in \mathbb{R}^n$

$$D^+ v(z) = \{ p \ :\ p \cdot q \geq \partial^+ v(z; q),\ \forall\, q \in \mathbb{R}^n \}.$$

Hence, for $p \in D^+ v(y(t))$, one has

$$p \cdot \dot{y}(t) + \ell(y(t)) = -p \cdot (-\dot{y}(t)) + \ell(y(t)) \leq -\partial^+ v(y(t); -\dot{y}(t)) + \ell(y(t)).$$

But, as in Claim 1 in the proof of Theorem 3.2.5, and since y is optimal, one gets

$$-\partial^+ v(y(t); -\dot{y}(t)) + \ell(y(t)) = h'(t) \leq 0,$$

which proves the claim.

Claim 2. $\dot{y}(t) = -\frac{p}{|p|}$ for all $p \in D^+ v(y(t)), p \neq 0$, a.e. t.

Recalling $|\dot{y}| \in B(0,1)$ and v being a subsolution of $|Dv| - \ell = 0$, we have for all $p \in D^+v(y(t))$, $|p| \leq \ell(y(t)) \leq -p \cdot \dot{y}(t) \leq |p|$, and hence, either $p = 0$ or $\dot{y}(t) = -\frac{p}{|p|}$.

To prove *(ii)* note that at all points of differentiability of v, one has $|Dv(z)| = \ell(z)$. Then for all $p \in D^*v(z)$, $|p| = \ell(z)$. And one has

$$h'(t) = \partial^+ v(y(t); \dot{y}(t)) + \ell(y(t)) \leq p \cdot \dot{y}(t) + \ell(y(t)), \quad \forall\, p \in D^+v(y(t)).$$

Then, for y solving (3.2.31), $p \neq 0$

$$h'(t) \leq -p \cdot \frac{p}{|p|} + \ell(y(t)) = -|p| + \ell(y(t)) = 0$$

which concludes the proof as it has been done for Theorem 3.2.5. □

3.2.3.3 Convergence of optimal trajectories to the argmin

In order to show stability of $\mathfrak{M}$, we need an assumption which prevents $\ell(\cdot)$ from approaching 0 when $\text{dist}(x, \mathfrak{M}) \to \infty$, that is,

- for all $\delta > 0$, there exists $\gamma = \gamma(\delta) > 0$ such that

$$\inf\{\ell(x) \,:\, \text{dist}(x, \mathfrak{M}) > \delta\} > \gamma(\delta). \tag{H}$$

If $\mathfrak{M}$ is bounded, then it is easy to see that this condition is equivalent to

$$\liminf_{|x| \to \infty} \ell(x) > 0,$$

which is also equivalent to Assumption (L3) in [52]. The last inequality, however, is impossible when $\mathfrak{M}$ is unbounded.

Remark 3.2.2. *An example of function with a unique global minimizer that does not satisfy hypothesis* (H) *is* $\ell(x) = |x|e^{-x^2}$. *In this case* $\mathfrak{M} = \{0\}$ *and* $\inf\{\ell(x) : |x| > \delta\} = 0$ *for all* δ.

A direct consequence of Theorem 3.2.5 is the following result.

Corollary 3.5. *Assume* (H) *holds besides the standing assumptions (B1,B2). Let* $y_x^{\alpha^*}(\cdot)$ *be an optimal trajectory and* $\delta > 0$. *If there exists* $\tau > 0$ *such that dist*$(y^*(\tau), \mathfrak{M}) > \delta$, *then, for* $\gamma(\cdot)$ *defined in* (H),

$$\rho_t^{\gamma(\delta/2)} \geq \frac{\delta}{t}, \qquad \forall\, t > \tau + \frac{\delta}{2}. \tag{3.2.32}$$

Proof. Set $y^*(\cdot) := y_x^{\alpha^*}(\cdot)$. Since it satisfies (3.2.29), we have $|\dot{y}^*(\cdot)| \leq 1$ and hence $y^*(\cdot)$ is Lipschitz continuous. Therefore, given $\delta > 0$, if there exists $\tau > 0$ such that $\mathrm{dist}(y^*(\tau), \mathfrak{M}) > \delta$, then

$$\begin{aligned}\delta < \mathrm{dist}(y^*(\tau), \mathfrak{M}) &\leq \mathrm{dist}(y^*(s), \mathfrak{M}) + |y^*(s) - y^*(\tau)| \\ &\leq \mathrm{dist}(y^*(s), \mathfrak{M}) + |s - \tau|\end{aligned}$$

which yields

$$\mathrm{dist}(y^*(s), \mathfrak{M}) > \frac{\delta}{2}, \quad \forall\, s \in]\tau - \delta/2, \tau + \delta/2[.$$

Hence one has

$$\ell(y^*(s)) \geq \inf\left\{\ell(x) \,:\, \mathrm{dist}(x, \mathfrak{M}) > \frac{\delta}{2}\right\}, \quad \forall\, s \in]\tau - \delta/2, \tau + \delta/2[,$$

and together with (H), one gets

$$\ell(y^*(s)) > \gamma(\delta/2), \quad \forall\, s \in]\tau - \delta/2, \tau + \delta/2[. \tag{3.2.33}$$

Therefore

$$|\{s \in [0, t] \,:\, y^*(s) \notin K_{\gamma(\delta/2)}\}| \geq |\,]\tau - \delta/2, \tau + \delta/2[\,|, \quad \forall\, t > \tau + \frac{\delta}{2}.$$

The latter writes as

$$t\, \rho_t^{\gamma(\delta/2)} \geq \delta$$

and concludes the proof. □

We are now ready to show stability properties of the set of global minimizers $\mathfrak{M}$ with respect to the optimal trajectories $y_x^{\alpha^*}(\cdot)$.

Theorem 3.2.6. *Assume* (H) *holds besides the standing assumptions (B1,B2). Then for* $y^*(\cdot)$ *as in* (3.2.29)*,*

(i) $\mathfrak{M}$ *is Lyapunov stable* [3]*,*

(ii) $\mathfrak{M}$ *is globally asymptotically stable* [4]*.*

Proof. Let $y^*(\cdot) := y_x^{\alpha^*}(\cdot)$ be an optimal trajectory, i.e., a solution of (3.2.29). We proceed by contradiction.

[3]This means that $\forall\, \varepsilon > 0$, $\exists\, \eta > 0$ such that $\mathrm{dist}(x, \mathfrak{M}) \leq \eta \Rightarrow \mathrm{dist}\left(y_x^{\alpha^*}(t), \mathfrak{M}\right) \leq \varepsilon$, $\forall\, t \geq 0$.

[4]This means that $\mathfrak{M}$ is Lyapunov stable and $\lim_{t\to+\infty} \mathrm{dist}\left(y_x^{\alpha^*}(t), \mathfrak{M}\right) = 0$ for all $x \in \mathbb{R}^n$.

Proof of (i). Let $\varepsilon > 0$ be fixed and suppose for all $\eta > 0$, $\exists\,\tau > 0$ such that $\mathrm{dist}(y^*(\tau), \mathfrak{M}) > \varepsilon$ and $\mathrm{dist}(x, \mathfrak{M}) < \eta$. Then from Corollary 3.5, one has

$$\rho_t^{\gamma(\varepsilon/2)} \geq \frac{\varepsilon}{t}, \quad \forall\, t > \tau + \frac{\varepsilon}{2}.$$

And from Theorem 3.2.4, one has

$$\frac{t\,\gamma(\varepsilon/2)}{\bar{\ell}} \rho_t^{\gamma(\varepsilon/2)} \leq \mathrm{dist}(x, \mathfrak{M}).$$

Therefore one gets

$$\frac{\varepsilon\,\gamma(\varepsilon/2)}{\bar{\ell}} \leq \mathrm{dist}(x, \mathfrak{M})$$

which contradicts $\mathrm{dist}(x, \mathfrak{M}) < \eta$ when we choose $\eta < \frac{\varepsilon\,\gamma(\varepsilon/2)}{\bar{\ell}}$. Hence we have, for all $\varepsilon > 0$, there exists $\eta > 0$ such that if $\mathrm{dist}(x, \mathfrak{M}) \leq \eta$ then $\mathrm{dist}(y^*(t), \mathfrak{M}) \leq \varepsilon$ for all t, which concludes the proof of the Lyapunov stability.

Proof of (ii). Suppose there exists a diverging sequence $\{\tau_k\}_{k\geq 0}$ and $\varepsilon > 0$ such that $\mathrm{dist}(y^*(\tau_k), \mathfrak{M}) > \varepsilon$. Without loss of generality, one can extract a subsequence (again denoted by τ_k) such that $\tau_{k+1} - \tau_k \geq \varepsilon$. Using Corollary 3.5, in particular (3.2.33), one has for all $k \geq 0$

$$\ell(y^*(s)) \geq \gamma(\varepsilon/2), \quad \forall\, s \in]\tau_k - \varepsilon/2, \tau_k + \varepsilon/2[$$

and therefore

$$|\{s \in [0,t]\,:\, y^*(s) \notin K_{\gamma(\varepsilon/2)}\}| > \sum_{\{k\geq 0\,:\,\tau_k \leq t - \frac{\varepsilon}{2}\}} |\,]\tau_k - \varepsilon/2, \tau_k + \varepsilon/2[\,| = N(t)\,\varepsilon,$$

where $N(t)$ is the number of distinct elements $\{\tau_k\}_{k\geq 0}$ that are in $[0, t + \varepsilon/2]$, i.e.

$$N(t) := \#\{\tau_k \,:\, \tau_k \leq t + \varepsilon/2,\ k \geq 0\}.$$

The previous inequality writes as

$$t\rho_t^{\gamma(\varepsilon/2)} > N(t)\,\varepsilon.$$

On the other hand, we know from Theorem 3.2.4, in particular (3.2.28), that

$$t\rho_t^{\gamma(\varepsilon/2)} \leq \frac{\bar{\ell}\,\mathrm{dist}(x, \mathfrak{M})}{\gamma(\varepsilon/2)},$$

and so we have $N(t) < \frac{\bar{\ell}\,\mathrm{dist}(x,\mathfrak{M})}{\varepsilon\,\gamma(\varepsilon/2)}$. But this cannot be true since $N(t) \to +\infty$ as $t \to +\infty$, and hence it concludes the proof. □

3.2.4 On reaching the argmin in finite time

Here we investigate whether the hitting time $t_x(\alpha^*)$ of an optimal trajectory with the target $\mathfrak{M}$ is finite or not.

In some control problems it may happen that an optimal trajectory remains arbitrarily close to a target without ever reaching it. Such a behavior has been observed in a linear-quadratic control problem studied in [113, §6.1] with the target is a singleton $\{x_\circ\}$ and the time t_ε of being ε-close to $x_\circ$ is shown to be $t_\varepsilon = C \ln\left(\frac{|x-x_\circ|}{\varepsilon}\right)$, where x is the initial state. Moreover, an optimal trajectory oscillates periodically around $x_\circ$ (see [113, p. 55]).

In our control problem described in Section 3.2.3.1 we expect that the hitting time is finite, except perhaps for some pathological choices of the running cost ℓ. Next we give a sufficient condition that strengthens the hypothesis (H): denote $d(x) := \mathrm{dist}(x,\mathfrak{M})$ and assume the following.

- There exists a continuous function $\gamma(s) > 0$ for all $s > 0$ and $\gamma(0) = 0$ such that, for some $r > 0$,

$$\begin{cases} \ell(x) \geq \gamma(d(x)), & \forall\, x \in \mathbb{R}^n, \\ \ell(x) = \gamma(d(x)), & \forall\, x \ \text{ s.t. } \ d(x) \leq r. \end{cases} \tag{L}$$

Proposition 3.2.2. *Assume* (L) *and* α^* *be an optimal control for problem* (3.2.24). *Then the hitting time* $t_x(\alpha^*) = d(x)$ *whenever* $d(x) \leq r$ *and it is finite for all* x.

Proof. Let us first note that the finiteness for all x follows from the property in the case $d(x) \leq r$, because by Theorem 3.2.6 *(ii)* there exists a finite time $\widetilde{t}_x$ such that $d(y_x^{\alpha^*}(\widetilde{t}_x)) \leq r$.

We assume that the initial position x satisfies $d(x) \leq r$ and aim to prove that

$$v(x) = \int_0^{d(x)} \gamma(s)\,\mathrm{d}s, \tag{3.2.34}$$

where $v(x)$ is the value function defined in (3.2.24). Denote by $V(x)$ the right-hand side of the last equality.

We first claim that $v(x) \leq V(x)$. Take z is in the set of projections of x onto $\mathfrak{M}$ and consider the straight line from x to z given by the trajectory $\overline{y}_x(t) = x - pt$, $t \geq 0$, where $p = \frac{x-z}{|x-z|}$. Note that $\overline{t}_x := \inf\{t \geq 0 \,:\, \overline{y}_x(s) \in \mathfrak{M}\} = d(x)$, and that $d(x-pt) \leq r$

for all $0 \leq t \leq \bar{t}_x$. Then, by (L),

$$v(x) \leq \int_0^{\bar{t}_x} \ell(\overline{y}_x(t))\,\mathrm{d}t = \int_0^{\bar{t}_x} \gamma(d(\overline{y}_x(t)))\,\mathrm{d}t =: J(x).$$

Observe now that $d(\overline{y}_x(t)) = \big||x-z|-t\big| = d(x)-t$. Therefore, using the change of variable $s := d(\overline{y}_x(t)) = d(x)-t$, we obtain

$$J(x) = \int_0^{d(x)} \gamma(d(\overline{y}_x(t)))\,\mathrm{d}t = \int_0^{d(x)} \gamma(s)\,\mathrm{d}s = V(x)$$

and this proves the claim.

Next we show that $v(x) \geq V(x)$. Since $v(x)$ is a continuous viscosity solution to (3.2.22), then using [153, Theorem 3.2 (ii)] it satisfies the upper optimality principle [153, Definition 3.1], that is,

$$v(x) \geq \inf_{\alpha} \int_0^t \ell(y_x^{\alpha}(s))\,\mathrm{d}s + v(y_x^{\alpha}(t)), \quad \forall t \geq 0,$$

where the dynamics of $y_x^{\alpha}(\cdot)$ is again (3.2.25) with $|\alpha(s)| \leq 1$. Using (L) and $v \geq 0$ we get

$$v(x) \geq \inf_{\alpha} \int_0^t \gamma(d(y_x^{\alpha}(s)))\,\mathrm{d}s, \quad \forall t \geq 0.$$

In particular, since $\gamma(s) = 0$ if and only if $s = 0$, we have

$$v(x) \geq \inf_{\alpha \in B(0,1)} \int_0^{t_x(\alpha)} \gamma(d(y_x^{\alpha}(s)))\,\mathrm{d}s \ =: \ W(x).$$

Then the function $W(x)$ solves in the viscosity sense the Dirichlet problem

$$\begin{cases} |\nabla W(x)| = \gamma(d(x)), & x \in \mathbb{R}^n \setminus \mathfrak{M} \\ W(x) = 0, & x \in \mathfrak{M}. \end{cases} \tag{3.2.35}$$

But $V(x) := \int_0^{d(x)} \gamma(s)\,\mathrm{d}s$ is also a viscosity solution of this Dirichlet problem because $|D^{\pm}V(x)| = |D^{\pm}d(x)|\gamma(d(x))$. We conclude using [131, Theorem 1 and Remark 3.1] that $V(x) = W(x)$ and hence $v(x) \geq V(x)$.

Finally we use in the integral of the formula (3.2.34) the same change of variable as above to get

$$v(x) = \int_0^{d(x)} \gamma(d(\overline{y}_x(t)))\,\mathrm{d}t = \int_0^{d(x)} \ell(\overline{y}_x(t))\,\mathrm{d}t.$$

This proves that $\overline{y}_x(t) := x - pt$ is an optimal trajectory and $d(x)$ is its hitting time. $\square$

3.2.5 Appendix: A counterexample to uniqueness

We shall construct a continuous family (indexed by $k \in \mathbb{R}$) of solutions $(U_k, v_k(\cdot)) \in \mathbb{R} \times C(\mathbb{R}^n)$ to the equation

$$U_k + \frac{1}{2}|Dv_k(x)|^2 = g(x), \quad \text{in } \mathbb{R}^n \tag{3.2.36}$$

for some function g satisfying the same assumptions as f at the end of §3.2.1.

Assuming the space dimension $n \geq 2$, we set $n = n_1 + n_1$ such that $n_1, n_2 \geq 1$. For any $x \in \mathbb{R}^n$, we write $x = (x_1, x_2)$ where we denote by $x_1 := \mathrm{Proj}_{n_1}(x)$ (respec. x_2) the projection of x on $\mathbb{R}^{n_1}$ (respec. on $\mathbb{R}^{n_2}$), and we set $\tilde{x}_1 = (x_1, 0_{n_2}) \in \mathbb{R}^n$ (respec. $\tilde{x}_2 = (0_{n_1}, x_2) \in \mathbb{R}^n$) where $0_{n_1}, 0_{n_2}$ are the zero elements of $\mathbb{R}^{n_1}$ and $\mathbb{R}^{n_2}$.

We will need the following Lemma[5].

Lemma 3.2.1. *Let $n_1, n_2 \geq 1$, and suppose $\{u_i\}_{i=1,2}$ are viscosity solutions respectively to*

$$|Du_i|^2 = f_i(x_i), \quad x_i \in \mathbb{R}^{n_i}. \tag{3.2.37}$$

Then, $\omega(x) := u_1(x_1) + u_2(x_2)$ is a viscosity solution to

$$|D\omega(x)|^2 = f_1(x_1) + f_2(x_2), \quad x = (x_1, x_2) \in \mathbb{R}^{n_1+n_2}.$$

Proof. We check that ω is a viscosity subsolution. The proof of ω being a supersolution follows analogously.

Denote by $n = n_1 + n_2$ and $g(x) := f_1(x_1) + f_2(x_2)$ for all $x = (x_1, x_2) \in \mathbb{R}^n$. Let $\varphi \in C^1(\mathbb{R}^n)$ and suppose there exists $x^\circ = (x_1^\circ, x_2^\circ) \in \mathbb{R}^n$ a local maximum point for $\omega - \varphi$, i.e., one has for some $r > 0$

$$(\omega - \varphi)(x) \leq (\omega - \varphi)(x^\circ), \quad \forall\, x \in B_r(x^\circ) \subset \mathbb{R}^n \tag{3.2.38}$$

where $B_r(x^\circ) = B_r(x_1^\circ) \times B_r(x_2^\circ)$ is a ball of radius r centered in x°. We need to check that

$$|D\varphi(x^\circ)|^2 \leq g(x^\circ). \tag{3.2.39}$$

We first note that, since x° is a local maximum point for $\omega - \varphi$, then x_1° (respec. x_2°) is a local maximum point for $u_1 - \varphi(\,\cdot\,, x_2^\circ)$ in $B_r(x_1^\circ)$ (respec. for $u_2 - \varphi(x_1^\circ, \,\cdot\,)$ in $B_r(x_2^\circ)$). It suffices indeed to evaluate (3.2.38) in (x_1, x_2°) for all $x_1 \in B_r(x_1^\circ)$ (respec. in (x_1°, x_2) for all $x_2 \in B_r(x_2^\circ)$).

Moreover, since u_1 and u_2 are viscosity solutions to (3.2.37), then one has in particular

[5] An analogous, but more general, result is [19, Lemma II.5.17, p.87].

that

$$|D_{x_i}\varphi(x_1^\circ, x_2^\circ)|^2 \leq f_i(x_i), \quad x_i \in B_r(x_i^\circ) \subset \mathbb{R}^{n_i}. \tag{3.2.40}$$

Finally, it suffices to observe that $|D\varphi|^2 = |D_{x_1}\varphi|^2 + |D_{x_2}\varphi|^2$ and to sum (3.2.40) side by side in order to obtain (3.2.39). This concludes the proof. □

We are now ready to construct the desired counterexample.

Consider the following equation with $n_1 \geq 2$

$$U + \frac{1}{2}|Du(x_1)|^2 = \tilde{f}(x_1), \quad \text{in } \mathbb{R}^{n_1} \tag{3.2.41}$$

where we define $\tilde{f}(x_1) := f(\tilde{x}_1) = f((x_1, 0_{n_2}))$ and f is as in §3.2.1. Then Theorem 3.2.1 provides at least one pair $(U, u(\cdot)) \in \mathbb{R} \times C(\mathbb{R}^{n_1})$ such that $u(\cdot)$ is a viscosity solution to (3.2.41). Now let us consider the following equation

$$-C_k + \frac{1}{2}|Dv_k(x_2)|^2 = 0, \quad \text{in } \mathbb{R}^{n_2} \tag{3.2.42}$$

for which any function $v_k(x_2) = -\sqrt{2\,C_k}\,|x_2|$, with $\{C_k\}_{k\in\mathbb{R}}$ a sequence of non-negative real numbers, is a viscosity solution.

It suffices now to observe (using Lemma 3.2.1) that $\{(\Lambda_k, \omega_k)\}_{k\in\mathbb{R}}$ such that $\omega_k(x) := u(x_1) + v_k(x_2)$ and $\Lambda_k := U - C_k$, for all $k \in \mathbb{R}$, is a viscosity solution to (3.2.36) with $g(x) := \tilde{f}(\mathrm{Proj}_{n_1}(x))$. And moreover ω_k is not differentiable in 0.

We refer to the more general results of [105] where it has been shown that the set of constants for which there exists a viscosity solution is a half line, see [105, eq. (2.9)].

3.3 Stochastic approximation

3.3.1 Introduction

Given a function f to be minimized, our goal is to construct a dynamics which converge to the global minimizer of the latter. It is well known that stochastic gradient descent converges in general to local minima and depends on the choice of the initial starting point. We would like here to construct trajectories which rather converge to the global minimum regardless of the initial position.

Our approach is based on stochastic optimal control theory and on ergodic Hamilton-Jacobi-Bellman equations. We rely in the first part on PDE methods to derive useful estimates, mainly semiconcavity, for the value function of the control problem. This will allow us to construct trajectories that first converge towards quasi-minimizers. And then we shall use probabilistic tools inspired from Laplace method to prove the convergence to the global minimum in the zero-noise limit. We shall also provide a rate of convergence for the latter.

The third part of this chapter, §3.3, is organized as follows. In **section 3.3.2** we introduce the control problem setting and derive the required estimates on its value function. This will be a key step for studying the corresponding ergodic PDE (Theorem 3.3.1). And in **section 3.3.3**, we study the small-noise limit. We first show the behavior of the ergodic constant in the vanishing viscosity regime (Proposition 3.3.3). Then we use the latter result together with occupational measures, to show the desired result on the convergence of trajectories towards the global minimum (Theorem 3.3.2).

In the sequel, we refer to the set of global minimizers of the function f again by

$$\mathfrak{M} := \{x \in \mathbb{R}^n \,:\, f(x) = \min_{z\in\mathbb{R}^n} f(z)\}. \tag{3.3.1}$$

And we recall the standing assumptions in this section

A1. $f : \mathbb{R}^n \to \mathbb{R}$ is continuous and

$$\exists\, \underline{f},\, \overline{f} \text{ s.t. } \ \underline{f} \leq f(x) \leq \overline{f}, \quad \forall\, x \in \mathbb{R}^n, \tag{3.3.2}$$

A2. f attains the minimum, i.e.,

$$\mathfrak{M} := \{x \in \mathbb{R}^n \,:\, f(x) = \underline{f} := \min_{z\in\mathbb{R}^n} f(z)\} \neq \emptyset \tag{3.3.3}$$

A3. f is C_1-Lipschitz continuous, i.e. $C_1 = \|\nabla f\|_\infty$,

A4. f is C_2-semiconcave, i.e., $D^2_{\xi\xi} f \leq C_2$ a.e. for all $\xi \in \mathbb{R}^n$ s.t. $|\xi| = 1$, where $D^2_{\xi\xi} f$ is the second order derivative of f in the direction ξ.

A5. $\nabla f(\cdot)$ is locally Lipschitz continuous,

We will also assume that the set of global minimizers $\mathfrak{M}$ as defined in (3.3.1) is closed and is nonempty, i.e. there exists at least one element $x_\circ \in \mathbb{R}^n$ such that $f(x_\circ) = \min_{z\in\mathbb{R}^n} f(z)$.

3.3.2 The stochastic control problem

The stochastic setting

Let $\nu = (\Omega, \{\mathcal{F}_s\}, \mathbb{P}, W.)$ be a reference probability system, where Ω is a sample space, $\{\mathcal{F}_s\}$ a filtration, $\mathbb{P}$ a probability measure, and $W.$ a $\mathbb{P}$-Brownian motion adapted to $\{\mathcal{F}_s\}$. Given $\varepsilon > 0$, we introduce a controlled stochastic process X_s solution to

$$\begin{aligned} \mathrm{d}X_s &= \alpha_s \,\mathrm{d}s + \sqrt{2\varepsilon}\,\mathrm{d}W_s, \\ X_0 &= x \in \mathbb{R}^n \end{aligned} \tag{3.3.4}$$

where the control α_s is $\mathbb{R}^n$-valued $\mathcal{F}_s$-progressively measurable process

A convergence result

We consider the following pay-off functional

$$J(x, \alpha.) := \mathbb{E}\left[\int_0^{+\infty} \left(\frac{1}{2}|\alpha_t|^2 + f(X_t)\right) e^{-\lambda t}\,\mathrm{d}t \,\middle|\, X_0 = x\right] \tag{3.3.5}$$

where f is again a bounded continuous functional satisfying (**??**), and $\lambda > 0$ is the discount factor. We also consider the corresponding control system which value function is given by

$$u_\lambda^\varepsilon(x) := \inf_{\alpha.\in\mathcal{A}} J(x, \alpha.), \text{ s.t. } (3.3.4). \tag{3.3.6}$$

Following the results in section 3.1 (see Remark 3.1.1), one can show that u_λ is a classical solution to the PDE

$$\lambda u_\lambda^\varepsilon - \varepsilon \Delta u_\lambda^\varepsilon + \frac{1}{2}|Du_\lambda^\varepsilon|^2 = f(x), \quad x \in \mathbb{R}^n. \tag{3.3.7}$$

Using classical estimates (see Lemma 3.1.1) one has the following results.

Lemma 3.3.1. *Let f be bounded. Then for every $x \in \mathbb{R}^n$, one has*

$$\underline{f} \;\leq\; \lambda u_\lambda^\varepsilon(x) \;\leq\; \overline{f} \tag{3.3.8}$$

Lemma 3.3.2. *Let f be bounded and let $C_1 = \sup|D_x f|$. Then for all $x \in \mathbb{R}^n$, one has*

$$|Du_\lambda^\varepsilon(x)| \leq \frac{C_1}{\lambda} \tag{3.3.9}$$

Lemma 3.3.3. *Let f be bounded and C_2-semiconcave, $C_1 = \sup |D_x f|$ and $n \geq 2$. Then u^ε is C_3-semiconcave, where C_3 is a positive constant independent of $\lambda \geq 0$ and of $\varepsilon \in (0,1)$.*

Theorem 3.3.1. *Assuming (A1,A3,A4) hold and using the notations of the previous lemmata, we have*

$$|Du_\lambda^\varepsilon(x)| \leq \sqrt{4\|f\|_\infty + 2\varepsilon n C_3}, \quad \forall\, x \in \mathbb{R}^n, \lambda > 0. \tag{3.3.10}$$

Moreover, $\lambda u_\lambda^\varepsilon(\cdot)$ converges uniformly on $\mathbb{R}^n$ to a constant U^ε as λ goes to 0_+. And for any sequence λ_k going to 0_+, there exists a subsequence (that we still denote by λ_k) such that $u_{\lambda_k}^\varepsilon(\cdot) - u_{\lambda_k}^\varepsilon(0)$ converges locally uniformly to a continuous viscosity solution $v(\cdot) - v(0)$ of

$$U^\varepsilon - \varepsilon \Delta v^\varepsilon + \frac{1}{2}|Dv^\varepsilon|^2 = f(x), \quad x \in \mathbb{R}^n. \tag{3.3.11}$$

We have in addition $Du_{\lambda_k}(\cdot) \to Dv^\varepsilon(\cdot)$ a.e. in $\mathbb{R}^n$, and hence $|Dv^\varepsilon(\cdot)|$ also satisfies the uniform bound in (3.3.10)*.*

Proof *of Theorem 3.3.1.* We first show a uniform gradient estimate. This is done using Lemma 3.3.1 and Lemma 3.3.3, together with the PDE (3.3.7). We then get, for $R := \sqrt{4\|f\|_\infty + 2\varepsilon n C_3}$,

$$|Du_\lambda^\varepsilon(x)| \leq R, \quad \text{for all } \ x \in \mathbb{R}^n. \tag{3.3.12}$$

Define $z_\lambda(\cdot) := \lambda u_\lambda^\varepsilon(\cdot)$ and $\varphi_\lambda(\cdot) := u_\lambda(\cdot) - u_\lambda^\varepsilon(0)$, and observe that for all $x, y \in \mathbb{R}^n$

$$\begin{aligned} |z_\lambda(0)| &\leq \|f\|_\infty, \\ |z_\lambda(x) - z_\lambda(0)| &\leq \lambda\, R\, |x|, \\ |\varphi_\lambda(x)| &\leq R\, |x|, \\ |\varphi_\lambda(x) - \varphi_\lambda(y)| &\leq R\, |x - y|. \end{aligned} \tag{3.3.13}$$

Hence, $\{\varphi_\lambda(\cdot)\}_{\lambda \in (0,1)}$ is a uniformly bounded and equi-continuous family on any balls of $\mathbb{R}^n$. So we can choose $\{\lambda_k\}_{k \in \mathbb{N}} \subset (0,1)$ such that as $k \to +\infty$, we have $\lambda_k \to 0$, $z_{\lambda_k}(0) \to U^\varepsilon$, $\varphi_{\lambda_k}(\cdot) \to v^\varepsilon(\cdot) - v^\varepsilon(0)$ in $C(\mathbb{R}^n)$, for some $(U^\varepsilon, v^\varepsilon) \in \mathbb{R} \times C(\mathbb{R})$ and where $v^\varepsilon(0)$ is a constant that guarantees $\varphi_{t_k}(0) = 0$. And from the second inequality in (3.3.13), we get $z_{\lambda_k}(x) \to U$ uniformly on balls of $\mathbb{R}^n$ as $k \to +\infty$. Finally, by the stability of the viscosity solutions, we find that v satisfies (3.3.11) in the viscosity sense.

The convergence of the gradient $Du_{\lambda_k}^\varepsilon(\cdot)$ to $Dv^\varepsilon(\cdot)$ is a direct consequence of [53, Theorem 3.3.3], recalling that $|\varphi_{\lambda_k}(x)| \leq R\,|x|$ and using the uniform semiconcavity estimate in Lemma 3.3.3. $\square$

3.3.3 Reaching the minima via stochastic optimal control

We define the *candidate* optimal strategy for the discounted control problem in feedback form as $\alpha^\lambda : x \mapsto \alpha^\lambda(x) := -\nabla u^\varepsilon_\lambda(x)$ where λ is the discount factor and the corresponding trajectory $X^{\lambda,\varepsilon}_\cdot$ is the one given by

$$\mathrm{d}X^{\lambda,\varepsilon}_s = -\nabla u^\varepsilon_\lambda(X^{\lambda,\varepsilon}_s)\,\mathrm{d}s + \sqrt{2\varepsilon}\,\mathrm{d}W_s, \quad X^{\lambda,\varepsilon}_0 = x \in \mathbb{R}^n. \tag{3.3.14}$$

Optimal Markov Control Policies

Very often, we require Markov controls to be bounded and Borel measurable and satisfy in addition local Lipchitz continuity in x and with at most a linear growth (see [85, §IV.3, p. 159]). In fact, for this class of controls, the SDE in (3.3.14) admits a pathwise unique solution and the reference probability system ν as defined in §3.3.2 can then be arbitrarily chosen (see [85, Remark IV.3.2, p. 160]).

Proposition 3.3.1. *The strategy defining the SDE* (3.3.14) *is the optimal one for the control problem* (3.3.6)*. It enjoys moreover pathwise uniqueness and the corresponding control problem is invariant w.r.t. the reference probability system.*

Proof. We are in the case of Markov control policies as defined above, thanks to the C^2 regularity and boundedness of the value function $u^\varepsilon_\lambda(\cdot)$ and its gradient (cf. Theorem 3.3.1). Therefore, given any $(\Omega, \mathcal{F}, \{\mathcal{F}_s\}, \mathbb{P})$ and $\mathcal{F}_s$-adapted brownian motion W_s, the stochastic differential equation (3.3.14) has a pathwise unique solution, and hence the reference probability system ν can be arbitrary chosen. Moreover the process $\alpha_s = -\nabla u^\varepsilon_\lambda(X^{\lambda,\varepsilon}_s)$ is $\mathcal{F}_s$-measurable and hence admissible.
Finally, a verification theorem (see [85, Corollary IV.5.1] and [117, Theorem 1]) insures that the candidate strategy defining the SDE (3.3.14) is the optimal one for the control problem (3.3.6). □

Proposition 3.3.2. *The SDE* (3.3.14) *admits a strong solution and enjoys pathwise uniqueness.*

Proof. This is a direct consequence of the the regularity of the function $v(\cdot)$ solution to (3.3.11). □

Before we move to results on global optimization via *small-discounted average* control problems, we study in the next upcoming section the asymptotic behavior of the ergodic constant U^ε as $\varepsilon \to 0$. This will play a key role in the proof of our convergence result of optimal trajectories towards the global minimum.

3.3.3.1 The ergodic constant

In this subsection, we study the behavior of the ergodic constant U^ε as the diffusion parameter ε goes to zero. But before we do so, let us recall some definitions (see, e.g., [129]) and known facts on Gibbs measure that we summarize in the next lemma.

Definition 3.3.1. *(i) A probability measure μ is said to be invariant for the process $\xi^x_\cdot$ when it satisfies $\int_{\mathbb{R}^n} \mathbb{E}[\varphi(\xi^x_t)]\, d\mu(x) = \int_{\mathbb{R}^n} \varphi(x)\, d\mu(x)$, for all bounded Borel-measurable functions φ and $t > 0$.*

(ii) An invariant measure is said to be ergodic when

$$\lim_{t\to+\infty} \frac{1}{t} \int_0^t \mathbb{E}[\varphi(\xi^x_t)]\, dt = \int_{\mathbb{R}^n} \varphi(z)\, d\mu(z)$$

for all φ s.t. $\int_{\mathbb{R}^n} |\varphi(z)|^2\, d\mu(z) < +\infty$. In this case, we also have

$$\lim_{\lambda\to 0} \lambda \int_0^{+\infty} \mathbb{E}[\varphi(\xi^x_s)]\, e^{-\lambda s}\, ds = \int_{\mathbb{R}^n} \varphi(z)\, d\mu(z)$$

This last statement is known as Abelian-Tauberian theorem.

Lemma 3.3.4. *Let $V : \mathbb{R}^n \to [0, +\infty)$ belongs to $C^{1+\alpha}(\mathbb{R}^n)$ for some $\alpha \in (0,1)$ and such that $e^{-V} \in L^1(\mathbb{R}^n)$ and $\{x : V(x) = \min_{y\in\mathbb{R}^n} V(y)\} = \{x_\circ\}$. Assume in addition that ∇V satisfies the following*

$$\begin{aligned} &\forall\, R > 0, \exists\, K_R > 0,\ \langle \nabla V(x) - \nabla V(y), x - y\rangle \le K_R |x-y|^2, \forall\, |x|, |y| \le R, \\ &\exists\, K > 0, \quad \langle \nabla V(x), x\rangle \le K(1 + |x|^2),\ \forall x \in \mathbb{R}^n. \end{aligned} \tag{3.3.15}$$

Then, for any $\sigma > 0$ the stochastic differential equation

$$d\xi^x_t = -\nabla V(\xi^x_t)\, dt + \sigma\, dW_t, \quad \xi^x_0 = x \in \mathbb{R}^n \tag{3.3.16}$$

is well defined and admits a unique invariant probability measure given by

$$\mu^\sigma(dx) = Z^{-1} e^{-2V(x)/\sigma^2} dx, \quad Z = \int_{\mathbb{R}^n} e^{-2V(x)/\sigma^2} dx \tag{3.3.17}$$

that is in addition ergodic. If moreover V satisfies: for all $\delta > 0$, there exists $\eta > 0$ such that

$$\inf\{\, V(x) : |x - x_\circ| \ge \delta \,\} > \sup\{\, V(x) : |x - x_\circ| \le \eta \,\} \tag{3.3.18}$$

then μ^σ converges, as $\sigma \to 0$, weakly to $\delta_{x_\circ}$, the Dirac measure with unit mass concentrated on $\{x_\circ\}$ the unique global minimizer of V.

***Proof*.** It is well known that (3.3.15) guarantees existence and uniqueness of a solution ξ_t^x continuous in t, which moreover defines a semigroup $\{T(t)\}$ such that $T(t)\varphi(x) = \mathbb{E}[\varphi(\xi_t^x)]$ for every bounded Borel measurable φ (see Theorem 2.4.3 and Theorem 2.5.2 in [129, Chapter 2]). Since $V \in C^{1+\alpha}(\mathbb{R}^n)$ and $e^{-V} \in L^1(\mathbb{R}^n)$, the probability measure μ^σ as defined in 3.3.17 is well defined. Then from [129, Theorem 8.1.26], it follows that μ^σ is the invariant measure of the semigroup $\{T(t)\}$, and [129, Theorem 8.1.15] insures its uniqueness, whereas ergodicity is given by [129, Theorem 8.1.11]. Convergence of μ^σ weakly to $\delta_{x_\circ}$, as $\sigma \to 0$, is shown in [13, Theorem 6.1] and is based on [96, Theorem 2.1]. □

Let us now recall the discounted optimal control problem

$$u_\lambda^\varepsilon(x) = \inf_{\alpha.} \mathbb{E}_x \left[\int_0^{+\infty} \left(\frac{1}{2}|\alpha_t|^2 + f(X_t) \right) e^{-\lambda t} \, \mathrm{d}t \right].$$

where $\mathrm{d}X_s = \alpha_s \mathrm{d}s + \sqrt{2\varepsilon}\, \mathrm{d}W_s,\ X_0 = x$.

Proposition 3.3.3. *Let U^ε be the ergodic constant in* (3.3.11)*, assume $\mathfrak{M}$ is closed and (A1–A5) hold. Then there exists $\widetilde{C}(\varepsilon, f, n)$ a positive constant that goes to* 0 *as $\varepsilon \to 0$ such that*

$$\min_{z \in \mathbb{R}^n} f(z) \le U^\varepsilon \le \widetilde{C}(\varepsilon, f, n) + \min_{z \in \mathbb{R}^n} f(z). \tag{3.3.19}$$

In particular we have $\lim_{\varepsilon \to 0} U^\varepsilon = \min_{z \in \mathbb{R}^n} f(z)$. If moreover $f \in C^2(\mathbb{R}^n)$, then $\forall\, \varepsilon > 0$

$$|U^\varepsilon - \min_{z \in \mathbb{R}^n} f(z)| \le C(f,n) \left(\varepsilon^{2\kappa} + \varepsilon^{(1-\kappa)/2} + \varepsilon^{1+\kappa} \right), \quad \forall\, \kappa \in (0,1), \tag{3.3.20}$$

where $C(f,n)$ is a positive constant.

Remark 3.3.1. *When $\varepsilon \in (0,1)$, we have $\varepsilon^{1+\kappa} \le \max(\varepsilon^{2\kappa}, \varepsilon^{(1-\kappa)/2})$ for all κ in $(0,1)$, and* (3.3.20) *writes as*

$$|U^\varepsilon - \min_{x \in \mathbb{R}^n} f(x)| \le C(f,n) \left(\varepsilon^{2\kappa} + \varepsilon^{(1-\kappa)/2} \right), \quad \forall\, \kappa \in (0,1).$$

***Proof*.** We proceed in three steps.

Step 1. (The bounds in (3.3.19)*)*

We have

$$\min_{z \in \mathbb{R}^n} f(z) \le \lambda\, \mathbb{E}_x \left[\int_0^\infty \left(\frac{1}{2}|\alpha_t|^2 + f(X_t) \right) e^{-\lambda t} \, \mathrm{d}t \right]$$

hence

$$\min_{z\in\mathbb{R}^n} f(z) \le \lambda u_\lambda^\varepsilon(x), \quad \forall\, x, \lambda > 0 \tag{3.3.21}$$

It suffices then to take the limit as λ goes to 0 using Theorem 3.3.1.

We now prove the second inequality.

Let $x_\circ \in \{x \,:\, f(x) = \min_{y\in\mathbb{R}^n} f(y)\}$ so $f(x_\circ) = \min_{y\in\mathbb{R}^n} f(y)$. Consider the function

$$V(x) := f(x) - f(x_\circ) + \frac{\gamma}{2}|x - x_\circ|^2.$$

where $\gamma > 0$ is sufficiently large. It is immediate to see that the function V satisfies the assumptions in Lemma 3.3.4. Let us fix $\kappa \in (0,1)$ and choose a control $\overline{\alpha}_s = -\varepsilon^\kappa \nabla V(Y_s^\varepsilon)$ where Y_s^ε is the unique solution to

$$dY_s^\varepsilon = -\varepsilon^\kappa \nabla V(Y_s^\varepsilon)\mathrm{d}s + \sqrt{2\varepsilon}\,\mathrm{d}W_s, \quad Y_0^\varepsilon = x. \tag{3.3.22}$$

We now estimate the term $\mathbb{E}_x\left[\int_0^\infty \left(\frac{1}{2}|\overline{\alpha}_s|^2\right) e^{-\lambda s}\,\mathrm{d}s\right]$. On the one hand we have

$$\begin{aligned}
&\mathbb{E}_x\left[\int_0^\infty \left(\frac{1}{2}|\overline{\alpha}_s|^2\right) e^{-\lambda s}\,\mathrm{d}s\right] \\
&\quad = \mathbb{E}_x\left[\int_0^\infty \frac{\varepsilon^{2\kappa}}{2}|\nabla f(Y_s^\varepsilon) + \gamma\,(Y_s^\varepsilon - x_\circ)|^2\, e^{-\lambda s}\,\mathrm{d}s\right] \\
&\quad \le \frac{1}{\lambda}\frac{\varepsilon^{2\kappa}}{2}C_1^2 + \varepsilon^{2\kappa}\,\gamma\,\mathbb{E}_x\left[\int_0^\infty \left(\nabla f(Y_s^\varepsilon)\cdot(Y_s^\varepsilon - x_\circ) + \frac{\gamma}{2}|Y_s^\varepsilon - x_\circ|^2\right) e^{-\lambda s}\,\mathrm{d}s\right].
\end{aligned} \tag{3.3.23}$$

On the other hand, Dynkin formula applied with the function

$$\varphi : (t, Y_t^\varepsilon) \mapsto \frac{1}{2}|Y_t^\varepsilon - x_\circ|^2 e^{-\lambda t}$$

yields, for some $t > 0$,

$$\begin{aligned}
&\mathbb{E}_x\left[\varphi(t, Y_t^\varepsilon)\right] - \varphi(0, x) \\
&\quad = \mathbb{E}_x\left[\int_0^t \left(-\frac{\lambda}{2}|Y_s^\varepsilon - x_\circ|^2 - \varepsilon^\kappa\,\nabla V(Y_s^\varepsilon)\cdot(Y_s^\varepsilon - x_\circ) + \varepsilon\, n\right) e^{-\lambda s}\,\mathrm{d}s\right] \\
&\quad \le \mathbb{E}_x\left[\int_0^t \big(-\varepsilon^\kappa\,\nabla V(Y_s^\varepsilon)\cdot(Y_s^\varepsilon - x_\circ) + \varepsilon\, n\big)\, e^{-\lambda s}\,\mathrm{d}s\right]
\end{aligned}$$

and since $\varphi \ge 0$, $\mathbb{E}_x\left[\varphi(t, Y_t^\varepsilon)\right] \ge 0$ and we have

$$0 \le \varphi(0, x) + \mathbb{E}_x\left[\int_0^t \big(-\varepsilon^\kappa\,\nabla V(Y_s^\varepsilon)\cdot(Y_s^\varepsilon - x_\circ) + \varepsilon\, n\big)\, e^{-\lambda s}\,\mathrm{d}s\right]$$

The latter holds for any $t > 0$, we have

$$\begin{aligned}
0 &\leq \varphi(0,x) + \mathbb{E}_x\left[\int_0^\infty \big(-\varepsilon^\kappa \nabla V(Y_s^\varepsilon)\cdot(Y_s^\varepsilon - x_\circ) + \varepsilon\, n \big)\, e^{-\lambda\, s}\,\mathrm{d}s\right] \\
&= \varphi(0,x) + \frac{\varepsilon\, n}{\lambda} + \mathbb{E}_x\left[\int_0^\infty -\varepsilon^\kappa \nabla V(Y_s^\varepsilon)\cdot(Y_s^\varepsilon - x_\circ)\, e^{-\lambda\, s}\,\mathrm{d}s\right]
\end{aligned}$$

and hence

$$\begin{aligned}
\varepsilon^{-\kappa}\varphi(0,x) + \frac{\varepsilon^{1-\kappa}\, n}{\lambda} &\geq \mathbb{E}_x\left[\int_0^\infty \nabla V(Y_s^\varepsilon)\cdot(Y_s^\varepsilon - x_\circ)\, e^{-\lambda\, s}\,\mathrm{d}s\right] \\
&= \mathbb{E}_x\left[\int_0^\infty \left(\nabla f(Y_s^\varepsilon)\cdot(Y_s^\varepsilon - x_\circ) + \gamma\,|Y_s^\varepsilon - x_\circ|^2\right)\, e^{-\lambda\, s}\,\mathrm{d}s\right] \\
&\geq \mathbb{E}_x\left[\int_0^\infty \left(\nabla f(Y_s^\varepsilon)\cdot(Y_s^\varepsilon - x_\circ) + \frac{\gamma}{2}|Y_s^\varepsilon - x_\circ|^2\right)\, e^{-\lambda\, s}\,\mathrm{d}s\right].
\end{aligned}$$

Using the latter together with (3.3.23), yields

$$\mathbb{E}_x\left[\int_0^\infty \left(\frac{1}{2}|\overline{\alpha}_s|^2\right) e^{-\lambda\, s}\,\mathrm{d}s\right] \leq \frac{1}{\lambda}\frac{\varepsilon^{2\kappa}}{2}C_1^2 + \varepsilon^\kappa\,\gamma\,\varphi(0,x) + \frac{1}{\lambda}\,\varepsilon^{1+\kappa}\,\gamma\, n.$$

Therefore, we have

$$\begin{aligned}
u_\lambda^\varepsilon(x) &\leq \mathbb{E}_x\left[\int_0^\infty \left(\frac{1}{2}|\overline{\alpha}_s|^2 + f(Y_s^\varepsilon)\right) e^{-\lambda\, s}\,\mathrm{d}s\right] \\
&\leq \frac{1}{\lambda}\frac{\varepsilon^{2\kappa}}{2}C_1^2 + \varepsilon^\kappa\,\gamma\,\varphi(0,x) + \frac{1}{\lambda}\,\varepsilon^{1+\kappa}\,\gamma\, n + \mathbb{E}_x\left[\int_0^\infty f(Y_s^\varepsilon) e^{-\lambda\, s}\,\mathrm{d}s\right]
\end{aligned}$$

and hence, setting

$$C(\varepsilon) \coloneqq \frac{C_1^2}{2}\,\varepsilon^{2\kappa} + \gamma\, n\,\varepsilon^{1+\kappa}, \tag{3.3.24}$$

we have

$$\lambda\, u_\lambda^\varepsilon(x) \leq C(\varepsilon) + \lambda\,\varepsilon^\kappa\,\gamma\,\varphi(x) + \lambda\int_0^\infty \mathbb{E}_x[f(Y_s^\varepsilon)]e^{-\lambda\, s}\,\mathrm{d}s \tag{3.3.25}$$

From Lemma 3.3.4, the process $Y_\cdot^\varepsilon$ as defined in (3.3.22) admits a unique invariant probability measure given by

$$\begin{aligned}
\mu(\mathrm{d}x;\varepsilon) &= Z^{-1}\,\exp\left(-\frac{f(x) - f(x_\circ) + \frac{\gamma}{2}|x - x_\circ|^2}{\varepsilon^{1-\kappa}}\right)\,\mathrm{d}x, \\
\text{with } Z &= \int_{\mathbb{R}^n} \exp\left(-\frac{f(x) - f(x_\circ) + \frac{\gamma}{2}|x - x_\circ|^2}{\varepsilon^{1-\kappa}}\right)\,\mathrm{d}x.
\end{aligned} \tag{3.3.26}$$

The latter being ergodic (again from Lemma 3.3.4), we can pass to the limit $\lambda \to 0$ in (3.3.25) and using Theorem 3.2.1 we get

$$U^\varepsilon \leq \widetilde{C}(\varepsilon, f) + \min_{z \in \mathbb{R}^n} f(z) \tag{3.3.27}$$

where

$$\widetilde{C}(\varepsilon, f) := C(\varepsilon) + \int_{\mathbb{R}^n} \big(f(x) - f(x_\circ)\big)\, \mu(\mathrm{d}x; \varepsilon).$$

This concludes the proof of the first statement in the proposition.

Step 2. (The convergence as $\varepsilon \to 0$)

The limit as $\varepsilon \to 0$ is justified on the one hand by the convergence $C(\varepsilon) \xrightarrow[\varepsilon \to 0]{} 0$ from the definition (3.3.24), and on the other hand by the weak convergence of $\mu(\cdot\,; \varepsilon)$ as $\varepsilon \to 0$ to Dirac measure $\delta_{x_\circ}(\cdot)$ given by Lemma 3.3.4 which then yields the convergence $\int_{\mathbb{R}^n} f(x)\mu(\mathrm{d}x;\varepsilon) \xrightarrow[\varepsilon \to 0]{} \int_{\mathbb{R}^n} f(x)\delta_{x_\circ}(x)\,\mathrm{d}x = f(x_\circ) = \min_{z \in \mathbb{R}^n} f(z)$, provided the function $V(\cdot)$ satisfies the hypothesis (3.3.18) which we now check. Indeed, recalling the definition $V(\cdot)$, we have

$$\frac{\gamma}{2}|x - x_\circ|^2 \leq V(x) = f(x) - f(x_\circ) + \frac{\gamma}{2}|x - x_\circ|^2.$$

Therefore, for $\delta > 0$ fixed, we have on one hand

$$\frac{\gamma}{2}\delta^2 \leq \inf\{\, V(x) \,:\, |x - x_\circ| \geq \delta \,\},$$

and on the other hand, we search for $\eta > 0$ such that

$$\sup\{\, V(x) \,:\, |x - x_\circ| \leq \eta \,\} \;<\; \frac{\gamma}{2}\delta^2.$$

And we have

$$\begin{aligned}\sup\{\, V(x) \,:\, |x - x_\circ| \leq \eta \,\} &\leq \sup\{\, f(x) - f(x_\circ) \,:\, |x - x_\circ| \leq \eta \,\} + \frac{\gamma}{2}\eta^2 \\ &\leq C_1\,\eta + \frac{\gamma}{2}\eta^2.\end{aligned}$$

It is then sufficient to search for $\eta > 0$ such that

$$C_1\,\eta + \frac{\gamma}{2}\eta^2 \;<\; \frac{\gamma}{2}\delta^2.$$

This holds true for any $\eta \in (0, \eta^*)$, where $\eta^* = \frac{-C_1 + \sqrt{C_1^2 + (\gamma\delta)^2}}{\gamma}$, and concludes the proof of convergence.

Step 3. (The rate of convergence)

For the rate of convergence, we need to estimate how fast the term

$$\int_{\mathbb{R}^n} \big(f(x) - f(x_\circ)\big)\, \mu(\mathrm{d}x;\varepsilon)$$

present in the constant $\widetilde{C}(\varepsilon, f)$ of (3.3.27) goes to zero. To do so, we apply the result in [13, Theorem 6.3], in particular [13, Example 6.3.1]. First, note that since f is differentiable and $x_\circ \in \operatorname*{argmin}_{x\in\mathbb{R}^n} f(x)$, one has $\nabla f(x_\circ) = 0$. Then, using a Taylor expansion of $V(\cdot)$ in a neighborhood of $x_\circ$, one gets

$$\frac{V(x) - V(x_\circ)}{|x - x_\circ|^2} = \frac{1}{2}\left(D^2 f(x_\circ) + \gamma I_n\right) \frac{x - x_\circ}{|x - x_\circ|} \cdot \frac{x - x_\circ}{|x - x_\circ|} + o(1)$$

where in addition $D^2 f(x_\circ) \geq 0$. Hence for $\gamma > 0$, $D^2 f(x_\circ) + \gamma I_n$ is a positive definite matrix, and setting

$$\phi(y) := \left(\frac{1}{2}\left(D^2 f(x_\circ) + \gamma I_n\right) y \cdot y\right)^{1/2}, \quad \forall\, y \in \mathbb{R}^n,$$

we have

$$\frac{V(x) - V(x_\circ)}{(\phi(x - x_\circ))^2} \xrightarrow[x\to x_\circ]{} 1.$$

This being satisfied, we can now apply the result in [13, Theorem 6.3] which insures that a random vector Z^ε with distribution $\mu(\cdot\,;\varepsilon)$ as defined in (3.3.26) satisfies

$$\frac{Z^\varepsilon - x_\circ}{(\varepsilon^{1-\kappa})^{1/2}} \xrightarrow[\varepsilon\to 0]{d} Z \tag{3.3.28}$$

where $\xrightarrow{d}$ means convergence in distribution and Z is a random vector with density $c\exp(-(\phi(z))^2)$ for some $c \in (0, +\infty)$. Using the latter in (3.3.27) together with (3.3.24), we can estimate the dependence on ε in the upper-bound of the ergodic constant U^ε as follows. We denote by $\xi_r(\cdot) : \mathbb{R}^n \to \mathbb{R}$ the function supported on $B(x_\circ, r)$, the ball centered in $x_\circ$ of radius $r > 0$, where it is equal to 1, and is 0 otherwise. And let $\{f_r\}_{r>0}$ be a sequence of functions such that $f_r(x) = \xi_r(x) f(x)$. Using Lipschitz continuity of f (hence also of f_r), one gets

$$\mathbb{E}\left[f_r(Z^\varepsilon) - f_r(x_\circ)\right] \leq \varepsilon^{\frac{1-\kappa}{2}}\, C_1\, \mathbb{E}\left[\xi_r(Z^\varepsilon)\frac{|Z^\varepsilon - x_\circ|}{\varepsilon^{\frac{1-\kappa}{2}}}\right]$$

The term in the expectation being bounded (thanks to the compactly supported function ξ_r), we can then use the above result (3.3.28) of convergence (in distribution) to pass to the limit in $\varepsilon \to 0$. Indeed, for any $\delta > 0$, there exists $\overline{\varepsilon} > 0$ such that for all $\varepsilon \in (0, \overline{\varepsilon})$, one has

$$\mathbb{E}\left[\xi_r(Z^\varepsilon)\frac{|Z^\varepsilon - x_\circ|}{\varepsilon^{\frac{1-\kappa}{2}}}\right] \leq \delta + \mathbb{E}[\,\xi_r(x_\circ)\,|Z|\,] = \delta + \mathbb{E}[\,|Z|\,],$$

since Z^ε converges in distribution to $x_\circ$ (as it is shown in *Step 2* of the proof) and $\xi_r(x_\circ) = 1$ for all $r > 0$. And we have (recalling the definition of Z above)

$$\mathbb{E}[\,|Z|\,] = c \int_{\mathbb{R}^n} |z|\, e^{-(\phi(z))^2}\, \mathrm{d}z =: R$$

which is a positive constant. Therefore, one gets

$$\mathbb{E}\left[f_r(Z^\varepsilon) - f_r(x_\circ)\right] \leq \varepsilon^{\frac{1-\kappa}{2}}\, C_1\, (R + \delta), \quad \forall\, \varepsilon \in (0, \overline{\varepsilon}).$$

The right-hand side being independent from r, we let $r \to +\infty$ in the left-hand side and recover

$$\int_{\mathbb{R}^n} \big(f(x) - f(x_\circ)\big)\, \mu(\mathrm{d}x; \varepsilon) = \mathbb{E}\left[f(Z^\varepsilon) - f(x_\circ)\right] \leq \varepsilon^{\frac{1-\kappa}{2}}\, C_1\, (R + \delta).$$

Going back to (3.3.27), we have

$$0 \leq U^\varepsilon - \min_{z \in \mathbb{R}^n} f(z) \leq \alpha\, \varepsilon^{2\kappa} + \beta\, \varepsilon^{1+\kappa} + \eta\, \varepsilon^{\frac{1-\kappa}{2}}, \quad \kappa \in (0,1)$$

where (using (3.3.24)) we set $\alpha := \frac{C_1^2}{2}$, $\beta := \gamma\, n$ and $\eta := C_1(R + \delta)$. This concludes the last statement, setting (as in (3.3.20)) the constant $C(f, n) := \max(\alpha, \beta, \eta)$. □

We are now ready to state and make precise the convergence result of optimal trajectories towards the global minimum.

3.3.3.2 The global minimum

Let $\varepsilon > 0$ be the diffusion coefficient, and recall the dynamics

$$\mathrm{d}X_s = \alpha_s \mathrm{d}s + \sqrt{2\varepsilon}\, \mathrm{d}W_s,\; X_0 = x \in \mathbb{R}^n \tag{3.3.29}$$

We are interested in this section in the asymptotic behavior of these trajectories in the context of optimal control, when the discount factor $\lambda \to 0$, and then when letting the diffusion coefficient ε goes to zero. We will show in particular how optimal trajectories converge to the global minimum of the function f.

We introduce occupational (random) measures that comply with the discounted optimal control problem and that we define by

$$\mu_{\lambda,x,\alpha}(Q) := \lambda \int_0^\infty \mathbb{1}_Q(X_s) e^{-\lambda s}\, \mathrm{d}s \tag{3.3.30}$$

where Q is any Borel set of $\mathbb{R}^n$, $X_\cdot$ is the trajectory (3.3.29) corresponding to the control α with initial position x and for any $z \in \mathbb{R}^n$, $\mathbb{1}_Q(z) = 1$ if $z \in Q$ and is 0 otherwise. It is clear that $\mu_{\lambda,x,\alpha}(Q) + \mu_{\lambda,x,\alpha}(Q^c) = 1$, where again Q^c is the complement of Q in $\mathbb{R}^n$.

Let $(X^*_\cdot, \alpha^*_\cdot) := (X^{\lambda,\varepsilon}_\cdot, \alpha^\lambda_\cdot)$ be an optimal pair state-control for the control problem (3.3.6) as defined in (3.3.14) and in Proposition 3.3.1. For $\delta \geq 0$ fixed, we define the set of quasi-minimizers (or quasi-optimal sublevel set) as follows

$$K_\delta := \{\, y \in \mathbb{R}^n \mid f(y) \leq \underline{f} + \delta \,\} \tag{3.3.31}$$

where we recall $\underline{f} := \min_{x \in \mathbb{R}^n} f(x)$. And denote by $\rho_\lambda^{\delta,\varepsilon}$ the (weighted) fraction of the time interval $(0, +\infty)$ during which the optimal trajectory $X^*_\cdot$ is far away from a minimizer of f, that is

$$\rho_\lambda^{\delta,\varepsilon} := \mu_{\lambda,x,\alpha^*}(K_\delta^c) = \lambda \int_0^\infty \mathbb{1}_{K_\delta^c}(X^*_s) e^{-\lambda s}\, \mathrm{d}s \tag{3.3.32}$$

where $K_\delta^c := \mathbb{R}^n \setminus K_\delta$. Note that $\rho_\lambda^{\delta,\varepsilon}$ is a random variable

$$\rho_\lambda^{\delta,\varepsilon} : \Omega \ni \omega \mapsto \lambda \int_0^\infty \mathbb{1}_{K_\delta^c}(X^*_s(\omega)) e^{-\lambda s}\, \mathrm{d}s = \rho_\lambda^{\delta,\varepsilon}(\omega) \in [0,1].$$

We can now state and prove the following convergence result.

Theorem 3.3.2. *In the situation of Proposition 3.3.3, we have*

$$\lim_{\lambda \to 0} \mathbb{P}\left(\rho_\lambda^{\delta,\varepsilon} > a\right) \leq \frac{1}{\delta\, a}\left(U^\varepsilon - \underline{f}\right), \quad \forall\, a > 0, \delta > 0, \varepsilon > 0 \tag{3.3.33}$$

where U^ε is the ergodic constant in (3.3.11)*. In particular, as the diffusion coefficient ε vanishes, we have*

$$\lim_{\varepsilon \to 0} \lim_{\lambda \to 0} \mathbb{P}\left(\rho_\lambda^{\delta,\varepsilon} > a\right) = 0, \quad \forall\, a > 0, \delta > 0, \tag{3.3.34}$$

with the same rate of convergence as in (3.3.20) *when $f \in C^2(\mathbb{R}^n)$.*

Proof. Note first that $\rho_\lambda^{\delta,\varepsilon} \geq 0$ and Markov inequality writes

$$\mathbb{P}(\rho_\lambda^{\delta,\varepsilon} > a) \leq \frac{\mathbb{E}[\rho_\lambda^{\delta,\varepsilon}]}{a}, \quad \forall\, a > 0, \delta > 0.$$

It suffices then to upperbound $\mathbb{E}[\rho_\lambda^{\delta,\varepsilon}]$. We have for $\delta > 0$

$$\begin{aligned}\min_{x\in\mathbb{R}^n} f(x) =: \underline{f} &= (1-\rho_\lambda^{\delta,\varepsilon})\underline{f} + \rho_\lambda^{\delta,\varepsilon}\underline{f}, \quad \text{a.s.}\\ &\le (1-\rho_\lambda^{\delta,\varepsilon})\underline{f} + \rho_\lambda^{\delta,\varepsilon}(\underline{f}+\delta), \quad \text{a.s.}\end{aligned}$$

Then, multiplying by $\lambda\, e^{-\lambda s}$ and integrating over $(0,+\infty)$, and recalling the definition (3.3.32) of $\rho_\lambda^{\delta,\varepsilon}$, yield

$$\begin{aligned}\underline{f} &\le (1-\rho_\lambda^{\delta,\varepsilon})\underline{f} + \rho_\lambda^{\delta,\varepsilon}(\underline{f}+\delta), \quad \text{a.s.}\\ &\le \lambda \int_0^\infty f(X_s^*)\, e^{-\lambda s}\,\mathrm{d}s, \quad \text{a.s.}\end{aligned}$$

where the second inequality holds because, on one hand we have by definition of the minimum $\underline{f} \le f(X_s^*)$ for any $s \ge 0$, in particular the inequality holds during the fraction of time $(1-\rho_\lambda^{\delta,\varepsilon})$, i.e. we have $(1-\rho_\lambda^{\delta,\varepsilon})\underline{f} \le (1-\rho_\lambda^{\delta,\varepsilon}) \int_0^\infty \lambda\, f(X_s^*)\, e^{-\lambda s}\,\mathrm{d}s$ almost surely, and on the other hand during the fraction of time $\rho_\lambda^{\delta,\varepsilon}$ we have $\underline{f}+\delta \le f(X_s^*)$ and hence $\rho_\lambda^{\delta,\varepsilon}(\underline{f}+\delta) \le \rho_\lambda^{\delta,\varepsilon} \int_0^\infty \lambda\, f(X_s^*)\, e^{-\lambda s}\,\mathrm{d}s$ almost surely.
We can now take the expectation and from the optimality of the pair $(X_\cdot^*, \alpha_\cdot^*)$ we have

$$\begin{aligned}\underline{f} \le (1-\mathbb{E}[\rho_\lambda^{\delta,\varepsilon}])\underline{f} + \mathbb{E}[\rho_\lambda^{\delta,\varepsilon}](\underline{f}+\delta) &\le \mathbb{E}\left[\lambda \int_0^\infty f(X_s^*)\, e^{-\lambda s}\,\mathrm{d}s\right]\\ &\le \mathbb{E}\left[\lambda \int_0^\infty \left(\frac{1}{2}|\alpha_s^*|^2 + f(X_s^*)\right) e^{-\lambda s}\,\mathrm{d}s\right] \;=\; \lambda\, u_\lambda^\varepsilon(x).\end{aligned}$$

which then yields

$$\underline{f} \le \underline{f} \;+\; \delta\,\mathbb{E}[\rho_\lambda^{\delta,\varepsilon}] \le \lambda u_\lambda^\varepsilon(x).$$

and together with Markov inequality we have for any $a > 0$

$$a\mathbb{P}(\rho_\lambda^{\delta,\varepsilon} > a) \le \mathbb{E}[\rho_\lambda^{\delta,\varepsilon}] \le \frac{1}{\delta}\left(\lambda u_\lambda^\varepsilon(x) - \underline{f}\right)$$

Using the convergence result in Theorem 3.2.1 together with the upperbound in Proposition 3.3.3, we have

$$0 \le \lim_{\lambda\to 0} \mathbb{P}(\rho_\lambda^{\delta,\varepsilon} > a) \le \frac{1}{a}\lim_{\lambda\to 0}\mathbb{E}[\rho_\lambda^{\delta,\varepsilon}] \le \frac{1}{\delta\, a}\left(U^\varepsilon - \underline{f}\right).$$

It is then immediate to recover the limit as the diffusion coefficient ε vanishes using Proposition 3.3.3, and the convergence rate from (3.3.20). □

Conclusion

With a stochastic discounted-optimal control problem and given a function f to be minimized, we constructed a trajectory (3.3.14) that converge, when the discount factor λ vanishes, to quasi-minimizers of f with a quantified error in (3.3.33). Then, in the small-noise limit, we recovered global minimization in (3.3.34) with an explicit rate of convergence (3.3.20). The latter is based on the behavior of the ergodic constant in the vanishing viscosity regime (3.3.19).

Chapter 4

The viscous ergodic problem

4.1 Introduction

This chapter is devoted to the problem of existence and uniqueness of solutions to some ergodic partial differential equations in the whole space domain $\mathbb{R}^m$ with unbounded data satisfying a subexponential growth. Such problems take the form of

$$\textit{Find } (c, u(\cdot)) \in \mathbb{R} \times \mathcal{X}(\mathbb{R}^m) \textit{ s.t.: } F(x, \nabla u(x), D^2 u(x)) = c, \quad \text{in } \mathbb{R}^m \tag{4.1.1}$$

where $\mathcal{X}$ is a functional space (part of the unknowns) and F is either

- a linear operator of the form $F := -\mathcal{L}u(x) + f(x)$, or
- a Bellman Hamiltonian of one of the two forms

$$F := \min_{\alpha \in A} \{ -\mathcal{L}_\alpha u(x) + f(x, \alpha) \} \quad \text{ or } \quad F := \max_{\alpha \in A} \{ -\mathcal{L}_\alpha u(x) + f(x, \alpha) \}$$

and where $\mathcal{L}$ is a diffusion operator

$$\mathcal{L}\varphi(x) := \text{trace}(a(x) D^2 \varphi(x)) + b(x) \cdot \nabla \varphi(x)$$

and $\mathcal{L}_\alpha$ is analogously defined with $b := b(x, \alpha)$ and $a := a(x, \alpha)$, and $\alpha \in A$ a compact subset of $\mathbb{R}^k$ for some $k > 0$. Such problems arise in ergodic stochastic control, weak KAM theory, homogenization, singular perturbations and asymptotic approximations in partial differential equations (long-time behavior, vanishing discount coefficient).

Throughout the chapter, we will make the following assumptions and refer to them wherever it is needed:

A0. The dimension $m \geq 2$.

A1. (i) $a = (a^{ij})$ is a continuous mapping on $\mathbb{R}^m$ such that $a(x) = \varrho(x)\varrho(x)^\top$ where ϱ is a continuous $m \times m_1$ matrix function (for some $m_1 \geq m$),

(ii) $b = (b^i) : \mathbb{R}^m \to \mathbb{R}^m$ is a locally bounded Borel-measurable vector field.

A2. For some $p > m$, $a^{ij} \in W^{p,1}_{\text{loc}}(\mathbb{R}^m)$ and $b^i \in L^p_{\text{loc}}(\mathbb{R}^m)$.

A3. There exist $\overline{\Lambda} \geq \underline{\Lambda} > 0$ such that $\forall\, x, \xi \in \mathbb{R}^m,\ \underline{\Lambda}\|\xi\|^2 \leq \xi a(x) \cdot \xi \leq \overline{\Lambda}\|\xi\|^2$, i.e. $a(\cdot)$ is positive, uniformly bounded and nondegenerate.

A4. $\lim\limits_{|x|\to\infty} b(x) \cdot x = -\infty$ (Recurrence condition).

A5. f is a Borel-measurable mapping on $\mathbb{R}^m$ with at most a polynomial growth, i.e. $\exists\, K_f > 0,$ s.t. $|f(x)| \leq K_f(1 + |x|^d)$ for all $x \in \mathbb{R}^m$ and for some $d \geq 1$.

A6. There exist $K_b > 0$ and $\beta \in [0, d]$ such that $|b(x)| \leq K_b(1 + |x|^\beta)$ for all $x \in \mathbb{R}^m$.

When the data will be depending on a (control) parameter α, the above conditions will be assumed to hold uniformly in α, assumption (A4) writes as $\lim\limits_{|x|\to\infty} \sup\limits_{\alpha\in A} b(x, \alpha) \cdot x = -\infty$ and in (A5) we will assume in addition f to be continuous in α (i.e. a Carathéodory condition: measurable in x, continuous in α).

Assumption (A4) is reminiscent of the existence of a Lyapunov function $w \in C^2(\mathbb{R}^m)$ s.t. $\lim\limits_{|x|\to\infty} w(x) = +\infty$ and $\lim\limits_{|x|\to\infty} \mathcal{L}w(x) = -\infty$. We will come back to this in §4.2.3.2.

The main difficulty and novelty in this setting is that we are looking for solutions in the whole space $\mathbb{R}^m$ while both b and f are unbounded. Usually, we refer to c as the *ergodic constant* and $u(\cdot)$ as the *corrector*. The differential operator $\mathcal{L}_\alpha$ can be interpreted as the infinitesimal generator of the controlled stochastic process

$$dX_t = b(X_t, \alpha_t)dt + \sqrt{2}\varrho(X_t, \alpha_t)dB_t \tag{4.1.2}$$

where B_t is a Wiener process while f is the running cost of the control problem. Similarly, the operator $\mathcal{L}$ would correspond to the same stochastic process where we drop the dependency on the parameter α (see the discussion in §4.2.3.2). Note that (4.1.2) should be understood in its weak sense (see e.g. [114, 115]).

The **main results** in the first part of this chapter (see Theorem 4.3.2, Theorem 4.4.1 and Theorem 4.4.2) can be informally stated as: *Under assumptions including (A0)-(A6), the following statements hold true:*

(i) (Existence) There exists a constant $c \in \mathbb{R}$ *such that the PDE in* (4.1.1) *admits an almost everywhere solution* $u(\cdot) \in W^{r,2}_{loc}(\mathbb{R}^m)$ *with* $r \in [1, +\infty)$ *and satisfying* $|u(x)| \leq K(1 + |x|^\kappa)$ *where* $K > 0$ *and* $\kappa = d + 1 - \beta$.

(ii) (Uniqueness) If we assume moreover that b is locally Lipschitz continuous with at most a linear growth (i.e. $\beta = 1$*), then* $u(\cdot)$ *is unique in* $W^{r,2}_{loc}(\mathbb{R}^m)$ *with* $r > \frac{m}{2}$*, up to an additive constant. That is, if* $(c, u(\cdot))$ *and* $(c, v(\cdot))$ *are two solutions in the sense of (i), then* $u(\cdot) - v(\cdot)$ *is a constant.*

We will then conclude by providing an analogous result in the manifold setting (see Theorem 4.4.3) and and by deriving a continuity estimate for the ergodic constant with respect to the data of the problem (see Proposition 4.4.2).

Then, in the second part of the present chapter, we will tackle the ergodic problem for Mean-Field Games and provide analogous results in the same setting.

Related results. There have been many contributions on the problem of ergodic HJB equations in various settings and with many different techniques since it is of interest not only of the community of PDEs but also of probabilists and (stochastic) control theorists. Indeed, the ergodic problem captures the asymptotic behavior of a system (e.g. the long-time behavior of a control problem, or the effective phenomena in homogenization) and hence plays the role of a model reduction technique that is of interest in many applications. In the context of stochastic control, such a problem arises for the first time in the pioneering work [119]. Then probably the first results linking homogenization to ergodic theory goes back to [32], and the ergodic problem as we have stated appears in the context of homogenization in [128]. Since then many results on the problem and related topics have been established.

- **In the linear case.** This corresponds to the *ergodic Poisson equation*, that is to find a pair $(c, \varphi(\cdot))$ where c is a constant, that solves $c + \mathcal{L}\varphi = g$ in the whole space. If one already knows what a possible ergodic constant c can be, then this boils down to the usual Poisson equation $\mathcal{L}\varphi = \widetilde{g}$ where $\widetilde{g} \coloneqq g - c$. In this case, the methods usually performed for the latter problem are mainly of stochastic analysis, based on Feynman-Kac representation and the theory of Dirichlet forms and semigroups [129]. With assumptions similar to ours, the problem $\mathcal{L}\varphi = \widetilde{g}$ is solved in [141, Theorem 1] (see also [142, 143] and more recently [149, 150] and the references therein) under the additional assumption $\int \widetilde{g} d\mu = 0$ where μ is the invariant measure associated to $\mathcal{L}$, i.e. the solution to $\mathcal{L}^*\mu = 0$ where $\mathcal{L}^*$ is the adjoint operator. In fact, with our result, we get $c = \int g d\mu$ and hence $c + \mathcal{L}\varphi = g$ becomes $\mathcal{L}\varphi = g - c = \widetilde{g}$ and our problem falls in the setting of [141]. Let us also mention that in chapter 2, we constructed in Proposition 2.3.2 the ergodic constant c and showed that it corresponds indeed to the mean of g w.r.t. μ using probabilistic techniques and without the need of proving the existence of the solution $\varphi(\cdot)$; see also equation (2.3.40). We also mention [132] where linear subelliptic operators are considered in the whole space $\mathbb{R}^m$ with possibly unbounded coefficients. The methods used in the latter are inspired by the lectures "Equations paraboliques et ergodicité" of P. L. Lions at Collège de France (2014–2015) [125].

- **In the nonlinear case.** We shall consider the particular case of Bellman equations. Most of the theory has been developed for the multidimensional torus where one enjoys compactness. Such a setting allows for a deeper analysis, in the sense of the underlying dynamical system, and is treated in [8]. There have been since then a wide literature on such a problem, mainly in the context of long-time behavior of HJB equation and of homogenization, both for the first-order and second-order equations, but also in the context of weak KAM theory, and that we do not review here, since it does not address the problem studied in this manuscript. We would also like to mention the recent work [38] where the link between PDEs and dynamical systems is brought into play. And probably the first results treating the second-order ergodic Bellman equation on the whole space $\mathbb{R}^m$ are [30], then [31]. By now, many results exist, addressing the problem under structural assumptions on the Hamiltonian. In [26] (see also [27]), the ergodic problem considered is of the form

$$-\Delta u + \frac{1}{\gamma}|Du|^{\gamma} = f(x) + c, \quad \text{in } \mathbb{R}^m. \tag{4.1.3}$$

Classical solutions are shown to exist using PDE methods, where f is locally Lipschitz continuous with a growth condition on its gradient and is moreover assumed coercive when $\gamma \geq 2$ or $f \sim |x|^{\beta}$ when $\gamma < 2$. These results are similar to those previously shown in [97, 98, 99, 100] using methods of stochastic control theory and probability tools. Similar arguments are used in [104] for quadratic Hamiltonian arising in risk-sensitive stochastic control problems. A study of the underlying (controlled) stochastic process can also be useful to derive helpful ergodic properties which then yield some compactness. This is done for example in the very recent paper [62] where an inward drift is assumed (similar to our assumption (A4)). Another approach that uses the stochastic ergodic control formulation together with PDE methods is the one in [65] where the problem considered is of the form (4.1.3) with an additional term of the form $-b(x) \cdot Du$, and the term $\frac{1}{\gamma}|Du|^\gamma$ is replaced by $H(Du)$ with H satisfying some regularity and growth assumptions. In the latter, the problem is approximated by a sequence of truncated problems (bounded with Neumann condition) as in [126]. The usual PDE method for dealing with the viscous ergodic HJB equation as being a limiting problem of either the long-time behavior of parabolic equations or to vanishing-discount coefficient in elliptic equations is described in detail in [4] (and references therein). On the other hand, [5] is devoted to uniqueness of classical solutions to HJB equation of the form (4.1.3) in the case where $\gamma \in (1, 2)$, and it relies on an infinite dimensional linear program for elliptic equations for measures which is an approach that is reminiscent of ours.

The usual method. The ergodic problem is studied as being a limiting problem of either the long-time behavior ($t \to +\infty$) of parabolic equations

$$\partial_t \omega + H(x, D\omega, D^2\omega) = 0, \quad \text{in } (0, +\infty) \times \mathbb{R}^m$$

or to vanishing-discount coefficient ($\delta \to 0$) in elliptic equations

$$\delta\omega + H(x, D\omega, D^2\omega) = 0, \quad \text{in } \mathbb{R}^m$$

The main questions then are the study of the limits $\lim\limits_{t\to+\infty} \frac{1}{t}\omega(x,t)$ (or $\lim\limits_{\delta\to 0} \delta\omega(x)$) and $\lim\limits_{t\to+\infty} \omega(x,t) - \omega(x_\circ, t)$ (or $\lim\limits_{\delta\to 0} \omega(x) - \omega(x_\circ)$) for some fixed $x_\circ$. In our setting these limits (in time or in the discount factor) are hard to obtain and remain, to our knowledge, an open question. However these methods are extremely powerful and provide a better insight on the problem (and motivates where the ergodic problem comes from). We refer to [4] (and references therein) for much more details on the latter.

Our method relies on duality tools together with the extension of the diffusion operator $\mathcal{L}$. The idea is to isolate the two terms c and f which make the PDE in (4.1.1) difficult to solve and consider them as (part of) objective functions in suitable optimization problems which are dual to each other. Then we interpret a solution $(c, u(\cdot))$ of (4.1.1) as a Lagrange multiplier of an optimization problem over the space of measures μ and whose admissible set is made of measures solving $\mathcal{L}^*\mu = 0$. And provided we can solve the latter equation, which is in fact a stationary *Fokker-Planck-Kolmogorov* equation, we can describe the admissible set of the optimization problem and hence recover existence and uniqueness of its corresponding dual variables i.e. the Lagrange multipliers, and which turn out to be the solution of the ergodic equation.

In fact, this method allows us to transpose to problems of the form (4.1.1) the information one can get from the study of the operator $\mathcal{L}$ and its adjoint $\mathcal{L}^*$ through a duality scheme for suitably chosen optimization problems.

This optimization view point is not totally new since it is briefly mentioned in [7, §6.6] and is also reminiscent of [77]. However, to our knowledge, this analysis has never been used to address a PDE problem such as the solvability of an ergodic HJB equation in our setting. Another interesting direction is the one considered in [5] where the problem of uniqueness of solutions to viscous HJB is addressed via similar duality methods, unlike in our case where we use duality to prove existence only and rely rather on Liouville type results [22] to prove uniqueness. We would like also to mention that our method allows to deal with the ergodic HJB equation under weak regularity assumptions, in particular the dependency on the space variable is assumed to be merely measurable and with a subexponential growth. Moreover our assumptions concern the coefficients of the diffusion operator (or the underlying stochastic differential equation) which is a way of presentation that is different from the classical references (amongst the abovementioned) that rather rely on structural assumptions on the Hamiltonian. Finally, the method can be extended to deal with ergodic Mean-Field-Games in the same setting as we shall do next.

This chapter is organized as follows. In **Section 4.2** we provide the main results from duality theory and also from diffusion operators, in particular we define the closed extension of an operator and which is the definition we shall consider for $\mathcal{L}$ and $\mathcal{L}_\alpha$ in the equation (4.1.1). Then in **Section 4.3** we apply the duality procedure for the linear case and show how the method applies for existence and uniqueness of a solution to (4.1.1). The techniques used for the linear case are instrumental for what follows. Indeed the same procedure will be adapted in **Section 4.4** to address the nonlinear case, i.e. the Bellman Hamiltonian, before we conclude with a similar result in the setting of manifolds and provide an estimate on the difference of ergodic constants. We will then move to the problem of ergodic Mean-Field Games which will be addressed in the devoted **Section 4.5**. Finally, in the conclusion in **Section 4.6** we discuss some remarks and possible further extensions of our method to tackle other problems in this direction.

4.2 Survey of known results

4.2.1 Duality theory

"Duality in mathematics is not a theorem, but a "principle". It has a simple origin, it is very powerful and useful, and has a long history going back hundreds of years. Over time it has been adapted and modified and so we can still use it in novel situations. It appears in many subjects in mathematics (geometry, algebra, analysis) and in physics. Fundamentally, duality gives two different points of view of looking at the same object. There are many things that have two different points of view and in principle they are all dualities."– Sir Michael F. Atiyah, in [14].

The results and remarks mentioned in this section are wellknown, and can be found in [46]. For the sake of a broad readability and self-containedness, we include the necessary results we will use, that we borrow again from [46].

Let (X, X^*) and (Y, Y^*) be paired spaces, i.e. such that each space of a pair is a locally convex topological vector space and is the topological dual of the other. We assume moreover that X and Y are Banach spaces that we endow with their respective strong topologies, while X^* and Y^* are endowed with the respective weak-$*$ topologies.

Let Q and K be closed convex subsets of X and Y respectively. We are interested in first order optimality conditions for the optimization problem

$$\min_{x\in Q} f(x), \quad \text{s.t.:} \quad G(x) \in K \tag{P}$$

where $f : X \to \mathbb{R}$ and $G : X \to Y$. The objective function in (P) can be reformulated as $f(x) + I_Q(x)$ while we minimize over the whole set X. We denote by $I_Q(\cdot)$ the indicator function ($I_Q(x) = 0$ if $x \in Q$, and $+\infty$ if $x \notin Q$). By

$$\Phi := \{x \in Q \ : \ G(x) \in K\} = Q \cap G^{-1}(K) \tag{4.2.1}$$

and

$$L(x, y^*) := f(x) + \langle y^*, G(x)\rangle, \quad (x, y^*) \in X \times Y^*, \tag{4.2.2}$$

we denote the feasible set and the Lagrangian of (P) respectively.

We embed the problem (P) into the family of optimization problems

$$\min_{x\in Q} f(x), \quad \text{s.t.:} \quad G(x) + y \in K \tag{P_y}$$

where $y \in Y$ is viewed as the parameter vector. Clearly for $y = 0$, the corresponding problem (P_0) coincides with the problem (P). We denote by $v(y)$ the corresponding value function

$$v(y) = val(P_y) = \inf_{x \in Q} \varphi(x, y)$$

where

$$\varphi(x, y) = f(x) + I_K(G(x) + y) \tag{4.2.3}$$

and $v(0) = val(P)$.

The (conjugate) dual of (P) can be written in the form (see [46, §2.5.3, p. 107]):

$$\max_{y^* \in Y^*} \left\{ \inf_{x \in Q} \; L(x, y^*) \; - \; \sigma(y^*; K) \right\} \tag{D}$$

where $\sigma(\cdot; K)$ is the support function of the set K which is the Legendre-Fenchel conjugate of the indicator function supported on K, i.e.

$$\sigma(y^*; K) = I_K^*(y^*) = \sup_{z \in K} \langle y^*, z \rangle. \tag{4.2.4}$$

Recall that $val(P) \geq val(D)$ (this can be easily obtained for example as a consequence of conjugate duality; see [46, eq. (2.268), p. 96], or by Lagrange duality; see [46, Proposition 2.156, p. 104]) and that if for some $x_o \in Q$, $y_o^* \in Y^*$ the equality of primal and dual objective functions holds, i.e.

$$f(x_o) + I_K(G(x_o)) = \inf_{x \in Q} \; L(x, y_o^*) - \sigma(y_o^*; K), \tag{4.2.5}$$

then $val(P) = val(D)$, and if the common value is finite, then $x_o \in Q$ and $y_o^* \in Y^*$ are optimal solutions of (P) and (D) respectively. The equality (4.2.5) can be written in the following equivalent form

$$\big(L(x_o, y_o^*) - \inf_{x \in Q} L(x, y_o^*)\big) + \big(I_K(G(x_o)) + I_K^*(y_o^*) - \langle y_o^*, G(x_o) \rangle\big) = 0. \tag{4.2.6}$$

Clearly, the first term in the left hand side is non-negative and the second term is also non-negative by the Young-Fenchel inequality. Moreover the equality

$$I_K(G(x_o)) + I_K^*(y_o^*) - \langle y_o^*, G(x_o) \rangle = 0$$

holds if and only if $y_o^* \in \partial I_K(G(x_o))$; the subdifferential of the indicator function evaluated in $G(x_o)$. Therefore, the equality of the objective functions (4.2.5) is equivalent to

$$x_o \in \operatorname*{argmin}_{x\in Q} L(x, y_o^*) \quad \text{and} \quad y_o^* \in \partial I_K(G(x_o)). \tag{4.2.7}$$

Proposition 4.2.1. *([46, Theorem 2.158, p. 109]) If conditions* (4.2.7) *are satisfied for some $x_\circ$ and $y_\circ^*$, then $x_\circ$ is an optimal solution of (P), $y_\circ^*$ is an optimal solution of (D), and there is no duality gap between (P) and (D).*

We assume that K is nonempty, closed and convex. Then $I_K(\cdot)$ is proper, lower semicontinuous and convex. Now using Young-Fenchel equality together with the definition of $N_K(\cdot)$; the normal cone[1] to K; we have $\partial I_K(G(x_o)) = N_K(G(x_o))$ and the optimality conditions (4.2.7) write

$$x_o \in \operatorname*{argmin}_{x\in Q} L(x, y_o^*) \quad \text{and} \quad y_o^* \in N_K(G(x_o)). \tag{4.2.8}$$

Note that existence of $y_o^* \in N_K(G(x_o))$ implies that $G(x_o) \in K$ and hence that x_o is a feasible point of the problem (P). Moreover, if K is a convex cone, the condition $y_o^* \in N_K(G(x_o))$ is equivalent to

$$G(x_o) \in K, \quad y_o^* \in K^- \quad \text{and} \ \langle y_o^*, G(x_o)\rangle = 0 \tag{4.2.9}$$

where K^- is the polar (negative dual) cone[2] of K, and is equal to the normal cone when K is a convex cone.

The convex case

In what follows, we consider the convex case, that is, we suppose $f(x)$ is convex, K is a convex, closed and nonempty cone, and $G(x)$ is $(-K)$-convex[3]. We assume moreover that $f(x)$ and $G(x)$ are continuous and the set Q is nonempty and closed. We have then the functions $f(x) + I_Q(x)$ and $I_K(G(x) + y)$ are lower semicontinuous and proper[4], and $\varphi(x, y) = f(x) + I_Q(x) + I_K(G(x) + y)$ is proper, lower semicontinuous and convex. We assume in addition that $G(x)$ is continuously differentiable, and we also make an assumption on the regularity of the value function insured by

$$0 \in int(G(x_\circ) + DG(x_\circ)[Q - x_\circ] - K), \quad \forall\, x_\circ \in \Phi \tag{4.2.10}$$

[1]Let $S \subset X$ be convex, then $N_S(x) := \{x^* \in X^* \ : \ \langle x^*, z - x\rangle \le 0 \quad \forall\, z \in S\}$. If $x \notin S$ then $N_S(x) = \emptyset$.

[2]Let C be a subset of X, then $C^- := \{x^* \in X^* \ : \ \langle x^*, x\rangle \le 0, \quad \forall\, x \in C\}$.

[3]See [46, Definition 2.163, p. 110] and [46, Definition 2.103, p. 72].

[4]A function f is said to be proper if Dom$f \neq \emptyset$ and $f(x) > -\infty$ for all x.

This is equivalent to the metric regularity of the multifunction $\mathcal{F}(x) := G(x) - K$ when G is $(-K)$-convex (see [46, p. 65]), and is also known as *Robinson's constraint qualification.*

Theorem 4.2.1. *([46, Theorem 3.6, p. 149]) Suppose that the problem (P) is convex, that x_o is an optimal solution of (P), and that the regularity condition* (4.2.10) *holds. Then the set of elements $y_o^* \in Y^*$ satisfying* (4.2.8) *is a nonempty, convex, bounded, and weakly-$*$ compact subset of Y^*, and is the same for any optimal solution.*

Proof. See the proof [46, Theorem 3.6, p. 149], noticing that when the problem (P) is convex, then $y_o^* \in Y^*$ satisfying the optimality conditions (4.2.8) is a Lagrange multiplier (See [46, Definition 3.5, p. 149]). □

A particular case. If we have $G(x) = (G_1(x), G_2(x))$ and $K = K_1 \times K_2 \subset Y_1 \times Y_2$ the Cartesian product of two Banach spaces, with K_1 and K_2 closed and convex, then the following lemma provides an equivalent formulation for Robinson's constraint qualification (4.2.10) at a feasible point $x_\circ \in \Phi$.

Lemma 4.2.1. *([46, Lemma 2.100, p. 70]) Let the constraints be given in the above product form, and assume that $DG_2(x_\circ)$ is onto. Then* (4.2.10) *which writes when $Q = X$ as*

$$0 \in int(G(x_\circ) + DG(x_\circ)X - K)$$

is equivalent to

$$0 \in int(G_1(x_\circ) + DG_1(x_\circ)[DG_2(x_\circ)^{-1}(K_2 - G_2(x_\circ))] - K_1). \tag{4.2.11}$$

Recall that by $[DG(x_\circ)]^{-1}$ we denote the multifunction with graph inverse to the one of $DG(x_\circ)$, i.e.,

$$[DG(x_\circ)]^{-1}(y) := \{h \in X \, : \, DG(x_\circ)h = y\}.$$

If we have $Q \subset X$ and in the particular case of:

$$Y_2 = X \;\text{ and }\; G_2(x) = x, \quad \forall\, x \in X, \tag{4.2.12}$$

then we set $\widetilde{K}_2 := K_2 \cap Q$ and (4.2.11) simplifies to (see [46, eq. (2.192), p. 71])

$$0 \in int(G_1(x_\circ) + DG_1(x_\circ)[\widetilde{K}_2 - x_\circ] - K_1). \tag{4.2.13}$$

The following is the last result we will need for existence.

Proposition 4.2.2. *([46, Proposition 3.3, p. 148]) If conditions* (4.2.8) *are satisfied for some x_o and y_o^*, then x_o is an optimal solution of (P), y_o^* is an optimal solution of (D), and there is no duality gap between (P) and (D).*

The general case

We drop here the assumption of convexity and are interested again in existence of dual variables and no duality gap between (P) and (D). But before we do so, we need the following notion of *calmness*.

Definition 4.2.1. *([46, Definition 2.146, p. 99]) We say that the problem (P_y) is calm if val(P_y) is finite and its optimal value function $v(\cdot)$ is subdifferentiable*[5] *at y, i.e., $\partial v(y) \neq \emptyset$.*

Hence when setting $y = 0$, the problem (P) is said to be *calm* if its optimal value $v(0) = val(P)$ is finite and $v(y)$ is subdifferentiable at $y = 0$. The following theorem then holds.

Theorem 4.2.2. *([46, Theorem 3.4, p. 148])*

(i) If (P) is calm, then there is no duality gap between (P) and (D), and a feasible point $x_\circ \in \Phi$ is an optimal solution of (P) if and only if there exists $y_\circ^* \in Y^*$ satisfying conditions (4.2.8).

(ii) If (P) is calm and $x_\circ$ is an optimal solution of (P), then the set of multipliers $y_\circ^*$ satisfying optimality conditions (4.2.8) *is nonempty and convex, and coincides with the set of optimal solutions of the dual problem (D), and hence is the same for any optimal solution of (P).*

It can be in general difficult to check subdifferentiability of the value function (and hence *calmness*). In the convex case, it turns out that Robinson's constraint qualification (4.2.10) is sufficient to guarantee the desired results in this section, whereas in the nonconvex case, one needs to rely on special structures of the problem (P) for which the computation of the subdifferential can be handled.

[5] A function $x^* \in X$ is said to be a *subgradient* of a (possibly nonconvex) function $f : X \to \mathbb{R} \cup \{+\infty\} \cup \{-\infty\}$ at a point x, if $f(x)$ is finite and $f(y) - f(x) \geq \langle x^*, y - x\rangle$, for all $y \in X$. The collection of all *subgradients* of f is called the *subdifferential* of f at x (See [46, §2.4.3, p. 81])

$$\partial f(x) := \{x^* \in X \,:\, f(y) - f(x) \geq \langle x^*, y - x\rangle, \quad \forall\, y \in X\}.$$

Uniqueness

To conclude this section, we present a further assumption under which we have uniqueness.

Proposition 4.2.3. *([46, Proposition 4.47, p. 297]) Suppose that y_o^* satisfies the optimality conditions* (4.2.8) *and that the strict constraint qualification*

$$0 \in int(G(x_o) + DG(x_o)Q - K_o) \tag{4.2.14}$$

where $K_o := \{y \in K \,:\, \langle y_o^*, y - G(x_o)\rangle = 0\}$ holds. Then y_o^* is unique.
In fact, $K_o = K \cap Ker\, y_o^*$ and the strict constraint qualification writes as

$$0 \in int(G(x_o) + DG(x_o)Q - K \cap Ker\, y_o^*). \tag{4.2.15}$$

Proof. See [46, Proposition 4.47, p. 297] where $K_\circ$ is defined as the set $\{y \in K \,:\, \langle y_o^*, y - G(x_o)\rangle = 0\}$. But if K is a convex cone (which is our case here), then by the first order optimality conditions (4.2.8) (see also (4.2.9)), one has $\langle y_o^*, G(x_o)\rangle = 0$, and hence $K_o = K \cap \operatorname{Ker} y_o^*$. □

4.2.2 Optimization in space of measures

We consider a particular case of the optimization problem (P) that we write in the context of functionals depending on a measure following the results in [134, 135, 136].

In this subsection, we choose Q and K as closed convex subsets of $\mathcal{M}^+(\mathbb{R}^m)$ and a Banach space Y, respectively, and we define $f : \mathcal{M}(\mathbb{R}^m) \to \mathbb{R}$ and $G : \mathcal{M}(\mathbb{R}^m) \to Y$ as Fréchet differentiable functions. The derivative of f is a linear functional $Df(\mu)[h]$ acting on $h \in \mathcal{M}(\mathbb{R}^m)$ and the derivative of G is a linear operator $DG(\mu)[h]$ mapping $\mathcal{M}$ into Y. The optimization problem writes as

$$\min f(\mu), \qquad \text{s.t.:} \qquad \mu \in Q \;\text{ and }\; G(\mu) \in K \tag{4.2.16}$$

Before we state a first-order optimality condition, we need to define a notion of regularity (also called *Constraint Qualification*) that is due to Robinson [147] (see also [46, §2.3.4, p. 67]) which is analogous to (4.2.10).

Definition 4.2.2. *([134, Definition 1.1]) A measure μ is called regular for Problem* (4.2.16) *if*

$$0 \in int(G(\mu) + DG(\mu)[Q - \mu] - K) \tag{4.2.17}$$

where $int(A)$ is the set of all $y \in A \subset Y$ such that $y + ty_1 \in A$ for all $y_1 \in Y$ and all sufficiently small positive t.

The following theorem is [68, Theorem 4.1] and gives first order necessary conditions for a minimum in Problem (4.2.16). When applied to the framework of measures, it is stated in [134].

Theorem 4.2.3. *([134, Theorem 1.1]) Assume that both $f : \mathcal{M}(\mathbb{R}^m) \to \mathbb{R}$ and $G : \mathcal{M}(\mathbb{R}^m) \to Y$ are continuous on Q and Fréchet differentiable at a regular $\mu_\circ \in Q$ such that $G(\mu_\circ) \in K$. Then, if $\mu_\circ$ is a local minimum point in Problem (4.2.16), the following (necessary) optimality condition is satisfied:*

$$Df(\mu_\circ)[h] \geq 0, \quad \forall\, h \in T_{Q \cap G^{-1}(K)}(\mu_\circ), \tag{4.2.18}$$

where $T_B(\mu)$ is the first order tangent set to a set B at a point μ in a Banach space and is defined as

$$T_B(\mu) = \liminf_{t \downarrow 0} \frac{B - \mu}{t}.$$

In order to make use of the latter theorem, we will need to determine what is the tangent set in the space of measures. In our case, we shall be interested in $\mathcal{M}^+(\mathbb{R}^m)$, the cone of finite non-negative measures.

Theorem 4.2.4. *([134, Theorem 2.1]) Let $\mu \in \mathcal{M}^+(\mathbb{R}^m)$. Then*

$$T_{\mathcal{M}^+(\mathbb{R}^m)}(\mu) = \{h \in \mathcal{M}(\mathbb{R}^m) \,:\, h^- \ll \mu\}, \tag{4.2.19}$$

where for a signed measure h, its Jordan decomposition is written as $h = h^+ - h^-$, and for $p, q \in \mathcal{M}^+(\mathbb{R}^m)$, $p \ll q$ refers to absolute continuity of p with respect to q.

A direct consequence of the latter theorem, is the case of measures with finite d-moment. It suffices indeed to replace $\mathcal{M}(\mathbb{R}^m)$ (respec. $\mathcal{M}^+(\mathbb{R}^m)$) with $\mathcal{M}_d(\mathbb{R}^m)$ (respec. $\mathcal{M}_d^+(\mathbb{R}^m)$ and obtain the following result.

Corollary 4.2.1. *Let $\mu \in \mathcal{M}_d^+(\mathbb{R}^m)$. Then*

$$T_{\mathcal{M}_d^+(\mathbb{R}^m)}(\mu) \supseteq \{h \in \mathcal{M}_d(\mathbb{R}^m) \,:\, h^- \ll \mu\}. \tag{4.2.20}$$

4.2.3 Diffusion operators

Let X be a Banach space with a norm $\|\cdot\|$. In the sequel, the term operator will refer to a linear transformation, not necessarily bounded, with domain and range subspaces of the same space X.

On unbounded operators

Let us denote by A a linear operator

$$A : D(A) \to X$$

where $D(A)$ is a linear manifold, the domain of the operator A. It is important to note that the domain is part of the definition of the operator.

Definition 4.2.3. *A linear operator $A : D(A) \to X$ is closed if for any sequence of vectors $f_n \in D(A)$ such that, as $n \to \infty$, $f_n \to f$ and $Af_n \to g$, it follows that $f \in D(A)$ and $Af = g$.*

Definition 4.2.4. *An operator B is an extension of A if $D(A) \subset D(B)$ and $Af = Bf$ for all $f \in D(A)$. We write $A \subset B$.*

Definition 4.2.5. *A linear operator $(A, D(A))$ is closable if it has a closed extension $(\overline{A}, D(\overline{A}))$ where*

$$D(\overline{A}) := \{f \in X \ : \ \exists\, f_n \in D(A),\ f_n \to f,\ Af_n \to g\}, \quad \textit{and} \ \ \overline{A}f := \overline{A}\lim_n f_n = \lim_n Af_n.$$

The closure $\overline{A}$ of the operator A is the smallest closed extension of A in the sense that if $A \subset B$ and B is closed, then $\overline{A} \subset B$.

Definition 4.2.6. *(See [76, Definition B.8, p. 537]) For a densely defined operator $(A, D(A))$ on X, we define the adjoint operator $(A^*, D(A^*))$ on X^* by*

$$D(A^*) := \{x^* \in X^* \ : \ \exists\, z^* \in X^* \ \textit{s.t.} \ \langle x^*, Ax\rangle = \langle z^*, x\rangle, \forall\, x \in D(A)\},$$
$$A^*x^* := z^* \ \textit{for} \ x \in D(A)$$

4.2.3.1 Semigroups of operators

Definition 4.2.7. *Let $\{T(t) \ : \ t \geq 0\}$ be a family of operators. We say that it is a semigroup if*

$$T(0) = I, \quad T(t+s) = T(t)T(s),\ \forall\, t, s \geq 0.$$

A semigroup is called strongly continuous, or C_0-semigroup, if for every $f \in X$, the function $T(\cdot)f : [0, +\infty) \to X$ is continuous.
And it is a contractive semigroup if it satisfies $\|T(t)\| \leq 1$ for all $t \geq 0$.

Definition 4.2.8. *The infinitesimal generator (or, shortly, the generator) if the semigroup* $\{T(t) \, : \, t \geq 0\}$ *is the operator defined by*

$$D(L) = \left\{ f \in X \, : \, \exists \lim_{h \to 0^+} \frac{T(h) - I}{h} f \right\}, \quad Lf = \lim_{h \to 0^+} \frac{T(h) - I}{h} f.$$

We refer to this operator by the pair $(L, D(L))$.

By definition, the vector Lf is the right derivative of the function $t \mapsto T(t)f$ at $t = 0$ and $D(L)$ is the subspace where such derivative exists. In general, $D(L)$ is not the whole space X, but it is dense, as the next proposition shows.

Proposition 4.2.4. *(See [130, Proposition 11.1.4, p. 138]) Let* $\{T(t) \, : \, t \geq 0\}$ *be a strongly continuous semigroup. The domain* $D(L)$ *of its generator is dense in* X.

Proposition 4.2.5. *(See [130, Proposition 11.1.6, p. 139]) The generator* L *of any strongly continuous semigroup is a closed operator.*

As a direct consequence, $D(L)$ is a Banach space with the graph norm $\|f\|_{D(L)} = \|f\| + \|Lf\|$.

4.2.3.2 Extension of diffusion operators

We resume in this subsection some known results from [42, §2] (see also the references therein). We suppose the dimension $m \geq 2$. Let $A = (a^{ij})$ be a continuous mapping on $\mathbb{R}^m$, and let $b = (b^i) : \mathbb{R}^m \to \mathbb{R}^m$ be a Borel-measurable vector field. Let us set

$$L_{A,b}\varphi = a^{ij}\partial_i\partial_j\varphi + b^i\partial_i\varphi, \quad \varphi \in C_0^\infty(\mathbb{R}^m), \tag{4.2.21}$$

where we use the standard summation rule for repeated indices. Suppose that μ is a locally finite (not necessarily non-negative) Borel measure on $\mathbb{R}^m$, i.e. a measure on the Borel σ-algebra $\mathcal{B}(\mathbb{R}^m)$ of $\mathbb{R}^m$, such that

$$L^*_{A,b}\mu = 0 \tag{4.2.22}$$

in the following sense:

$$a^{ij}, b^i \in L^1_{\mathrm{loc}}(\mu) \quad \text{and} \quad \int_{\mathbb{R}^m} L_{A,b}\varphi \, d\mu = 0, \quad \forall \, \varphi \in C_0^\infty(\mathbb{R}^m) \tag{4.2.23}$$

Measures μ satisfying (4.2.22) are called *infinitesimally* invariant, or simply invariant if there is no confusion. And define

$$\mathcal{M}_{\text{ell}}^{A,b} := \big\{\mu \mid \mu \text{ a probability measure on } \mathbb{R}^m \text{ satisfying } (4.2.22)\big\}. \tag{4.2.24}$$

where the subscript "ell" stands for *elliptic*. In [42], it is shown that the question whether or not $\mathcal{M}_{\text{ell}}^{A,b}$ contains at most one element turns out to be related to the question whether $\mu \in \mathcal{M}_{\text{ell}}^{A,b}$ is invariant for the C_0-semigroup generated by the closure of the operator $(L_{A,b}, C_0^\infty(\mathbb{R}^m))$.
In particular, and under assumptions that we will shortly made precise, if $\mathcal{M}_{\text{ell}}^{A,b} = \{\mu\}$ a singleton, then μ allows to define a new operator $(\overline{L}_{A,b}^{\mu}, D(\overline{L}_{A,b}^{\mu}))$ which is the closed extension of $(L_{A,b}, C_0^\infty(\mathbb{R}^m))$ on $L^1(\mathbb{R}^m, \mu)$. The latter operator will play a key role in our main result on the existence of solutions to (4.1.1). We recall the usual notations: when a measure μ has a density ρ with respect to Lebesgue measure that we denote by dx, then μ is absolutely continuous with respect to dx, we write $\mu \ll dx$ and $\rho = \frac{d\mu}{dx}$ which is Radon-Nikodym derivative of μ with respect to dx.

Theorem 4.2.5. *(Regularity –[42, Theorem 2.1]) Let μ be a locally finite and non-negative Borel measure satisfying* (4.2.22)*. Assume (A0), (A1), (A2) and (A3) for some $p > m$. Then $\mu \ll dx$ with $\frac{d\mu}{dx} \in W^{p,1}_{loc}(\mathbb{R}^m)\big(\subset C^{1-\frac{m}{p}}(\mathbb{R}^m)\big)$. If ρ denotes the continuous version of $\frac{d\mu}{dx}$, then for all compact $K \subset \mathbb{R}^m$, $\exists\ c_K \in]0, \infty[$ s.t.: $\sup_K \rho \leq c_K \inf_K \rho$. In particular, either $\rho \equiv 0$ or $\rho(x) > 0$, $\forall\ x \in \mathbb{R}^m$.*

Proof. The theorem stated in this form is [42, Theorem 2.1] and is a combination of the results [39, Corollary 2.10 & Corollary 2.11] which are slightly more general. $\square$

Theorem 4.2.6. *(Existence –[42, Theorem 2.2]) Assume (A0), (A1), (A2) and (A3). And assume in addition that there exists a function $\omega \in C^2(\mathbb{R}^m)$ s.t.*

$$\omega(x) \to +\infty \ \text{ and } \ L_{A,b}\omega(x) \to -\infty \quad \text{ as } |x| \to \infty. \tag{4.2.25}$$

Then $\mathcal{M}_{ell}^{A,b}$ as defined in (4.2.24) *is non-empty.*

Proof. The theorem stated in this form is [42, Theorem.2.2] and it relies on [41, Theorem 1.2]. $\square$

Corollary 4.2.2. *Assume (A0), (A1), (A2), (A3) and (A4). Then $\omega(x) := |x|^2$ fulfills* (4.2.25) *and the conclusion of Theorem 4.2.6 holds.*

Proof. See [41, Corollary 1.4 & Corollary 1.3(ii)]. $\square$

Let us consider now the situation of theorem 4.2.6. Fix $\mu \in \mathcal{M}_{\text{ell}}^{A,b}$. As observed in [42, §2.3], by Theorem 4.2.5, μ is equivalent to Lebesgue measure, and therefore is strictly positive on all non-empty open subsets of $\mathbb{R}^m$. So $C_0^\infty(\mathbb{R}^m)$ can be identified with a subset of $L^1(\mathbb{R}^m, \mu)$, since each corresponding μ-class has a unique continuous μ-version. Hence the operator $(L_{A,b}, C_0^\infty(\mathbb{R}^m))$ is well defined on $L^1(\mathbb{R}^m, \mu)$. The following theorem is a collection of results from [42] (See [154] for weaker assumptions).

Theorem 4.2.7. *Assume (A0), (A1), (A2), (A3) and (A4). Then $\mathcal{M}_{ell}^{A,b} = \{\mu\}$ is a singleton and the following statements hold true*

(i) there exists a closed extension of the operator $(L_{A,b}, C_0^\infty(\mathbb{R}^m))$ on $L^1(\mathbb{R}^m, \mu)$;

(ii) its closure $(\overline{L}_{A,b}^\mu, D(\overline{L}_{A,b}^\mu))$ on $L^1(\mathbb{R}^m, \mu)$ generates a C_0-semigroup $(T_t^\mu)_{t\geq 0}$ on $L^1(\mathbb{R}^m, \mu)$;

(iii) $(T_t^\mu)_{t\geq 0}$ is the only C_0-semigroup on $L^1(\mathbb{R}^m, \mu)$ which has a generator extending $(L_{A,b}, C_0^\infty(\mathbb{R}^m))$;

(iv) $(T_t^\mu)_{t\geq 0}$ is contractive, and μ is $(T_t^\mu)_{t\geq 0}$-invariant in the sense

$$\int_{\mathbb{R}^m} T_t^\mu f d\mu = \int_{\mathbb{R}^m} f d\mu, \quad \forall\, f \in L^\infty(\mathbb{R}^m, \mu). \tag{4.2.26}$$

Proof. The proof relies on the results in [42] which use mostly [154].

First, and from [42, Lemma 2.5], we have $(L_{A,b}, C_0^\infty(\mathbb{R}^m))$ is closable on $L^1(\mathbb{R}^m, \mu)$, which proves *(i)*.

Now, from Theorem 4.2.6, we have $\mathcal{M}_{\text{ell}}^{A,b} \neq \emptyset$, i.e. $\#\mathcal{M}_{\text{ell}}^{A,b} \geq 1$. Let $\mu \in \mathcal{M}_{\text{ell}}^{A,b}$. From Corollary 4.2.2, $\omega(x) = |x|^2$ is a a Lyapunov function, that is, [42, Proposition 2.9 (3)] holds and which insures, thanks to [42, Theorem 2.8 (1)], that μ is maximally dissipative. Now using the main result [42, Theorem 3.1], we have $\#\mathcal{M}_{\text{ell}}^{A,b} = 1$, i.e. $\mathcal{M}_{\text{ell}}^{A,b} = \{\mu\}$.

Moreover, since μ is maximally dissipative, [42, Proposition 2.6 (2)] is satisfied and is equivalent to [42, Proposition 2.6 (1)] that is, our statement *(ii)*, and is also equivalent to [42, Proposition 2.6 (3)] which corresponds to *(iii)*.

Finally, the last result in [42, Proposition 2.6] together with [42, Theorem 2.12] yield *(iv)*. □

Thanks to this result, we can now define on a larger space the operator $\mathcal{L}$ in the problem (4.3.1). This is an important step when dealing with unbounded right-hand side terms f in (4.3.1), since there cannot exist any solution in $C_0^\infty(\mathbb{R}^m)$, i.e. compactly supported.

In fact, the differential operator $(\mathcal{L}, D(\mathcal{L}))$ in (4.1.1) should be understood in the sense of the closed extension $(\overline{L}^{\mu}_{A,b}, D(\overline{L}^{\mu}_{A,b}))$ provided by Theorem 4.2.7, where $D(\overline{L}^{\mu}_{A,b})$ is the closure of $C_0^{\infty}(\mathbb{R}^m)$ in $L^1(\mathbb{R}^m, \mu)$. More precisely, we have $C_0^{\infty}(\mathbb{R}^m) \subset D(\overline{L}^{\mu}_{A,b}) \subset L^1(\mathbb{R}^m, \mu)$ with dense inclusions.

Remark 4.2.1. *If the dimension $m = 1$, then Theorem 4.2.7 fails, and we are not able to provide a closed extension to the diffusion operator (see [42, Appendix A] and a counter example in [154, Example 1.12]).*

In the following, we state from [40] a theorem which makes $D(\overline{L}^{\mu}_{A,b})$ more precise. In fact, for every $r \in [1, +\infty)$, the restriction of $\{T_t^{\mu}\}_{t\geq 0}$, whose generator is $\overline{L}^{\mu}_{A,b}$, to $L^r(\mu)$ is a strongly continuous semigroup on $L^r(\mu)$ (see [40, Lemma 5.1.4, p. 180]). Its generator will be denoted by $(L^{\mu,r}_{A,b}, D(L^{\mu,r}_{A,b}))$, where

$$D(L^{\mu,r}_{A,b}) = \{f \in D(L^{\mu}_{A,b} \cap L^r(\mu)) : \ L^{\mu}_{A,b} f \in L^r(\mu)\}.$$

Theorem 4.2.8. *([40, Theorem 5.2.7, p. 190]) In the situation of Theorem 4.2.7, one has for any $r \in [1, +\infty)$*

$$\begin{aligned} &(L^{\mu,r}_{A,b}, D(L^{\mu,r}_{A,b})) \subset \{f \in L^r(\mu) \cap W^{r,2}_{loc}(\mathbb{R}^m) \ : \ L_{A,b} f \in L^r(\mu)\} \\ &\quad \textit{and} \ \ L^{\mu,r}_{A,b} f = L_{A,b} f \quad \textit{for all } f \in D(L^{\mu,r}_{A,b}). \end{aligned} \tag{4.2.27}$$

Observe that for existence of solutions, we only need the first statement in Theorem 4.2.7 together with Theorem 4.2.8 in order to give a sense to the problem (4.1.1). Besides, the other results in Theorem 4.2.7 allow us to interpret the closed extension of the generator as a generator of a stochastic process and hence the invariant measure μ will be the corresponding one to the stochastic process.

Indeed, it is well-known (see for example [140, Theorem 5.2.1, p. 66]) that when b, ϱ are locally Lipschitz with a linear at most a linear growth, the stochastic differential equation

$$dX_t = b(X_t)dt + \sqrt{2}\varrho(X_t)dB_t, \quad t \in [0, T], \quad X_0 = Z \tag{4.2.28}$$

when Z is a random variable which is independent of the σ-algebra generated by $B_s(\cdot)$, $s \geq 0$ and such that $\mathbb{E}[|Z|^2] < \infty$, has a unique t-continuous solution $X_t(\omega)$ with the property that $X_t(\omega)$ is adapted to the filtration $\mathcal{F}_t^Z$ generated by Z and $B_s(\cdot)$; $s \leq t$, and $\mathbb{E}[\int_0^T |X_t|^2 dt] < \infty$. Now let $P(t, \cdot, \cdot)$ be the transition function of the homogeneous Markov process X_t, and let $f \in C_b(\mathbb{R}^m)$ the set of continuous and bounded functions.

Then the operator

$$(P_t f)(x) := \mathbb{E}[f(X_t)|X_0 = x] = \int_{\mathbb{R}^m} f(y)P(t,x,dy) \tag{4.2.29}$$

defines a C_0-semigroup on continuous bounded functions. To the latter, we can define its infinitesimal generator acting again on $C_b(\mathbb{R}^m)$ (see [129] for a general study of such semigroup). And using a convolution argument with standard mollifiers, each $f \in C_b(\mathbb{R}^m)$ can be approximated with a sequence of compactly supported functions $\{f_n\} \subset C_0^\infty(\mathbb{R}^m)$ which is bounded with respect to the sup-norm and such that f_n tends to f almost everywhere in $\mathbb{R}^m$. Therefore, the infinitesimal generator of the C_0-semigroup $(P_t)_{t\geq 0}$ defined above coincides with $(L_{A,b}, D(L_{A,b}))$ and hence the closed extension $(\overline{L}^\mu_{A,b}, D(\overline{L}^\mu_{A,b}))$ proved in Theorem 4.2.7 is also a closed extension for the generator of the C_0-semigroup $(P_t)_{t\geq 0}$ (thanks to (*ii*) and (*iii*) in Theorem 4.2.7) provided we assume that $|b|, |\varrho| \in L^1(\mathbb{R}^m, |\mu|)$ and $\|\mu\| < \infty$. And the latter assumptions turn out to be true thanks to Lemma 4.2.2 and the fact that μ is a probability measure. Hence μ coincides except on a set of measure zero with the invariant measure of the stochastic process (4.2.28) (we consider here (4.2.28) with $Z(\cdot) = x \in \mathbb{R}^m$ deterministic).

Here and in what follows we denote by μ both the invariant measure in Theorem 4.2.7 and the invariant probability measure of the SDE (4.2.28).

We conclude this section by recalling a result from [157].

Lemma 4.2.2. *Assuming (A1) and (A4), the invariant measure μ exists and has finite moments of any order $\ell \geq 1$, i.e. $\int_{\mathbb{R}^m} |x|^\ell \, d\mu(x) < +\infty$.*

***Proof*.** This is a particular case of the more general result in [157, Theorem 6] (see in particular [157, eq. (28) in §6]). Indeed, the main assumption in [157] is

$$\exists\, M_0 \geq 0,\ r \geq 0 \ \text{ s.t.} \quad \langle b(x), x\rangle \leq -r, \quad \forall\, |x| \geq M_0 \tag{4.2.30}$$

Then introduce the following constants

$$\lambda_- := \inf_{y\neq 0} \langle \varrho\varrho^*(x)\frac{x}{|x|}, \frac{y}{|x|}\rangle, \qquad \lambda_+ := \sup_{x\neq 0} \langle \varrho\varrho^\top(x)\frac{x}{|x|}, \frac{x}{|x|}\rangle$$

$$\tilde{\Lambda} := \sup_x \frac{\operatorname{trace}(\varrho\varrho^\top(x))}{m}, \qquad r_0 := [r - (m\tilde{\Lambda} - \lambda_-)/2]\lambda_+^{-1}$$

In this context, it is shown that the invariant measure has finite moments of order $\ell \in (2k+2, 2r_0 - 1)$, where again $k \in (0, r_0 - \frac{3}{2})$. In our case, assumption (A4) guarantees a constant r (in (4.2.30)) as large as we want. Then it is enough to use Hölder inequality together with the fact that $\mu(\mathbb{R}^m) = 1$ to prove finite moments of any order $\ell \geq 1$. $\square$

Remark 4.2.2. *Some of the assumptions can be weakened, for example*

- *Sobolev regularity of the coefficients in (A2) can be expressed in a local way, i.e.:* $\forall U_R$ *a ball of radius* $R > 0$ *in* $\mathbb{R}^m$, $\exists p_R > m$ *such that* $b \in L^{p_R}(U_R)$ *and* $a \in W^{p_R,1}(U_R)$. *See [41] and [40].*
- *The recurrence condition (A4) can be replaced by the following:* $\exists M \geq 0, r > 0$ *such that* $\langle b(x), x\rangle \leq -r$ *for all* $|x| \geq M$. *In this case, one cannot insure Lemma 4.2.2, but rather prove finite moments of some orders only as in the proof of the latter Lemma. See [157]. This can be indeed enough if the order of the polynomial growth of our data falls in the range of order for which the moments are finite. However, it may not be enough to get uniqueness.*
- *The growth condition (A5) can be replaced by an integrability condition with respect to the invariant measure, i.e.* $f \in L^1(\mathbb{R}^m, d\mu)$. *For example, one can still handle the case* $|f(x)| \leq K\, e^{\gamma x}$, *provided we assume a condition on the drift* $b(x)$ *stronger than assumption (A4), mainly we need* $b(x)$ *to be of the form* $-\widetilde{\gamma}\, x$ *for some* $\widetilde{\gamma} > \gamma \geq 0$. *Indeed, if* $b(x) = -\widetilde{\gamma}\, x$ *and* $a(x) = I$ *the identity matrix, then the stochastic process is an Ornstein-Uhlenbeck whose invariant (Gibbs) measure behaves as* $e^{-\widetilde{\gamma}x}$ *and allows to perform the subsequent computations.*

4.3 Ergodic Poisson equation

We address in this section the problem of existence and uniqueness of solutions to the so-called *ergodic Poisson equation* by analogy with the Poisson equation $\mathcal{L}\varphi = f$, also known as the linear *cell problem* when it arises in homogenization theory, that is

$$\begin{aligned} &\textit{Find } \mathcal{X} \text{ a functional space, and } (c, u(\cdot)) \textit{ in } \mathbb{R} \times \mathcal{X} \\ &\text{such that } c + \mathcal{L}(x, \nabla u(x), D^2 u(x)) = f(x) \quad \textit{in } \mathbb{R}^m \end{aligned} \tag{4.3.1}$$

where $\mathcal{L}\varphi(x) := \mathcal{L}(x, \nabla\varphi(x), D^2\varphi(x))$ is a linear differential operator given by

$$\mathcal{L}\varphi(x) = \mathrm{trace}(a(x)D^2\varphi(x)) + b(x) \cdot \nabla\varphi(x), \quad \varphi \in C_0^\infty(\mathbb{R}^m). \tag{4.3.2}$$

We write the functional space $\mathcal{X}$ as an unknown because it is yielded by the procedure that we will later follow. We denote by $C_0^\infty(\mathbb{R}^m)$ the set of all real-valued, infinitely differentiable functions on $\mathbb{R}^m$ with compact support. And we define as usual $W^{p,k}(\mathbb{R}^m)$, for $p \geq 1, k > 0$, the Sobolev space of all functions on $\mathbb{R}^m$ with generalized derivatives up to order k in $L^p(dx)$, where dx denotes Lebesgue measure on $\mathbb{R}^m$. $W^{p,k}_{\text{loc}}(\mathbb{R}^m)$ denotes the corresponding local Sobolev space, i.e. $f \in W^{p,k}_{\text{loc}}(\mathbb{R}^m)$ if $\zeta f \in W^{p,k}(\mathbb{R}^m)$ for all $\zeta \in C_0^\infty(\mathbb{R}^m)$.

For the sake of simplicity of notation, we denote again by $(\mathcal{L}, D(\mathcal{L}))$ the closed extension $(\overline{L}^\mu_{A,b}, D(\overline{L}^\mu_{A,b}))$ as given in Theorem 4.2.7 and Theorem 4.2.8. The functional space $\mathcal{X}$ is a subset of $D(\mathcal{L})$ that we will make precise.

4.3.1 Useful reformulation and main results I

Let us denote by $\mathcal{M}(\mathbb{R}^m)$ the space of totally finite signed Borel measures on $\mathbb{R}^m$ and equipped with the *Total Variation* norm[6] $\|\mu\| = \mu^+(\mathbb{R}^m) + \mu^-(\mathbb{R}^m)$, where $\mu = \mu^+ - \mu^-$ is the Jordan decomposition of μ. It is known that $(\mathcal{M}(\mathbb{R}^m), \|\cdot\|)$ is a Banach space (see e.g. [74, §IV.2.16]), and hence also a locally convex topological vector space when equipped with its norm topology.
We also denote by $\mathcal{M}_d(\mathbb{R}^m)$ (respectively, $\mathcal{M}_d^+(\mathbb{R}^m)$) the subset of signed (resp. non-negative) totally finite Borel measures with finite moments of order d, where we recall d is the growth order of f as in assumption (A5). And since the latter two subsets are closed, they are also Banach spaces.

Let us define the duality product in $\mathcal{M}_d(\mathbb{R}^m)$ by

$$\langle h(\cdot), \mu\rangle = \int_{\mathbb{R}^m} h(x) d\mu(x), \quad \text{for all } \ \mu \in \mathcal{M}_d(\mathbb{R}^m)$$

where h is a Borel measurable function with at most a polynomial growth of order d.

Recall that a linear functional h on the normed space $(\mathcal{M}(\mathbb{R}^m), \|\cdot\|)$ is continuous if and only if it is bounded on the unit ball, i.e. if

$$\|h\|_* := \sup_{\|\mu\|\leq 1} \langle h, \mu\rangle \ < \infty$$

And so, the topological dual space $(\mathcal{M}(\mathbb{R}^m))^*$ (i.e. set of continuous linear functionals,

[6] To check it is a norm, the only technical step is in the triangle inequality; to prove that $\|\mu + \nu\| \leq \|\mu\| + \|\nu\|$ we need to consider a Hahn decomposition $\mathbb{R}^m = A \uplus B$ for $\mu + \nu$.

equipped with the dual norm $\|\cdot\|_*$) is again a Banach space. It is easy to see that Borel-measurable functions with at most a polynomial growth of order d are in $(\mathcal{M}_d(\mathbb{R}^m))^*$.

It can be quite hard to deal with $(\mathcal{M}(\mathbb{R}^m))^*$ which can indeed be seen as the bidual of the space of continuous and bounded functions. But we will see that we can avoid these difficulties provided we find a subset of the latter, which will turn out to be more convenient to work with. We refer the interested reader to the work of S. Kaplan on the bidual of the space of continuous functions [106, 107].

Consider now the following (*primal*) optimization problem

$$\min_{q\in\mathcal{M}_d^+(\mathbb{R}^m)} \big\{\langle f(\cdot), q\rangle, \quad \text{s.t.:}\;\; 1-\langle 1, q\rangle = 0 \text{ and } \; q\in \operatorname{Ker}(\mathcal{L}^*)\big\} \tag{$\mathfrak{P}$}$$

The constraint $q \in \operatorname{Ker}(\mathcal{L}^*)$ is analogue to (4.2.22) and should be understood in the sense of (4.2.23) as in §4.2.3.2. Since q is non-negative, and if $\|q\| \neq 0$, we have $\frac{1}{\|q\|}q \in \mathcal{M}_{\text{ell}}^{A,b}$ as defined in (4.2.24). In fact, the feasible set satisfies

$$\{q\in\mathcal{M}_d^+(\mathbb{R}^m)\;:\;1-\langle 1,q\rangle = 0 \text{ and } \; q\in\operatorname{Ker}(\mathcal{L}^*)\}\subseteq\mathcal{M}_{\text{ell}}^{A,b}. \tag{4.3.3}$$

since $\mathcal{M}_{\text{ell}}^{A,b}$ does not have any restriction on the moments.

For simplicity of notation, we set

$$X = \mathcal{M}_d(\mathbb{R}^m) \quad\text{and}\quad Q = \mathcal{M}_d^+(\mathbb{R}^m)$$

$$G_1 : X \to \mathbb{R}, \quad \text{s.t.} \quad G_1(q) = 1 - \langle 1, q\rangle$$

$$G_2 : X \to X, \quad \text{s.t.} \quad G_2(q) = q$$

$$G = (G_1, G_2) \quad\text{and}\quad Y = \mathbb{R}\times X$$

$$K_1 = \{0\},\; K_2 = \operatorname{Ker}(\mathcal{L}^*) \quad\text{and}\quad K = K_1\times K_2 \subset Y$$

The (*primal*) problem ($\mathfrak{P}$) can be written in the more compact form

$$\min_{q\in Q}\big\{\langle f(\cdot), q\rangle, \quad \text{s.t.:}\;\; G(q)\in K\big\} \tag{$\mathfrak{P}$}$$

Let us denote by $\sigma(\cdot; K) : Y^* \to \mathbb{R}\cup\{+\infty\}$ the support function of the set K, as defined in (4.2.4). And Y^* is the (topological) dual of Y.

To the (*primal*) problem ($\mathfrak{P}$), we associate the (conjugate or *dual*) problem

$$\max_{y^* \in Y^*} \left\{ \inf_{q \in Q} L(q, y^*) - \sigma(y^*; K) \right\} \tag{$\mathfrak{D}$}$$

where the Lagrangian $L : X \times Y^* \to \mathbb{R}$ is defined by

$$L(q, y^*) = \langle f(\cdot), q \rangle + \langle y^*, G(q) \rangle_{Y^*, Y} \tag{4.3.4}$$

and $\langle \cdot, \cdot \rangle_{Y^*, Y}$ is the duality product in Y. We are now ready to state our main results.

Assume (A0)-(A6) hold. Then we have the following statements.

Lemma 4.1. *The problem* ($\mathfrak{D}$) *is equivalent to*

$$\max_{\substack{c \in \mathbb{R} \\ u \in \mathcal{X}}} \left\{ c, \quad s.t.: \; c + \mathcal{L}u(x) - f(x) \leq 0, \quad a.e. \; in \, \mathbb{R}^m \right\} \tag{4.3.5}$$

where, setting $\kappa = d - 1 + \beta$, *we have*

$$\mathcal{X} = D(\mathcal{L}) \cap \{ u : \mathbb{R}^m \to \mathbb{R}, \textit{Borel-measurable} \mid \exists\, C > 0, \; |u(x)| \leq C(1 + |x|^\kappa) \}, \tag{4.3.6}$$

that is, the two optimization problems have the same set of optimal solutions and the same optimal value.

Theorem 4.3.1. *The set of solutions of* (4.3.5) *with* $\mathcal{X}$ *as in* (4.3.6) *is nonempty, convex, bounded, and weakly-$*$ compact subset of* Y^*.

Theorem 4.3.2. *Let* μ *be the unique invariant probability measure associated to* $\mathcal{L}^*$. *Then for* $c = \langle f(\cdot), \mu \rangle$, *there exists* $u(\cdot) \in W^{r,2}_{loc}(\mathbb{R}^m)$ *for any* $r \in [1, +\infty)$, *satisfying* $|u(x)| \leq K(1 + |x|^\kappa)$ *where* $\kappa = d - 1 + \beta$, *and which solves*

$$c + \mathcal{L}(x, \nabla u(x), D^2 u(x)) = f(x) \quad a.e. \; in \; \mathbb{R}^m.$$

When $r > \frac{m}{2}$, $u(\cdot)$ *is continuous and pointwise twice differentiable almost everywhere. If moreover the vector field* b *is globally Lipchitz continuous and* $\beta = 1$ *in (A6), then* $u(\cdot)$ *with such a polynomial growth is unique in any* $W^{r,2}_{loc}(\mathbb{R}^m)$, $r > \frac{m}{2}$ *in the sense: if* $(c, u_1(\cdot))$ *and* $(c, u_2(\cdot))$ *are two solutions, then* $u_1(\cdot) - u_2(\cdot) \equiv$ *constant.*

Remark 4.3.1. *In fact, the PDE is solved on spt(μ) (the support of the unique invariant measure μ). But thanks to Theorem 4.2.5, we have* $\mu \ll dx$ *and spt(μ)* $= \mathbb{R}^m$.

4.3.2 Proof of the main results I

Our strategy follows three steps:

Step 1. Show that (4.3.5) can be represented as a a dual problem ($\mathfrak{D}$), i.e. the set of solutions of ($\mathfrak{D}$) coincides with the set of solutions of (4.3.5) and hence we have **Lemma 4.1**,

Step 2. Study the problem ($\mathfrak{D}$) as a dual to ($\mathfrak{P}$). Thus, the set of solutions of ($\mathfrak{D}$) is the set of Lagrange multipliers of ($\mathfrak{P}$), which is **Theorem 4.3.1**,

Step 3. Make use of the optimality conditions together with the zero duality gap to finally get the solvability of the PDE as in **Theorem 4.3.2**.

Proof *of Lemma 4.1.*
We need to show that the *dual* problem

$$\max_{y^*\in Y^*} \left\{ \inf_{q\in Q} L(q,y^*) - \sigma(y^*;K) \right\} \tag{$\mathfrak{D}$}$$

where the Lagrangian $L : X \times Y^* \to \mathbb{R}$ is defined by

$$L(q,y^*) = \langle f(\cdot), q\rangle + \langle y^*, G(q)\rangle_{Y^*,Y} \tag{4.3.7}$$

is equivalent to (4.3.5)

$$\max_{\substack{c\in\mathbb{R}\\ u\in\mathcal{X}}} \big\{ c, \quad \text{s.t.:} \;\; c + \mathcal{L}u(x) - f(x) \le 0, \quad \text{a.e. in } \mathbb{R}^m \big\}. \tag{4.3.8}$$

Recall

$$X = \mathcal{M}_d(\mathbb{R}^m) \quad \text{and} \quad Y = \mathbb{R}\times X$$

so $Y^* = \mathbb{R}\times X^*$ and we denote the elements $y^* \in Y^*$ by

$$y^* = (c,w) \in \mathbb{R}\times(\mathcal{M}_d(\mathbb{R}^m))^* = \mathbb{R}\times X^* = Y^*.$$

We also recall $K = K_1 \times K_2 = \{0\}\times \mathrm{Ker}(\mathcal{L}^*)$.

The set $(\mathcal{M}_d(\mathbb{R}^m))^*$ contains Borel-measurable functions with a polynomial growth of order at most d. We also recall $G(q) = (G_1(q),\ G_2(q)) = (1-\langle 1,q\rangle,\ q) \in \mathbb{R}\times X$ and q is chosen in the subset $Q = \mathcal{M}_d^+(\mathbb{R}^m) \subset X$ (in fact, one can minimize over the whole set X and consider instead the objective function $\langle f(\cdot), q\rangle + I_Q(q)$ as we discussed in §4.2.1).

Therefore, the problem $(\mathfrak{D})$ writes

$$\max_{\substack{c\in\mathbb{R}\\ w\in X^*}} \Big\{ \inf_{q\in Q}\langle f(\cdot),q\rangle + c(1-\langle 1,q\rangle) + \langle w(\cdot),q\rangle - \sigma((c,w);K)\Big\} \tag{$\mathfrak{D}$}$$

Since K is a cone, then $\sigma(y^*;K)=0$ if $y^*\in K^-$ and $+\infty$ if not. Hence the problem $(\mathfrak{D})$ becomes

$$\begin{aligned}
&\max_{\substack{c\in\mathbb{R}\\ w\in K_2^-}} \Big\{ \inf_{q\in Q}\langle f(\cdot),q\rangle + c(1-\langle 1,q\rangle) + \langle w(\cdot),q\rangle\Big\}\\
&\quad= \max_{\substack{c\in\mathbb{R}\\ w\in K_2^-}} \Big\{ c + \inf_{q\in Q}\langle f-c+w,q\rangle\Big\},\\
&\quad= \max_{\substack{c\in\mathbb{R}\\ w\in K_2^-}} \Big\{ c - \sup_{q\in Q}\langle -f+c-w,q\rangle\Big\}\\
&\quad= \max_{\substack{c\in\mathbb{R}\\ w\in K_2^-}} \Big\{ c - \sigma(-f+c-w;Q)\Big\}\\
&\quad= \max_{\substack{c\in\mathbb{R}\\ w\in K_2^-}} \Big\{ c\,, \quad \text{s.t.: } -f+c-w\in Q^- \Big\}, \quad (\text{Recall: } Q=\mathcal{M}_d^+(\mathbb{R}^m) \text{ is a cone})
\end{aligned} \tag{4.3.9}$$

We have set $K_2=\mathrm{Ker}(\mathcal{L}^*)$, so $K_2^-=(\mathrm{Ker}(\mathcal{L}^*))^\perp=\mathrm{cl}(\mathrm{range}(\mathcal{L}))$, that is, for any $w\in K_2^-$, there exists $-u\in D(\mathcal{L})$ such that $w=-\mathcal{L}u$.

And $Q^-=(\mathcal{M}_d^+(\mathbb{R}^m))^-$ is made of Borel-measurable functions ψ with a polynomial growth of order at most d and such that $\langle\psi(\cdot),q\rangle\le 0$ for any $q\in\mathcal{M}_d^+(\mathbb{R}^m)$, and hence $\psi\le 0$ a.e., that is, we have necessarily $-f+c+\mathcal{L}u\le 0$ and $-f+c+\mathcal{L}u$ with a polynomial growth of order at most d. Since f satisfies the latter condition (by assumption (A5)) and c is a constant, we need $\mathcal{L}u$ to satisfy this condition. By assumption (A3), the matrix function a is uniformly bounded, and by assumption (A6) the drift vector field has a polynomial growth of order β. Hence, setting κ as the polynomial growth of u, then it necessarily satisfies $\kappa-1+\beta\le d$ where $\kappa-1$ corresponds to the growth of ∇u. So a sufficient condition to have $\mathcal{L}u$ with a polynomial growth of order at most d is to have u satisfying a polynomial growth of order at most $\kappa=d+1-\beta$ (note that $\kappa\ge 1$ since $\beta\in[0,d]$). Therefore, $u\in D(\mathcal{L})$ and with at most a polynomial growth of order κ. Hence the dual problem $(\mathfrak{D})$ becomes

$$\max_{\substack{c\in\mathbb{R}\\ u\in\mathcal{X}}} \big\{ c\,, \quad \text{s.t.: } c+\mathcal{L}u-f\le 0 \quad a.e. \text{ in } \mathbb{R}^m\big\} \tag{$\mathfrak{D}$}$$

with $\mathcal{X}=D(\mathcal{L})\cap\{u:\mathbb{R}^m\to\mathbb{R}, \textit{Borel-measurable} \mid \exists\, C>0,\ |u(x)|\le C(1+|x|^\kappa)\}$. $\square$

Remark 4.3.2. *We can now see the problem* (4.3.8) *(equivalently* (4.3.5)*) as the dual of the primal problem* $(\mathfrak{P})$*, that is,*

$$\min_{q \in \mathcal{M}_d^+(\mathbb{R}^m)} \big\{ \langle f(\cdot), q \rangle, \quad \textit{s.t.:}\ \ 1 - \langle 1, q \rangle = 0 \ \textit{and} \ \ q \in \textit{Ker}(\mathcal{L}^*) \big\} \tag{$\mathfrak{P}$}$$

also represented by

$$\min_{q \in Q} \big\{ \langle f(\cdot), q \rangle, \quad \textit{s.t.:}\ \ G(q) \in K \big\}. \tag{$\mathfrak{P}$}$$

This will be used in the next proof.

Proof *of Theorem 4.3.1.*
The theorem is a consequence of Theorem 4.2.1. In fact, $(c, w = -\mathcal{L}u)$ are the Lagrange multipliers whose existence need to be proved. Note that we are in the *Particular case* for the constraints (see (4.2.12)) at the end of §4.2.1. And thanks to Theorem 4.2.7,

$$\mathrm{Ker}(\mathcal{L}^*) = \{\lambda\, \mu \ : \ \lambda \in \mathbb{R}\}$$

i.e. $\mathrm{Ker}(\mathcal{L}^*)$ is a one-dimensional linear space.

Before using Theorem 4.2.1, Let us check the assumptions in our setting.
The set $K = K_1 \times K_2$ is a nonempty, closed and convex cone. And both $g(q)$ and $G(q)$ are linear, and continuous, and G is $(-K)$-convex. So the problem $(\mathfrak{P})$ is convex. It remains to check the regularity condition (4.2.10) or equivalently the two conditions in Lemma 4.2.1 (see also (4.2.13)).

First, we have $G_1(q) = 1 - \langle 1, q \rangle$, $G_2(q) = q$, and $K_1 = \{0\}$ and $K_2 = \mathrm{Ker}(\mathcal{L}^*)$, hence also $\widetilde{K}_2 := K_2 \cap Q$, are closed and convex. Moreover $\mathrm{Ker}(\mathcal{L}^*) \cap Q = \{\lambda\, \mu \ : \ \lambda \geq 0\}$ where μ is as given in Theorem 4.2.7, and $DG_1(q)h = -\langle 1, h \rangle$ for all $h \in \mathcal{M}_d(\mathbb{R}^m)$. So for $q \in \mathcal{M}_d(\mathbb{R}^m)$ we have

$$\begin{aligned} &G_1(q) + DG_1(q)[\widetilde{K}_2 - q] - K_1 \\ &\qquad = \{1 - \langle 1, q \rangle - \langle 1, \lambda\, \mu - q \rangle \ : \ \lambda \geq 0\} \\ &\qquad = \{1 - \lambda \ : \ \lambda \geq 0\} = (-\infty\, , \, 1] \end{aligned}$$

where we have used $\langle 1, \mu \rangle = 1$. Therefore we have

$$0 \in \mathit{int}(G_1(q) + DG_1(q)[\widetilde{K}_2 - q] - K_1)$$

and the required condition (4.2.13) (equivalently (4.2.10)) is satisfied.

Finally, we need to check if the problem $(\mathfrak{P})$ has an optimal solution. In fact, the equality constraint $G_1(q) \in K_1$, which is $1 - \langle 1, q \rangle = 0$, together with $q \in Q = \mathcal{M}_d^+(\mathbb{R}^m)$, yields that the problem ($\mathfrak{P}$) can be written as

$$\begin{aligned} &\min_{q \in \mathcal{P}_d(\mathbb{R}^m)} \left\{ \langle f(\cdot), q\rangle, \quad \text{s.t.:} \quad q \in \operatorname{Ker}(\mathcal{L}^*) \right\} \\ = &\min_{q \in \mathcal{M}(\mathbb{R}^m)} \left\{ \langle f(\cdot), q\rangle, \quad \text{s.t.:} \quad q \in \operatorname{Ker}(\mathcal{L}^*) \cap \mathcal{P}_d(\mathbb{R}^m) \right\} \end{aligned} \tag{$\mathfrak{P}$}$$

where $\mathcal{P}_d(\mathbb{R}^m)$ is the set of probability measures with d-finite moments. Therefore, $\operatorname{Ker}(\mathcal{L}^*) \cap \mathcal{P}_d(\mathbb{R}^m) = \operatorname{Ker}(\mathcal{L}^*) \cap \mathcal{P}(\mathbb{R}^m) \cap \mathcal{M}_d(\mathbb{R}^m)$ and $\operatorname{Ker}(\mathcal{L}^*) \cap \mathcal{P}(\mathbb{R}^m) = \mathcal{M}_{\text{ell}}^{A,b} = \{\mu\}$. And by Lemma 2.3.1, μ has finite moments of any order, i.e $\mu \in \mathcal{M}_d(\mathbb{R}^m)$. And hence $\operatorname{Ker}(\mathcal{L}^*) \cap \mathcal{P}_d(\mathbb{R}^m) = \{\mu\}$ which means that the feasible set of the problem $\mathfrak{P}$ is a singleton, and yields

$$\min_{q \in \mathcal{M}_d^+(\mathbb{R}^m)} \left\{ \langle f(\cdot), q\rangle, \quad \text{s.t.:} \;\; 1 - \langle 1, q\rangle = 0 \text{ and } \; q \in \operatorname{Ker}(\mathcal{L}^*) \right\} = \langle f(\cdot), \mu\rangle \tag{$\mathfrak{P}$}$$

Finally, applying Theorem 4.2.1 insures that the set Λ_o of Lagrange multipliers (c, w) is a nonempty, convex, bounded, and weakly-$*$ compact subset of Y^*. The latter satisfy the conditions (4.2.8), hence using proposition 4.2.2, the set Λ_o is the set of solutions of ($\mathfrak{D}$). And we conclude the proof using Lemma 4.1. □

Proof *of Theorem 4.3.2.*
Step 1. (Existence)
Using the conclusion of Theorem 4.3.1 and for μ the optimal solution to ($\mathfrak{P}$), the condition (4.2.8) is satisfied. Therefore, Proposition 4.2.2 insures that (c, w) is a solution to the dual problem ($\mathfrak{D}$). Moreover, using the formulation (4.2.9), we have in particular

$$G(\mu) \in K, \quad (c, w) \in \mathbb{R} \times K_2^-, \quad \text{and } c(1 - \langle 1, \mu\rangle) + \langle w(\cdot), \mu\rangle = 0. \tag{4.3.10}$$

And for $w \in K_2^-$, we can choose $w = -\mathcal{L}u$ as stated in the proof of Theorem 4.3.1. And the equality in the right hand side writes $c - \langle c + \mathcal{L}u(\cdot), \mu\rangle = 0$, and since there is no duality gap (by Proposition 4.2.2), $c = \langle f(\cdot), \mu\rangle$. Hence, we have $\langle c + \mathcal{L}u(\cdot) - f(\cdot), \mu\rangle = 0$. On the other hand, μ is a non-negative measure, and (c, w) solves the dual problem and hence $c + \mathcal{L}u - f \leq 0$ almost everywhere, i.e. does not change sign almost everywhere. Therefore $\langle c + \mathcal{L}u(\cdot) - f(\cdot), \mu\rangle = 0$ implies that $c + \mathcal{L}u - f = 0$, for μ-almost every $x \in \operatorname{supp}(\mu)$. But μ is absolutely continuous with respect to Lebesgue measure and is supported on the the whole $\mathbb{R}^m$ thanks to Theorem 4.2.5, therefore (c, u) solves the PDE $c + \mathcal{L}u - f = 0$ almost everywhere in $\mathbb{R}^m$.

Step 2. (Regularity)

As we have seen at the end of §4.2.3.2, we can consider $\mathcal{L}$ as the generator of the strongly continuous semigroup $\{T^{\mu}\}_{t\geq 0}$ when restricted to the weighted Lebesgue space $L^r(\mu)$ for any $r \in [1, +\infty)$. In this case, and thanks to Theorem 4.2.8, $D(\mathcal{L})$ is a subset of $W^{r,2}_{\rm loc}(\mathbb{R}^m)$. Now if $2r > m$, then a classical embedding theorem (see, e.g., [1, Chapter 5]) states that $W^{r,2}(\Omega) \subset C(\Omega)$ for any Ω bounded subset of $\mathbb{R}^m$ satisfying the *cone property*[7]. Now by using smooth cut-off functions $\zeta \in C_0^{\infty}(\mathbb{R}^m)$ with a support U bounded subset of $\mathbb{R}^m$, we have $\zeta\, u \in W^{r,2}(U) \subset C(U)$. We conclude that for any $r > \frac{m}{2}$, the solution $u(\cdot) \in W^{r,2}_{\rm loc}(\mathbb{R}^m)$ is a continuous function and, from Step 1, it satisfies the polynomial growth, that is, there exists a constant $K > 0$ such that for any $x \in \mathbb{R}^m$, $|u(x)| \leq K(1 + |x|^{\kappa})$ where $\kappa = d - 1 + \beta$ as in Lemma 4.1.

Note also that the range $2r > m$ is the one where $W^{r,2}_{\rm loc}$ functions are not only continuous but also pointwise twice differentiable almost everywhere (see, e.g., [50, Appendix C]).

Step 3. (Uniqueness)

First, we need to check that the Lagrange multipliers $y^*_\circ = (c, \omega = -\mathcal{L}u)$ which existence is proved in Theorem 4.3.1 (see the proof of Theorem 4.3.1) is in fact unique. This is a direct consequence of Proposition 4.2.3. Indeed, the strict constraint qualification (4.2.14) is clearly satisfied noticing that (in the notation of Proposition 4.2.3) we have $x_\circ = \mu$, so $G(x_\circ) = (0, \mu)$, $DG(x_\circ)$ is a nonzero constant and independent of $x_\circ$, $Q = \mathcal{M}^+_d(\mathbb{R}^m)$, and $K = \{0\} \times \{\lambda\,\mu\,:\,\lambda \in \mathbb{R}\}$. Hence any element of K can be written as a pair $(0, \lambda\,\mu)$ for some $\lambda \in \mathbb{R}$. It is therefore immediate to see that such a pair $(0, \lambda\mu)$ is in $\operatorname{Ker} y^*_\circ$: indeed $\langle (c, -\mathcal{L}u)\,,\,(0, \lambda\mu)\rangle_{Y^*,Y} = -\lambda\,\langle \mathcal{L}u\,,\,\mu\rangle_{X^*,X} = 0$ since by definition we have $\mathcal{L}^*\mu = 0$, and hence $K = \operatorname{Ker} y^*_\circ$. Thus, we have $K_\circ = K$ where we recall from Proposition 4.2.3 (and the comment after the latter) that $K_\circ := K \cap \operatorname{Ker} y^*_\circ \subseteq K$.
Therefore $(c, \omega = -\mathcal{L}u)$ is unique. However, note that this in fact does not tell us anything on uniqueness of the ergodic constant, because the only one we are dealing with is $c = \langle f, \mu\rangle$ (which is unique by definition of c as the objective function in the optimization problem $(\mathfrak{D})$). The latter preliminary result of uniqueness shall be used instead to show uniqueness of $u(\cdot)$ as we will now do.

To prove that $u(\cdot)$ is unique, we need to assume in addition that b is locally Lipschitz continuous with at most a linear growth, i.e. $\beta = 1$ and hence $\kappa = d$. This setting will allow us to apply the Liouville type result in [22].
So we need to show that if $u_1, u_2 \in W^{r,2}_{\rm loc}$ with a polynomial growth of order at most d are

[7] Ω has the *cone property* if there exists a finite cone C (i.e. C is an intersection of a cone and an open ball) and such that each point $x \in \Omega$ is the vertex of a finite cone C_x contained in Ω and congruent to C (i.e. C_x is obtained from C by a rigid motion).

such that $\mathcal{L}u_1 = \mathcal{L}u_2 = \omega$, then $u_1(\cdot) - u_2(\cdot) \equiv$ *constant.* Note also that when $2r > m$, $W^{r,2}_{\mathrm{loc}}$ functions are continuous and pointwise twice differentiable almost everywhere (see, e.g., [50, Appendix C]). Therefore, $v := u_1 - u_2$ is a viscosity solution to $-\mathcal{L}v(x) = 0$ in $\mathbb{R}^m$.

We make the following

<u>*claim:*</u> *there exist a function $\psi \in C^\infty(\mathbb{R}^m)$ and $R_o > 0$ such that*

$$-\mathcal{L}\psi(x) \geq 0 \quad \text{in } \overline{B(0,R_o)}^C, \quad \psi(x) \to +\infty \text{ when } |x| \to +\infty \tag{4.3.11}$$

and such that

$$\lim_{|x|\to+\infty} \frac{v(x)}{\psi(x)} = 0 \tag{4.3.12}$$

Hence, using a Liouville type result [22, Theorem 2.1], we deduce that v is constant, i.e. $u_1(\cdot) - u_2(\cdot) \equiv$ constant.

Proof of the claim:

We check that $\psi(x) := |x|^d \log(|x|)$ satisfies (4.3.11) and (4.3.12).

Using the polynomial growth of u_1 and u_2, (4.3.12) is immediate.

To check the validity of (4.3.11), we compute $-\mathcal{L}\psi(x)$ and make use of assumption (A6). We have $\omega(x) \to +\infty$ when $|x| \to +\infty$ and

$$\begin{aligned}
\nabla\psi(x) &= |x|^{d-2}\big(d\log(|x|) + 1\big)x \\
D^2\psi(x) &= |x|^{d-2}\big[(d-2)\big(d\log(|x|)+1\big) + d\big]\frac{x \otimes x}{|x|^2} + |x|^{d-2}\big(d\log(|x|)+1\big)I_m
\end{aligned}$$

where I_m is the identity matrix of dimension m. Therefore, one has

$$-b(x) \cdot \nabla\psi(x) = -|x|^{d-2}\big(d\log(|x|)+1\big)\langle b(x), x\rangle$$

and

$$\begin{aligned}
-\mathrm{trace}(A(x)D^2\psi(x)) &= -|x|^{d-2}\big(d\log(|x|)+1\big)\mathrm{trace}(A(x)) \\
&\quad - |x|^{d-2}\big[(d-2)\big(d\log(|x|)+1\big)+d\big]\mathrm{trace}\left(A(x)\frac{x\otimes x}{|x|^2}\right) \\
&\geq -m\overline{\Lambda}|x|^{d-2}\big(d\log(|x|)+1\big) - \overline{\Lambda}|x|^{d-2}\big[(d-2)\big(d\log(|x|)+1\big)+d\big] \\
&\geq -m\overline{\Lambda}|x|^{d-2}\big(d\log(|x|)+1\big) - d\,\overline{\Lambda}|x|^{d-2}\big[\big(d\log(|x|)+1\big)+1\big]
\end{aligned}$$

Hence, one gets (using (A4))

$$\begin{aligned}
-\mathcal{L}\omega(x) &\geq -|x|^{d-2}\big(d\log(|x|)+1\big)\langle b(x),x\rangle \\
&\qquad - m\overline{\Lambda}|x|^{d-2}\big(d\log(|x|)+1\big) - d\,\overline{\Lambda}|x|^{d-2}\big(d\log(|x|)+1\big) - d\,\overline{\Lambda}|x|^{d-2} \\
&\geq -|x|^{d-2}\big(d\log(|x|)+1\big)\Big(\langle b(x),x\rangle + (m+d)\overline{\Lambda}\Big) - d\,\overline{\Lambda}|x|^{d-2} \\
&\geq -|x|^{d-2}\Big[\big(d\log(|x|)+1\big)\Big(\langle b(x),x\rangle + (m+d)\overline{\Lambda}\Big) + d\,\overline{\Lambda}\Big] \ \to +\infty \text{ as } |x| \to +\infty.
\end{aligned}$$

In particular, there exists $R_o > 0$ such that (4.3.11) is satisfied.

□

4.4 Ergodic Bellman equation

4.4.1 The primal problem

Armed with the result in the linear case, we are interested now in a class of nonlinear equations, usually called *ergodic (stationary) Hamilton-Jacobi-Bellman* (HJB) equations and which are of the type

$$\begin{aligned}
&\textit{Find } \mathcal{X} \ \textit{ a functional space, and } \ (c,u(\cdot)) \ \textit{in} \ \mathbb{R}\times\mathcal{X}(\mathbb{R}^m) \\
&\textit{such that} \quad H(x,\nabla u(x),D^2u(x)) = c \quad \textit{in} \ \mathbb{R}^m
\end{aligned} \tag{4.4.1}$$

where the Bellman Hamiltonian takes one of the forms

$$H := \min_{\alpha\in A}\{\,-\mathcal{L}_\alpha u(x) + f(x,\alpha)\,\} \quad \text{or} \quad H := \max_{\alpha\in A}\{\,-\mathcal{L}_\alpha u(x) + f(x,\alpha)\,\}$$

and for each $\alpha \in A$ compact subset of $\mathbb{R}^k$ with $k > 0$, the linear differential operator $\mathcal{L}_\alpha\varphi(x) := \mathcal{L}_\alpha(x,\nabla\varphi(x),D^2\varphi(x))$ is defined as in the previous sections by

$$\mathcal{L}_\alpha\varphi(x) = \operatorname{trace}(a(x,\alpha)D^2\varphi(x)) + b(x,\alpha)\cdot\nabla\varphi(x), \quad \varphi\in C_0^\infty(\mathbb{R}^m). \tag{4.4.2}$$

As for the linear case, we write in the problem (4.4.1) the functional space $\mathcal{X}$ as an unknown because it is yielded by the procedure that we will follow, in the light of the previous sections §4.3.

Note that unlike the differential operator (4.3.2), the coefficients of $\mathcal{L}_\alpha$ as defined in (4.4.2) depend on a parameter α. This is also the case with the right-hand side function f in (4.3.1) which now in (4.4.1) also depends on α. Such an equation arises for example in the theory of stochastic ergodic control where α stands for the control parameter, $a^{i,j}$ and b^i describe the diffusion and the drift respectively of the controlled dynamics, f is the running cost which depends both on the state x and on the control α and finally c is the ergodic constant that captures the long-time average of the value function. This equation can also be encountered in stochastic control problems with singular perturbations or again in the theory of homogenization.

In what follows, we will first deal with the case where the Hamiltonian is given by

$$H(x, \nabla u(x), D^2u(x)) = \min_{\alpha\in A}\{ -\mathcal{L}_\alpha u(x) + f(x,\alpha) \}.$$

Using the notation $a \wedge b = \min(a,b)$, $a \vee b = \max(a,b)$, we will refer to the *primal* problem by $(\mathfrak{P}^\wedge)$ and its *dual* by $(\mathfrak{D}^\wedge)$. We will then recover the case where we have in the Hamiltonian a max instead of a min, and use the notation $(\mathfrak{P}^\vee)$ and $(\mathfrak{D}^\vee)$.

Before we go any further, let us state and prove a result that we will need in the sequel. It is an *exchange property* whose proof is similar to the one of Proposition 2.5.1 in §2.5.4.

Proposition 4.4.1. *Let f satisfies (A5). The following holds for any $q \in \mathcal{M}_d^+(\mathbb{R}^m)$*

$$\int_{\mathbb{R}^m} \min_{\alpha\in A} f(x,\alpha)\, dq(x) = \min_{\boldsymbol{\alpha}(\cdot)\in\mathcal{A}} \int_{\mathbb{R}^m} f(x, \boldsymbol{\alpha}(x))\, dq(x) \tag{4.4.3}$$

where A is a compact subset of $\mathbb{R}^k$, for some $k > 0$, and $\mathcal{A}$ is the set of measurable functions $\boldsymbol{\alpha}(\cdot) : \mathbb{R}^m \to A$. And the same holds true with max *instead of* min*.*

Remark 4.4.1. *In the context of stochastic control, the set $\mathcal{A}$ needs to be the one of progressively measurable functions. In fact, these are the admissible controls.*

Proof. We repeat mutatis mutandis the arguments in the proof of Proposition 2.5.1. Let $q \in \mathcal{M}_d^+(\mathbb{R}^m)$ be arbitrarily fixed and $f : \mathbb{R}^m \times A \to \mathbb{R}$ satisfies (A5).
Step 1. (the inequality "$\leq$")
For any $\varepsilon > 0$, there exists $\boldsymbol{\alpha}^\varepsilon(\cdot) \in \mathcal{A}$ such that

$$\begin{aligned}\min_{\boldsymbol{\alpha}(\cdot)\in\mathcal{A}} \int_{\mathbb{R}^m} f(x,\boldsymbol{\alpha}(x))\,\mathrm{d}q(x) + \varepsilon \ &\geq \int_{\mathbb{R}^m} f(x, \boldsymbol{\alpha}^\varepsilon(x))\,\mathrm{d}q(x)\\ &\geq \int_{\mathbb{R}^m} \min_{\alpha\in A} f(x,\alpha)\,\mathrm{d}q(x)\end{aligned}$$

which proves the desired inequality.
Step 2. (the inequality "$\geq$")

Let $(I_i)_{i\in\mathbb{Z}}$ a sequence of open intervals in $\mathbb{R}^m$ such that $I_i \cap I_j = \emptyset$ whenever $i \neq j$ and $\mathbb{R}^m = \cup_{i\in\mathbb{Z}} \overline{I_i}$, where $\overline{I_i}$ is the closure of I_i. We define $D_n = \cup_{i=-n}^{n} \overline{I_i}$ and let $x \mapsto F_n(x;\alpha)$ be a sequence of functions defined as

$$F_n(x;\alpha) = \mathbb{1}_{D_n}(x) f(x,\alpha(x)), \quad \forall\, x \in \mathbb{R}^m, \alpha \in \mathcal{A}, n \in \mathbb{N}.$$

where $\mathbb{1}_D(\cdot)$ is the indicator function of a set D which is 1 if $x \in D$ and 0 otherwise. It is clear that for any arbitrarily fixed $\alpha \in \mathcal{A}$, the sequence $\{F_n(\cdot;\alpha)\}_{n\in\mathbb{N}}$ is uniformly integrable over $\mathbb{R}^m$, that is, $\forall\,\varepsilon > 0$, $\exists\,\delta > 0$ s.t.

$$\text{if } D \subset \mathbb{R}^m \text{ is s.t. } q(D) = \int_{\mathbb{R}^m} \mathbb{1}_D(x)\,\mathrm{d}q(x) < \delta, \text{ then } \int_D |f(x,\alpha(x))|\,\mathrm{d}q(x) < \varepsilon,\ \forall\, n \in \mathbb{N}.$$

This is true since $|F_n(x;\alpha)| \leq |f(x,\alpha(x))| \leq C(1+|x|^d)$ uniformly in $\alpha \in \mathcal{A}$ from assumption (A5), and $q \in \mathcal{M}_d^+$ has finite moments of order less or equal d. Therefore since we have $\lim_{n\to+\infty} D_n = \mathbb{R}^m$, then $F_n(\cdot\,;\alpha(\cdot)) \xrightarrow[n\to+\infty]{} f(\cdot\,,\alpha(\cdot))$ for any $\alpha \in \mathcal{A}$, and then Vitali's convergence theorem insures that

$$\sum_{i=-n}^{n} \int_{I_i} f(x,\alpha(x))\,\mathrm{d}q(x) = \int_{\mathbb{R}^m} F_n(x;\alpha)\,\mathrm{d}q(x) \xrightarrow[n\to+\infty]{} \int_{\mathbb{R}^m} f(x,\alpha(x))\,\mathrm{d}q(x).$$

Armed with this result, we can now consider the *truncated* minimization problem $\mathfrak{f}_i(x) := \min_{\alpha\in A} f(x,\alpha)$ where $x \in I_i$. Since A is compact and $\mathfrak{f}_i(x) \in f(\{x\}\times A)$ with $\mathfrak{f}_i$ measurable and $f(x,\alpha)$ is measurable in x and continuous in α, then a classical selection theorem (see [94, Theorem 7.1, p. 66]) implies the existence of a measurable selector $\overline{\alpha}_i$ for which the minimization is achieved, i.e.

$$\forall\, i \in \mathbb{Z},\ \exists\, \overline{\alpha}_i \in \mathcal{A},\ \text{s.t. } \forall\, x \in I_i,\ \mathfrak{f}_i(x) = \min_{\alpha\in A} f(x,\alpha) = f(x,\overline{\alpha}(x)).$$

Now consider $\overline{\alpha} \in \mathcal{A}$ defined as $x \mapsto \overline{\alpha}(x) = \{\overline{\alpha}_i(x),\ \text{if } x \in I_i,\ \forall\, i \in \mathbb{Z}\}$. Therefore one has

$$\begin{aligned}
\int_{\mathbb{R}^m} \min_{\alpha\in A} f(x,\alpha)\,\mathrm{d}q(x) &= \sum_{i\in\mathbb{Z}} \int_{I_i} \min_{\alpha\in A} f(x,\alpha)\,\mathrm{d}q(x) \\
&= \sum_{i\in\mathbb{Z}} \int_{I_i} f(x,\overline{\alpha}_i(x))\,\mathrm{d}q(x) \\
&= \int_{\mathbb{R}^m} f(x,\overline{\alpha}(x))\,\mathrm{d}q(x) \\
&\geq \min_{\alpha(\cdot)\in\mathcal{A}} \int_{\mathbb{R}^m} f(x,\alpha(\cdot))\,\mathrm{d}q(x).
\end{aligned}$$

This yields the second desired inequality and concludes the proof.

Note finally that the same holds true if write $-f$ instead of f. Then one gets (4.5.3) with max instead of min. $\square$

The *exchange property* proved in Proposition 4.4.1 will be much needed in the sequel. It insures that we can exchange the minimization over the parameters α and the duality product in $\mathcal{M}_d(\mathbb{R}^m)$ provided we define the second argument in f as measurable functions $\alpha(\cdot) \in \mathcal{A}$ instead of vectors $\alpha \in A$, that is,

$$\min_{\alpha(\cdot)\in\mathcal{A}} \langle\, f(\cdot\,,\alpha(\cdot))\;,\; q\,\rangle = \langle\, \min_{\alpha\in A} f(\cdot\,,\alpha)\;,\; q\,\rangle$$

In the next section we will conduct the duality procedure as in §4.3 (in particular as in §4.3.1). We state our *primal* problem as follows

$$\min_{q\in\mathcal{M}_d^+(\mathbb{R}^m)} \left\{ \min_{\alpha(\cdot)\in\mathcal{A}} \quad \langle f(\cdot\,,\alpha(\cdot)),q\rangle, \quad \text{s.t. } 1-\langle 1,q\rangle = 0 \text{ and } q\in \mathrm{Ker}(\mathcal{L}_\alpha^*)\right\} \tag{$\mathfrak{P}^\wedge$}$$

where we recall $\langle f(\cdot\,,\alpha(\cdot)),q\rangle = \int_{\mathbb{R}^m} f(x,\alpha(x))\mathrm{d}q(x)$, and we will use the same notation as in §4.3.1 that we recall here for the reader's convenience and taking into account the dependency on the parameter α

$$X = \mathcal{M}_d(\mathbb{R}^m) \quad \text{and} \quad Q = \mathcal{M}_d^+(\mathbb{R}^m)$$

$$G_1 : X \to \mathbb{R}, \quad \text{s.t.} \quad G_1(q) = 1 - \langle 1,q\rangle$$

$$G_2 : X \to X, \quad \text{s.t.} \quad G_2(q) = q$$

$$G = (G_1, G_2) \quad \text{and} \quad Y = \mathbb{R}\times X$$

$$K_1 = \{0\},\; K_2(\alpha) = \mathrm{Ker}(\mathcal{L}_\alpha^*) \quad \text{and} \quad K_\alpha = K_1 \times K_2(\alpha) \subset Y$$

The *primal* problem then writes

$$\min_{q\in Q} \left\{ \min_{\alpha(\cdot)\in\mathcal{A}} \quad \langle f(\cdot\,,\alpha(\cdot)),q\rangle, \quad \text{s.t. } G(q)\in K_\alpha\right\} \tag{$\mathfrak{P}^\wedge$}$$

Setting $F(G(q),\alpha) := I_{K_\alpha}(G(q))$ the indicator function which is 0 if $G(q)\in K_\alpha$ and $+\infty$ otherwise, we can finally write the *primal* problem as

$$\min_{q\in Q} \left\{ \min_{\alpha(\cdot)\in\mathcal{A}} \quad \langle f(\cdot\,,\alpha(\cdot)),q\rangle + F(G(q),\alpha)\right\}. \tag{$\mathfrak{P}^\wedge$}$$

We will also need an assumption that will play a crucial role in the validity of our method for solving the problem (4.4.1). Besides the standing assumptions (A0) and (A1-A6) that we assume to hold uniformly in the parameter function α, we denote again by $(\mathcal{L}_\alpha, D(\mathcal{L}_\alpha))$ its closed extension as given by Theorem 4.2.7 and Theorem 4.2.8 and we assume the following holds true

(A*) The domain $D(\mathcal{L}_\alpha)$ of the closed extension is nonempty and is independent of α.

Such an assumption can be encountered in [85, §III.6, p. 130]. It means that there exists $\widetilde{\alpha}(\cdot) \in \mathcal{A}$ such that for all $\alpha(\cdot) \in \mathcal{A}$, one has $D(\mathcal{L}_\alpha) = D(\mathcal{L}_{\widetilde{\alpha}})$, and $\mathcal{L}_{\widetilde{\alpha}}$ falls in the framework of the previous sections, in particular it satisfies Theorem 4.2.8. The nonemptiness assumption is trivial otherwise the PDE problem (4.4.1) does not make sense. We will hereafter denote by $D(\mathcal{L}_0)$ the latter domain.

The next result shows that the *primal* problem enjoys *calmness* (see Definition 4.2.1).

Lemma 4.2. *The problem* $(\mathfrak{P}^\wedge)$ *is calm and admits an optimal solution* $(q_\circ, \alpha_\circ)$.

Proof. We need to check that the value function is subdifferentiable in 0 and that an optimal solution $(q_\circ, \alpha_\circ)$ exists (this shows in particular that $v(0) < +\infty$).
Step 1. ($\partial v(0) \neq \emptyset$)
Using the above notations, let $y \in Y$ such that $y := (\lambda, z)$ where $\lambda \in \mathbb{R}$ and $z \in X$. We define the value function $v(y)$ as in §4.2.1, and we have

$$\begin{aligned} v(y) &= \inf_{q\in Q} \min_{\alpha(\cdot)\in\mathcal{A}} \langle f(\cdot\,, \alpha(\cdot)), q\rangle, \quad \text{s.t. } G(q) + y \in K_\alpha \\ &= \inf_{q\in Q} \min_{\alpha(\cdot)\in\mathcal{A}} \langle f(\cdot\,, \alpha(\cdot)), q\rangle + I_{K_\alpha}(G(q) + y) \\ &\geq \inf_{q\in Q} \min_{\alpha(\cdot)\in\mathcal{A}} \{\, \langle f(\cdot\,, \alpha(\cdot)), q\rangle + I_{K_\alpha}(G(q))\,\} + \inf_{q\in Q} \min_{\alpha(\cdot)\in\mathcal{A}} \{I_{K_\alpha}(G(q) + y) - I_{K_\alpha}(G(q))\} \end{aligned}$$

where in the last inequality we used "$\min(A + B) \geq \min A + \min B$". Note that the first term in the right hand-side is $v(0)$ and hence, one gets, for any $y \in Y$

$$v(y) - v(0) \geq \inf_{q\in Q} \min_{\alpha(\cdot)\in\mathcal{A}} \{I_{K_\alpha}(G(q) + y) - I_{K_\alpha}(G(q))\} \tag{4.4.4}$$

Recalling the definition of the subdifferential (see §4.2.1), one has

$$I_{K_\alpha}(G(q) + y) - I_{K_\alpha}(G(q)) \geq \langle y^*, y\rangle_{Y^*,Y} \quad \text{for all} \quad y^* \in \partial I_{K_\alpha}(G(q)). \tag{4.4.5}$$

It suffices then to have $\partial I_{K_\alpha}(G(q))$ nonempty for any $\alpha(\cdot) \in \mathcal{A}$, in order to show that $\partial v(0)$ is nonempty. Hence, letting $\alpha(\cdot) \in \mathcal{A}$ be arbitrarily fixed, we first need to have

$G(q) \in K_\alpha$, and noting that (recalling $y := (\lambda, z) \in \mathbb{R} \times X$)

$$I_{K_\alpha}(G(q)+y) - I_{K_\alpha}(G(q)) = I_{\{0\}}(G_1(q)+\lambda) - I_{\{0\}}(G_1(q)) \\ + I_{\mathrm{Ker}(\mathcal{L}_\alpha^*)}(G_2(q)+z) - I_{\mathrm{Ker}(\mathcal{L}_\alpha^*)}(G_2(q)),$$

it suffices that the polar (negative dual) cones $\{0\}^-$ and $\big(\mathrm{Ker}(\mathcal{L}_\alpha^*)\big)^-$ are nonempty, since $\{0\}$ and $\mathrm{Ker}(\mathcal{L}_\alpha^*)$ are nonempty, closed and convex cones (the same argument is used when deriving the equivalent optimality conditions (4.2.7), (4.2.8) and (4.2.9) in §4.2.1). This holds true, since $\{0\}^- = \mathbb{R}$ and $\big(\mathrm{Ker}(\mathcal{L}_\alpha^*)\big)^- = \mathrm{cl}(\mathrm{range}(\mathcal{L}_\alpha))$ are nonempty. Indeed, for $z^* \in X^*$ to be in $\mathrm{cl}(\mathrm{range}(\mathcal{L}_\alpha))$ it suffices that there exists $u \in D(\mathcal{L}_\alpha)$ such that $z^* = \mathcal{L}_\alpha u$. But $D(\mathcal{L}_\alpha) = D(\mathcal{L}_0)$ is nonempty (thanks to (A*)), and hence there exists $z^* = \mathcal{L}_\alpha u$ for $u \in D(\mathcal{L}_0)$. So there exists $y^* = (\lambda^*, z^*) \in \{0\}^- \times \big(\mathrm{Ker}(\mathcal{L}_\alpha^*)\big)^- \subset Y^*$ and y^* depends on $\alpha(\cdot)$ (in fact only z^* depends on $\alpha(\cdot)$), satisfying (4.4.5).

To sum up, for any $\alpha(\cdot) \in \mathcal{A}$, there exists $q \in Q$ satisfying $G(q) \in K_\alpha$ (indeed $\{q \in Q \,:\, G(q) \in K_\alpha\} = \{\mu_\alpha\}$ a singleton, as shown in §4.2.3.2), and moreover there exists $y^* \in \{0\}^- \times \big(\mathrm{Ker}(\mathcal{L}_\alpha^*)\big)^-$ satisfying (4.4.5). The set $\mathcal{A}$ being closed and recalling (4.4.4), we conclude that there exists $y_\circ^* \in Y^*$ such that $v(y) - v(0) \geq \langle y_\circ^*, y\rangle$, i.e. $\partial v(0) \neq \emptyset$.

Step 2. (There exists an optimal solution)

Recall that the feasible set of our *primal* problem ($\mathfrak{P}^\wedge$) is $\{q \in Q \,:\, G(q) \in K_\alpha\} = \{\mu_\alpha\}$ a singleton, where $\mu_\alpha \in \mathcal{P}_d(\mathbb{R}^m)$. Hence, ($\mathfrak{P}^\wedge$) equivalently writes as

$$\min_{\alpha(\cdot) \in \mathcal{A}} \langle f(\cdot\,, \alpha(\cdot)), \mu_\alpha\rangle.$$

We proceed using a fixed-point approach: we first fix $\alpha_1(\cdot) \in \mathcal{A}$, hence also μ_{α_1}, and then show that $\alpha_2(\cdot) \in \operatorname*{argmin}_{\alpha(\cdot)\in\mathcal{A}} \langle f(\cdot\,, \alpha(\cdot)), \mu_{\alpha_1}\rangle$ exists. The next step is then to consider the corresponding unique invariant probability measure μ_{α_2} in the objective function, and repeat the process. We get a fixed point when $\alpha_\circ(\cdot) \in \operatorname*{argmin}_{\alpha(\cdot)\in\mathcal{A}} \langle f(\cdot\,, \alpha(\cdot)), \mu_{\alpha_\circ}\rangle$.

Let $\alpha_1(\cdot) \in \mathcal{A}$ be arbitrarily fixed, and let μ_{α_1} be the corresponding unique invariant probability measure. Using (4.5.3) from Proposition 4.4.1, one has

$$\min_{\alpha(\cdot)\in\mathcal{A}} \langle f(\cdot\,, \alpha(\cdot)), \mu_{\alpha_1}\rangle \;=\; \langle \min_{\alpha\in A} f(\cdot\,, \alpha)\,, \mu_{\alpha_1}\,\rangle = \int_{\mathbb{R}^m} \min_{\alpha\in A} f(x,\alpha)\,\mathrm{d}\mu_{\alpha_1}(x).$$

The minimization problem is then reduced to a finite dimensional optimization problem that is, to minimize $f(x,\alpha)$ over $\alpha \in A \subset \mathbb{R}^k$, for each $x \in \mathbb{R}^m$. the function $\alpha \mapsto f(x,\alpha)$ being continuous over a compact set A, a minimizer α_x to the latter finite dimensional optimization problem exists. We then define $\alpha_\circ : \mathbb{R}^m \ni x \mapsto \alpha_x \in A$ a measurable function, and we have $\alpha_\circ(\cdot) \in \underset{\alpha(\cdot)\in\mathcal{A}}{\operatorname{argmin}} \langle f(\cdot\,,\alpha(\cdot)), \mu_{\alpha_1}\rangle$. But $\alpha_\circ(\cdot)$ is independent of μ_{α_1} since it is obtained from the minimization of $f(x,\alpha)$ for $\alpha \in A$. Hence, one gets the desired fixed point by considering $\mu_{\alpha_\circ}$ which is the corresponding unique invariant probability measure, and $(\mu_{\alpha_\circ}, \alpha_\circ)$ is an optimal solution for $(\mathfrak{P}^\wedge)$. $\square$

4.4.2 The dual problem

In order to deduce the corresponding *dual* problem, we follow a parametric (conjugate) duality scheme as in [46, §2.5.3, p. 107]. Therefore we embed the problem $(\mathfrak{P}^\wedge)$ in a family of parameterized problems, where $y \in Y$ is the parameter vector and consider the function

$$\phi(q,y) = \min_{\alpha(\cdot)\in\mathcal{A}} \big\{ \langle f(\cdot\,,\alpha(\cdot)), q\rangle + F(G(q)+y,\alpha) \big\}.$$

It is clear that when setting $y=0$, we recover the objective function in $(\mathfrak{P}^\wedge)$.

Lemma 4.3. *ϕ is lower semi-continuous.*

Proof. We have $q \mapsto \langle f(\cdot\,,\alpha), q\rangle$ and $y \mapsto F(y,\alpha)$ are lower semi-continuous (l.s.c), and $q \mapsto G(q)$ is continuous. And $y \mapsto F(y,\alpha)$ is l.s.c. if and only if K_α is closed, and this holds in our setting. $\square$

We also consider the following (Lagrangian) function, $L : X \times Y^* \times \mathcal{A} \to \mathbb{R}$, analogue to (4.3.4) and s.t.

$$L(q,y^*,\alpha) \coloneqq \langle f(\cdot\,,\alpha(\cdot)), q\rangle + \langle y^*, G(q)\rangle_{Y^*,Y}. \tag{4.4.6}$$

Using the Legendre-Fenchel transform, we have

$$\begin{aligned}
\phi^*(q^*,y^*) &= \sup_{q\in Q, y\in Y} \{ \langle q^*, q\rangle + \langle y^*, y\rangle - \phi(q,y) \} \\
&= \sup_{q\in Q, y\in Y} \left\{ \langle q^*, q\rangle + \langle y^*, y\rangle - \min_{\alpha(\cdot)\in\mathcal{A}} \{ \langle f(\cdot\,,\alpha(\cdot)), q\rangle + F(G(q)+y,\alpha)\} \right\} \\
&= \sup_{q\in Q, y\in Y} \left\{ \max_{\alpha(\cdot)\in\mathcal{A}} \{\langle q^*, q\rangle + \langle y^*, y\rangle - \big(\langle f(\cdot\,,\alpha(\cdot)), q\rangle + F(G(q)+y,\alpha)\big) \} \right\} \\
&= \max_{\alpha(\cdot)\in\mathcal{A}} \left\{ \sup_{q\in Q, y\in Y} \{ \langle q^*, q\rangle + \langle y^*, y\rangle - \big(\langle f(\cdot\,,\alpha(\cdot)), q\rangle + F(G(q)+y,\alpha)\big) \} \right\} \\
&= \max_{\alpha(\cdot)\in\mathcal{A}} \Bigg\{ \sup_{q\in Q} \{ \langle q^*, q\rangle - \langle f(\cdot\,,\alpha(\cdot)), q\rangle - \langle y^*, G(q)\rangle_{Y^*,Y} \} + \\
&\qquad\qquad + \sup_{y\in Y}\{ \langle y^*, G(q)+y\rangle - F(G(q)+y,\alpha) \} \Bigg\}
\end{aligned}$$

$$= \max_{\alpha(\cdot)\in\mathcal{A}} \left\{ \sup_{q\in Q} \{ \langle q^*, q\rangle - L(q,y^*,\alpha) + F^*(y^*,\alpha) \} \right\}$$
$$= \sup_{q\in Q} \left\{ \langle q^*, q\rangle + \max_{\alpha(\cdot)\in\mathcal{A}} \{-L(q,y^*,\alpha) + F^*(y^*,\alpha)\} \right\}$$
$$= \sup_{q\in Q} \left\{ \langle q^*, q\rangle - \min_{\alpha(\cdot)\in\mathcal{A}} \{L(q,y^*,\alpha) - F^*(y^*,\alpha)\} \right\}$$

The *dual* of the parameterized *primal* problem is then obtained as

$$\max_{y^*\in Y^*} \{ \langle y^*, y\rangle - \phi^*(0,y^*) \}$$

which writes

$$\max_{y^*\in Y^*} \left\{ \langle y^*, y\rangle + \inf_{q\in Q} \min_{\alpha(\cdot)\in\mathcal{A}} \{L(q,y^*,\alpha) - F^*(y^*,\alpha)\} \right\}$$

Finally, the *dual* problem associated to $(\mathfrak{P}^\wedge)$ is obtained by setting $y=0$, and writes as

$$\max_{y^*\in Y^*} \left\{ \inf_{q\in Q} \min_{\alpha(\cdot)\in\mathcal{A}} \{L(q,y^*,\alpha) - F^*(y^*,\alpha)\} \right\}. \tag{$\mathfrak{D}^\wedge$}$$

We will now make $(\mathfrak{D}^\wedge)$ more explicit.

Lemma 4.4. *The problem* $(\mathfrak{D}^\wedge)$ *is equivalent to*

$$\max_{\substack{c\in\mathbb{R}\\ u\in\mathcal{X}}} \left\{ c, \ s.t.: \ c - H(x,\nabla u, D^2u) \le 0, \ a.e. \ in \ \mathbb{R}^m \right\} \tag{$\mathfrak{D}^\wedge$}$$

where $H(x,\nabla u(x), D^2u(x)) = \min_{\alpha\in A}\{ -\mathcal{L}_\alpha u(x) + f(x,\alpha) \}$ *and* $\mathcal{X}$ *is such that*

$$\mathcal{X} = D(\mathcal{L}_0) \cap \{u : \mathbb{R}^m \to \mathbb{R}, \textit{Borel-meas.} \mid \exists\, C>0, \ |u(x)| \le C(1+|x|^\kappa)\} \tag{4.4.7}$$

with $\kappa = d+1-\beta$*, that is, the two optimization problems have the same set of optimal solutions and the same optimal value.*

Remark 4.4.2. *Assumption (A*) together with Theorem 4.2.8 insure that* $D(\mathcal{L}_0) \subset W^{r,2}_{loc}(\mathbb{R}^m)$*.*

Proof. Recalling that F is an indicator function, its conjugate is the support function as defined in (4.2.4), that is,

$$F^*(y^*,\alpha) = I^*_{K_\alpha}(y^*) = \sigma(y^*;K_\alpha)$$
$$= \begin{cases} 0, & \text{if } y^* \in (K_\alpha)^- \\ +\infty, & \text{otherwise} \end{cases}$$

And recalling the definition $K_\alpha = \{0\} \times \mathrm{Ker}(\mathcal{L}_\alpha)$, we have

$$\begin{aligned} y^* \in (K_\alpha)^- &\Leftrightarrow (c,\omega) \in \Big(\{0\} \times \mathrm{Ker}(\mathcal{L}_\alpha)\Big)^- \\ &\Leftrightarrow (c,\omega) \in \mathbb{R} \times (\mathrm{Ker}(\mathcal{L}_\alpha))^\perp \\ &\Leftrightarrow (c,\omega) \in \mathbb{R} \times \mathrm{cl}(\mathrm{range}(\mathcal{L}_\alpha)) \end{aligned}$$

Since we are working with $\mathcal{L}_\alpha$ in its closed extension, we have

$$\begin{aligned} \omega \in \mathrm{cl}(\mathrm{range}(\mathcal{L}_\alpha)) &\Leftrightarrow \exists\, u \in D(\mathcal{L}_\alpha),\ \text{s.t.}\ \ \omega = -\mathcal{L}_\alpha u \\ &\Leftrightarrow \exists\, u \in D(\mathcal{L}_0),\ \text{s.t.}\ \ \omega = -\mathcal{L}_\alpha u \end{aligned}$$

where the last equivalence is obtained thanks to the assumption (A*) which guarantees that $D(\mathcal{L}_\alpha) = D(\mathcal{L}_0)$ for all $\alpha(\cdot) \in \mathcal{A}$. Note however that ω still depends on α through its definition as $\omega = -\mathcal{L}_\alpha u$. The fact that u belongs to a domain which is independent of α is important in this scheme, since the maximization over y^* is not in the same order as the minimization over α. Indeed, our *dual* problem now writes

$$\max_{y^*\in Y^*} \inf_{q\in Q} \min_{\alpha(\cdot)\in\mathcal{A}} \{L(q,y^*,\alpha) \quad \text{s.t.}\ \ y^* = (c, -\mathcal{L}_\alpha u) \text{ and } (c,u) \in \mathbb{R} \times D(\mathcal{L}_0)\}, \qquad (\mathfrak{D}^\wedge)$$

and the new variables on which we perform the maximization are now (c,u) and they belong to $\mathbb{R} \times D(\mathcal{L}_0)$. The latter being independent of $\alpha(\cdot)$, we can write the *dual* problem as

$$\max_{\substack{c\in\mathbb{R} \\ u\in D(\mathcal{L}_0)}} \inf_{q\in Q} \min_{\alpha(\cdot)\in\mathcal{A}} \{L(q,y^*,\alpha), \quad \text{s.t.}\ \ y^* = (c, -\mathcal{L}_\alpha u)\,\}. \qquad (\mathfrak{D}^\wedge)$$

Recalling the definition (4.4.6) of L and the notations introduced earlier, we have

$$\begin{aligned} L(q,y^*,\alpha) &= \langle f(\cdot\,,\alpha(\cdot)), q\rangle + \langle y^*, G(q)\rangle_{Y^*,Y} \\ &= \langle f(\cdot\,,\alpha(\cdot)), q\rangle + c(1 - \langle 1, q\rangle) + \langle -\mathcal{L}_\alpha u(\cdot), q\rangle \\ &= c + \langle f(\cdot\,,\alpha(\cdot)) - \mathcal{L}_\alpha u(\cdot) - c, q\rangle \end{aligned}$$

hence we have, using the *exchange property* in Proposition 4.4.1,

$$\begin{aligned} &\min_{\alpha(\cdot)\in\mathcal{A}} \{L(q,y^*,\alpha), \quad \text{s.t.}\ \ y^* = (c, -\mathcal{L}_\alpha u)\,\} \\ &\qquad = c + \min_{\alpha(\cdot)\in\mathcal{A}} \{\, \langle f(\cdot\,,\alpha(\cdot)) - \mathcal{L}_\alpha u(\cdot) - c, q\rangle \,\} \\ &\qquad = c + \langle \min_{\alpha\in A}\{f(\cdot\,,\alpha) - \mathcal{L}_\alpha u(\cdot)\} - c, q\rangle \\ &\qquad = c + \langle H(\cdot\,,\nabla u, D^2 u) - c, q\rangle \end{aligned}$$

and the *dual* problem writes

$$\max_{\substack{c\in\mathbb{R}\\ u\in D(\mathcal{L}_0)}} \left\{ c + \inf_{q\in Q}\langle H(\cdot\,,\nabla u, D^2u) - c, q\rangle \right\} \tag{$\mathfrak{D}^\wedge$}$$

Noticing that $-\inf_{q\in Q}\langle H(\cdot\,,\nabla u, D^2u) - c, q\rangle = \sup_{q\in Q}\langle c - H(\cdot\,,\nabla u, D^2u), q\rangle$ is the support function $\sigma(c - H(\cdot\,,\nabla u, D^2u); Q)$ which is 0 if $\langle c - H(\cdot\,,\nabla u, D^2u), q\rangle \leq 0$ for all $q \in Q$ and $+\infty$ otherwise. But since Q is made of non-negative measures with finite moment of order d, we need u to have a polynomial growth of order at most $\kappa = d+1-\beta$ (see the proof of Lemma 4.1) and $c - H(x, \nabla u, D^2u) \leq 0$ a.e. on each support of $q \in Q$, hence in $\mathbb{R}^m$. The *dual* problem finally writes

$$\max_{\substack{c\in\mathbb{R}\\ u\in\mathcal{X}}} \left\{\, c\,,\ \text{s.t.:}\ \ c - H(x,\nabla u, D^2u) \leq 0,\ \textit{a.e.}\ \text{in}\ \mathbb{R}^m \,\right\} \tag{$\mathfrak{D}^\wedge$}$$

and the functional space $\mathcal{X}$ is now

$$\mathcal{X} = D(\mathcal{L}_0) \cap \{u : \mathbb{R}^m \to \mathbb{R},\ \textit{Borel-meas.} \mid \exists\, C > 0,\ |u(x)| \leq C(1+|x|^\kappa)\}$$

where $\kappa = d+1-\beta$, which then concludes the proof. □

In the case where the Hamiltonian is given by

$$H(x, \nabla u(x), D^2u(x)) = \max_{\alpha\in A}\{\, -\mathcal{L}_\alpha u(x) + f(x,\alpha) \,\}.$$

the same proof as before can again be conducted, with minor modification in the duality procedure which we will now present.

The *primal* problem ($\mathfrak{P}^\wedge$) will in this case take the form

$$\min_{q\in\mathcal{M}_d^+(\mathbb{R}^m)} \left\{ \min_{\alpha(\cdot)\in\mathcal{A}} \quad \langle -f(\cdot\,,\alpha(\cdot)), q\rangle, \quad \text{s.t.}\ 1 - \langle 1, q\rangle = 0 \text{ and } q \in \mathrm{Ker}(\mathcal{L}_\alpha^*) \right\} \tag{$\mathfrak{P}^\vee$}$$

which writes as

$$\min_{q\in Q} \left\{ \min_{\alpha(\cdot)\in\mathcal{A}} \quad \langle -f(\cdot\,,\alpha(\cdot)), q\rangle, \quad \text{s.t.}\ G(q) \in K_\alpha \right\} \tag{$\mathfrak{P}^\vee$}$$

Note that the only difference is that instead of f we now consider $-f$. Then we define

$$\phi(q,y) = \min_{\alpha(\cdot)\in\mathcal{A}} \{\, \langle -f(\cdot\,,\alpha(\cdot)), q\rangle + F(G(q)+y, \alpha) \,\}\,.$$

and the Lagrangian in this case writes as

$$L(q, y^*, \alpha) \coloneqq \langle -f(\cdot\,, \alpha(\cdot)), q\rangle + \langle y^*, G(q)\rangle_{Y^*,Y}. \tag{4.4.8}$$

We compute in the same way as before the Legendre-Fenchel transform $\phi^*(q^*, y^*)$ and recover the *dual* problem similar to $(\mathfrak{D}^\wedge)$ and which is given by

$$\max_{y^*\in Y^*} \inf_{q\in Q} \min_{\alpha(\cdot)\in\mathcal{A}} \{L(q, y^*, \alpha) - F^*(y^*, \alpha)\}. \tag{$\mathfrak{D}^\vee$}$$

The following is an analogue of Lemma 4.4.

Lemma 4.5. *The problem* $(\mathfrak{D}^\vee)$ *is equivalent to*

$$\max_{\substack{c\in\mathbb{R}\\ u\in\mathcal{X}}} \left\{ -c,\ \textit{s.t.:}\ \ H(x, \nabla u, D^2u) - c \le 0,\ \textit{a.e. in } \mathbb{R}^m \right\} \tag{$\mathfrak{D}^\vee$}$$

where $H(x, \nabla u(x), D^2u(x)) = \max_{\alpha\in A}\{ -\mathcal{L}_\alpha u(x) + f(x, \alpha) \}$ *and* $\mathcal{X}$ *is such that*

$$\mathcal{X} = D(\mathcal{L}_0) \cap \{u : \mathbb{R}^m \to \mathbb{R}, \textit{Borel-meas.} \mid \exists\, C > 0,\ |u(x)| \le C(1 + |x|^\kappa)\} \tag{4.4.9}$$

with $\kappa = d + 1 - \beta$*, that is, the two optimization problems have the same set of optimal solutions and the same optimal value.*

We keep the *primal* problem $(\mathfrak{P}^\vee)$ and the *dual* problem $(\mathfrak{D}^\vee)$ written in this formulation because it will be needed when we will set the optimality conditions in the next section.

Proof. The proof follows the one of Lemma 4.4. The main difference is in the choice of the representation of the dual variable $y^* \in (K_\alpha)^-$ which we now write as

$$\begin{aligned} y^* \in (K_\alpha)^- &\Leftrightarrow (-c, -\omega) \in \Big(\{0\} \times \mathrm{Ker}(\mathcal{L}_\alpha)\Big)^- \\ &\Leftrightarrow (-c, -\omega) \in \mathbb{R} \times (\mathrm{Ker}(\mathcal{L}_\alpha))^\perp \\ &\Leftrightarrow (-c, -\omega) \in \mathbb{R} \times \mathrm{cl}(\mathrm{range}(\mathcal{L}_\alpha)) \end{aligned}$$

We set again as in Lemma 4.4,

$$\omega \in \mathrm{cl}(\mathrm{range}(\mathcal{L}_\alpha)) \Leftrightarrow \exists\, u \in D(\mathcal{L}_0),\ \text{s.t.}\ \ \omega = -\mathcal{L}_\alpha u.$$

And the *dual* problem $(\mathfrak{D}^{\vee})$ writes as

$$\max_{y^*\in Y^*} \inf_{q\in Q} \min_{\alpha(\cdot)\in\mathcal{A}} \{L(q, y^*, \alpha) \quad \text{s.t.} \;\; y^* = (-c, \mathcal{L}_\alpha u) \text{ and } (c, u) \in \mathbb{R} \times D(\mathcal{L}_0)\}, \qquad (\mathfrak{D}^{\vee})$$

Recalling the definition (4.4.8) of L and the notations introduced earlier, we have

$$\begin{aligned} L(q, y^*, \alpha) &= \langle -f(\cdot\,, \alpha(\cdot)), q\rangle + \langle y^*, G(q)\rangle_{Y^*,Y} \\ &= \langle -f(\cdot\,, \alpha(\cdot)), q\rangle - c(1 - \langle 1, q\rangle) + \langle \mathcal{L}_\alpha u(\cdot), q\rangle \\ &= -c - \langle f(\cdot\,, \alpha(\cdot)) - \mathcal{L}_\alpha u(\cdot) - c, q\rangle \end{aligned}$$

hence we have, using the *exchange property* in Proposition 4.4.1,

$$\begin{aligned} &\min_{\alpha(\cdot)\in\mathcal{A}} \{L(q, y^*, \alpha), \quad \text{s.t.} \;\; y^* = (-c, \mathcal{L}_\alpha u)\,\} \\ &\qquad = -c + \min_{\alpha(\cdot)\in\mathcal{A}} \{\, -\langle f(\cdot\,, \alpha(\cdot)) - \mathcal{L}_\alpha u(\cdot) - c, q\rangle\,\} \\ &\qquad = -c - \max_{\alpha(\cdot)\in\mathcal{A}} \{\, \langle f(\cdot\,, \alpha(\cdot)) - \mathcal{L}_\alpha u(\cdot) - c, q\rangle\,\} \\ &\qquad = -c - \langle \max_{\alpha\in A}\{f(\cdot\,, \alpha) - \mathcal{L}_\alpha u(\cdot)\} - c, q\rangle \\ &\qquad = -c - \langle H(\cdot\,, \nabla u, D^2 u) - c, q\rangle \end{aligned}$$

and the *dual* problem writes

$$\max_{\substack{c\in\mathbb{R} \\ u\in D(\mathcal{L}_0)}} \left\{ -c + \inf_{q\in Q} \{\, -\langle H(\cdot\,, \nabla u, D^2 u) - c, q\rangle\,\} \right\}. \qquad (\mathfrak{D}^{\vee})$$

Recalling the definition (4.2.4) of the support function of a set, the *dual* problem becomes

$$\max_{\substack{c\in\mathbb{R} \\ u\in D(\mathcal{L}_0)}} \{\, -c - \sigma(H(\cdot\,, \nabla u, D^2 u) - c\,; Q)\,\}. \qquad (\mathfrak{D}^{\vee})$$

The conclusion then follows as in the end of the proof of Lemma 4.4. $\square$

4.4.3 Main results II: ergodic HJB equation

4.4.3.1 The optimality conditions

We first consider the case where the Hamiltonian is given by

$$H(x, \nabla u(x), D^2 u(x)) = \min_{\alpha\in A}\{\, -\mathcal{L}_\alpha u(x) + f(x, \alpha)\,\}.$$

We check that the optimality conditions as stated in §4.2.1, in particular (4.2.8) and

(4.2.9), still hold in our framework. In order to do so, we start from the *duality gap* (or *duality inequality*) which states that the value of the *dual* problem $(\mathfrak{D}^\wedge)$ is less or equal than the value of the *primal* problem $(\mathfrak{P}^\wedge)$. Recalling the definition (4.4.6) of the Lagrangian function L and the value of the *dual* problem being less or equal the value of the *primal* problem (see §4.2.1), we have

$$\begin{aligned}&\max_{y^*\in Y^*}\min_{q\in Q}\min_{\alpha(\cdot)\in\mathcal{A}}\{L(q,y^*,\alpha)-F^*(y^*,\alpha)\}\\&\qquad\le\min_{q\in Q}\min_{\alpha(\cdot)\in\mathcal{A}}\{\langle f(\cdot\,,\alpha(\cdot)),q\rangle+F(G(q),\alpha)\}\\&\qquad\le\min_{q\in Q}\min_{\alpha(\cdot)\in\mathcal{A}}\{L(q,y^*,\alpha)+F(G(q),\alpha)-\langle y^*,G(q)\rangle_{Y^*,Y}\},\ \forall\, y^*\in Y^*.\end{aligned}$$

Let us denote by $(q_\circ,\alpha_\circ)$ an optimal solution in the *primal* problem $(\mathfrak{P}^\wedge)$ and by $y^*_\circ$ an optimal solution in the *dual* problem $(\mathfrak{D}^\wedge)$. We then have

$$\begin{aligned}\min_{q\in Q}\min_{\alpha(\cdot)\in\mathcal{A}}\{L(q,y^*_\circ,\alpha)-F^*(y^*_\circ,\alpha)\}&\le L(q_\circ,y^*_\circ,\alpha_\circ)+F(G(q_\circ),\alpha_\circ)-\langle y^*_\circ,G(q_\circ)\rangle_{Y^*,Y}\\&=\langle f(\cdot\,,\alpha_\circ(\cdot)),q_\circ\rangle+F(G(q_\circ),\alpha_\circ)\end{aligned}$$

Optimality conditions are then obtained when we reach equality in the above inequality. We can then characterize the optimal *primal* and *dual* solutions and provide a no-*duality gap* condition. Suppose the left hand side minimization in the above inequality is reached in the pair of optimal solutions $(q_\circ,\alpha_\circ)$. Therefore, we firstly need to have $F^*(y^*_\circ,\alpha_\circ)=0$ i.e.

$$y^*_\circ\in(K_{\alpha_\circ})^- \tag{4.4.10}$$

since F is an indicator function and hence F^* is a support function which is either 0 if $y^*_\circ\in(K_{\alpha_\circ})^-$ or $+\infty$ otherwise. Then, and secondly, since $L(q_\circ,y^*_\circ,\alpha_\circ)=\langle f(\cdot\,,\alpha_\circ(\cdot)),q_\circ\rangle+\langle y^*_\circ,G(q_\circ)\rangle$ then from the optimality of $(q_\circ,\alpha_\circ)$ we have

$$\langle y^*_\circ,G(q_\circ)\rangle_{Y^*,Y}=0. \tag{4.4.11}$$

And finally the inequality is reduced to

$$-F^*(y^*_\circ,\alpha_\circ)\le F(G(q_\circ),\alpha_\circ)-\langle y^*_\circ,G(q_\circ)\rangle_{Y^*,Y}$$

which is the Fenchel-Young inequality. The latter is an equality if and only if we have

$$y^*_\circ\in\partial F(G(q_\circ),\alpha_\circ)=\partial I_{K_{\alpha_\circ}}(G(q_\circ))=N_{K_{\alpha_\circ}}(G(q_\circ)) \tag{4.4.12}$$

And since K_{α_0} is a convex cone, then $y^*_\circ \in N_{K_{\alpha_\circ}}(G(q_\circ))$ is equivalent to

$$G(q_\circ) \in K_{\alpha_\circ}, \quad y^*_\circ \in (K_{\alpha_\circ})^- \;\text{ and }\; \langle y^*_\circ, G(q_\circ)\rangle_{Y^*,Y} = 0. \tag{4.4.13}$$

To sum up, we have the following sufficient optimality conditions which also guarantee the absence of the *duality gap*

$$\begin{cases} (q_\circ, \alpha_\circ) \in \underset{q\in Q, \alpha(\cdot)\in\mathcal{A}}{\operatorname{argmin}} \; L(q, y^*_\circ, \alpha) \\ G(q_\circ) \in K_{\alpha_\circ}, \quad y^*_\circ \in (K_{\alpha_\circ})^- \;\text{ and }\; \langle y^*_\circ, G(q_\circ)\rangle_{Y^*,Y} = 0. \end{cases} \tag{4.4.14}$$

They are indeed analogue to (4.2.8) and (4.2.9).

And the same optimality conditions (4.4.14) hold when the Hamiltonian is given by

$$H(x, \nabla u(x), D^2u(x)) = \max_{\alpha\in A}\{\, -\mathcal{L}_\alpha u(x) + f(x,\alpha)\,\},$$

provided we write $-f$ (instead of f) in the above computations and make use of definition (4.4.8) for the Lagrangian function L.

4.4.3.2 Main results II

We are now ready to state and prove the existence and uniqueness results for a solution to the ergodic HJB equation as given in our initial problem (4.4.1), assuming (A0-A6) and (A*) hold true.

Theorem 4.4.1. *There exists a pair $(c, u(\cdot)) \in \mathbb{R} \times W^{r,2}_{loc}(\mathbb{R}^m)$ for any $r \geq 1$, such that $|u(x)| \leq K(1 + |x|^\kappa)$ where $\kappa = d - 1 + \beta$, and which solves*

$$H(x, \nabla u(x), D^2u(x)) = c, \quad \textit{a.e. in } \mathbb{R}^m$$

where $H(x,p,P) = \min_{\alpha\in A}\{\, -b(x,\alpha)\cdot p - trace(a(x,\alpha)P) + f(x,\alpha)\,\}$. Moreover, the latter constant c is given by $c = \langle f(\cdot\,, \alpha(\cdot))\,,\, \mu_\alpha\rangle$ where

$$\alpha(x) \in \underset{\alpha\in A}{\operatorname{argmin}}\{\, -\mathcal{L}_\alpha u(x) + f(x,\alpha)\,\}, \quad \textit{a.e. in } \mathbb{R}^m$$

and μ_α is the unique invariant probability measure associated to $\mathcal{L}^*_\alpha$.

When $r > \frac{m}{2}$, $u(\cdot)$ is continuous and pointwise twice differentiable almost everywhere. If in addition the vector field b is locally Lipschitz continuous in x uniformly in α, and $\beta = 1$ in (A6), then $u(\cdot)$ with such a polynomial growth is unique in any $W^{r,2}_{loc}(\mathbb{R}^m)$,

$r > \frac{m}{2}$, in the sense: if $(c, u_1(\cdot))$ and $(c, u_2(\cdot))$ are two solutions, then $u_1(\cdot) - u_2(\cdot) \equiv$ constant.

Theorem 4.4.2. *The same conclusions of Theorem 4.4.1 are still valid when the Hamiltonian is $H(x,p,P) = \max_{\alpha \in A}\{ -b(x,\alpha)\cdot p - \mathrm{trace}(a(x,\alpha)P) + f(x,\alpha) \}$, except $\alpha(x)$ that is defined by*

$$\alpha(x) \in \operatorname*{argmax}_{\alpha \in A}\{ -\mathcal{L}_\alpha u(x) + f(x,\alpha) \}, \quad a.e.\ in\ \mathbb{R}^m.$$

Remark 4.4.3. *In fact $u(\cdot)$ is an L-viscosity solution (see e.g. [50, 70]), which is as expected as when we consider C-viscosity solutions for the continuous (and bounded) case. Recall that in our setting, the vector field b and the function f are assumed to be measurable (and unbounded) in x.*

Proof *of Theorem 4.4.1.* We follow the same scheme as for the linear case taking advantage of the previous results. We recall the two optimization problems from §4.4.1 and §4.4.2:

- The primal problem

$$\min_{q \in \mathcal{M}_d^+(\mathbb{R}^m)} \left\{ \min_{\alpha(\cdot) \in \mathcal{A}} \quad \langle f(\cdot\,, \alpha(\cdot)), q \rangle, \quad \text{s.t. } 1 - \langle 1, q \rangle = 0 \text{ and } q \in \mathrm{Ker}(\mathcal{L}_\alpha^*) \right\} \quad (\mathfrak{P}^\wedge)$$

- The dual problem from Lemma 4.4

$$\max_{\substack{c \in \mathbb{R} \\ u \in \mathcal{X}}} \left\{ c, \text{ s.t.: } \ c - H(x, \nabla u, D^2 u) \le 0,\ a.e. \text{ in } \mathbb{R}^m \right\}. \qquad (\mathfrak{D}^\wedge)$$

Step 1. (On the primal problem)

As we assume the standing assumptions hold uniformly in α, the diffusion operator $\mathcal{L}_\alpha$ satisfies the results in §4.2.3.2, in particular Theorem 4.2.7 holds for each $\alpha \in \mathcal{A}$, and hence the set $\mathcal{M}_{\mathrm{ell}}^{A,b}$ as defined in (4.2.24) is again a singleton for each α. Moreover, Lemma 4.2 insures existence of an optimal solution $(q_\circ, \alpha_\circ)$ to the problem $(\mathfrak{P}^\wedge)$.

Step 2. (On the dual problem)

The problem $(\mathfrak{P}^\wedge)$ being *calm* thanks to Lemma 4.2, Theorem 4.2.2 insures (i) a *no-duality gap* between $(\mathfrak{P}^\wedge)$ and $(\mathfrak{D}^\wedge)$, and (ii) existence of multipliers $y_\circ^* := (c_\circ, -\mathcal{L}_{\alpha_\circ} u_\circ)$ satisfying (4.4.14), which are in addition the solutions to the dual problem $(\mathfrak{D}^\wedge)$ thanks to Proposition 4.2.1.

Step 3. (On the PDE problem)

Setting $(q_\circ, \alpha_\circ)$ an optimal solution to $(\mathfrak{P}^\wedge)$, $(c_\circ, u_\circ) \in \mathbb{R} \times \mathcal{X}$ an optimal solution to

$(\mathfrak{D}^\wedge)$ and $y_\circ^* := (c_\circ, -\mathcal{L}_{\alpha_\circ} u_\circ)$, we have from the previous step $(q_\circ, \alpha_\circ)$ and $y_\circ^*$ satisfy the optimality conditions (4.4.14). Moreover no-*duality gap* yields $c_\circ = \langle f(\cdot\,, \alpha_\circ(\cdot)), q_\circ\rangle$ and the condition $\langle y_\circ^*, G(q_\circ)\rangle_{Y^*,Y} = 0$ writes as $c_\circ(1 - \langle 1, q_\circ\rangle) + \langle -\mathcal{L}_{\alpha_\circ} u_\circ(\cdot), q_\circ\rangle = 0$. In fact, since $(q_\circ, \alpha_\circ)$ in an optimal solution to the primal problem $(\mathfrak{P}^\wedge)$, one has $1 - \langle 1, q_\circ\rangle = 0$ and $\langle \mathcal{L}_{\alpha_\circ} u_\circ(\cdot), q_\circ\rangle = 0$. In particular, when setting q to its optimal value $q_\circ$ in $(\mathfrak{P}^\wedge)$, one has

$$\alpha_\circ(\cdot) \in \underset{\alpha(\cdot)\in\mathcal{A}}{\operatorname{argmin}} \left\{ \langle -\mathcal{L}_\alpha u_\circ(\cdot) + f(\cdot\,, \alpha(\cdot)) - c_\circ, q_\circ\rangle \right\}$$

which yields thanks to the *exchange property* (4.5.3)

$$\alpha_\circ(x) \in \underset{\alpha\in A}{\operatorname{argmin}} \left\{ -\mathcal{L}_\alpha u_\circ(x) + f(x, \alpha) \right\}, \quad q_\circ\text{-a.e. } x \in \mathbb{R}^m,$$

and implies

$$H(x, \nabla u_\circ(x), D^2 u_\circ(x)) = -\mathcal{L}_{\alpha_\circ} u_\circ(x) + f(x, \alpha_\circ(x)), \quad q_\circ\text{-a.e. } x \in \mathbb{R}^m.$$

Moreover, $(c_\circ, u_\circ)$ solves $(\mathfrak{D}^\wedge)$, in particular the constraint is satisfied, that is

$$c_\circ - H(x, \nabla u_\circ(x), D^2 u_\circ(x)) \le 0, \quad \text{a.e. in } \mathbb{R}^m$$

i.e. it does not change sign almost everywhere. Therefore, the equation

$$\langle c_\circ - \{-\mathcal{L}_{\alpha_\circ} u_\circ(\cdot) + f(\cdot\,, \alpha_\circ)\}, q_\circ\rangle = 0 \tag{4.4.15}$$

implies $c_\circ - \{-\mathcal{L}_{\alpha_\circ} u_\circ + f(\cdot\,, \alpha_\circ)\} = 0$ $q_\circ$-almost everywhere. But $q_\circ$ is absolutely continuous with respect to Lebesgue measure and is supported in the whole $\mathbb{R}^m$ thanks to Theorem 4.2.5, hence the result almost everywhere in $\mathbb{R}^m$:

$$(c_\circ, u_\circ) \text{ solves (4.4.1) where } \mathcal{X} \text{ is as in (4.4.7)}$$

Step 4. (Uniqueness of $u_\circ(\cdot)$*)*
As in *Step 3* in the proof of Theorem 4.3.2, we have uniqueness of the Lagrange multiplier $y_\circ^*$ thanks to Proposition 4.2.3.

To prove now that $u_\circ(\cdot)$ is unique, we need to assume in addition that b is locally Lipschitz continuous with at most a linear growth, i.e. $\beta = 1$ and hence $\kappa = d$. This setting will allow us to apply the Liouville type result in [22].

Suppose $(c_\circ, u_1(\cdot)), (c_\circ, u_2(\cdot))$ are two solutions with a polynomial growth of order at most d. Then we have, using the inequality "$\min(A - B) \le \min(A) - \min(B)$"

$$\min_{\alpha\in A}\{ -\mathcal{L}_\alpha(u_1 - u_2) \} \le \min_{\alpha\in A}\{-\mathcal{L}_\alpha u_1 + f(\cdot\,,\alpha)\} - \min_{\alpha\in A}\{-\mathcal{L}_\alpha u_2 + f(\cdot\,,\alpha)\} = 0$$

Therefore uniqueness of a solution $(c_\circ, u(\cdot))$ is reduced to proving that there cannot exist non-constant sub-solutions to the static HJB equation $\min_{\alpha\in A}\{-\mathcal{L}_\alpha u\} = 0$, i.e. whether Liouville property holds for the latter static HJB. This is answered positively in [22, Theorem 2.1] using the Lyapunov function ψ as in *Step 3* in the proof of Theorem 4.3.2.

$\square$

Proof *of Theorem 4.4.2.* The same proof as for Theorem 4.4.1 still works when the Hamiltonian is now given by a max (instead of a min), provided we make some minor modifications. Indeed, Lemma 4.2 holds true also in this case, since the only change is in the sign in front of f in the objective function of $(\mathfrak{P}^\vee)$. We recall the two optimization problems at the end of §4.4.2:

- The primal problem

$$\min_{q\in\mathcal{M}_d^+(\mathbb{R}^m)} \left\{ \min_{\boldsymbol{\alpha}(\cdot)\in\mathcal{A}} \quad \langle -f(\cdot\,,\boldsymbol{\alpha}(\cdot)), q\rangle, \quad \text{s.t. } 1 - \langle 1, q\rangle = 0 \text{ and } q \in \operatorname{Ker}(\mathcal{L}^*_{\boldsymbol{\alpha}}) \right\} \tag{$\mathfrak{P}^\vee$}$$

- The dual problem from Lemma 4.5

$$\max_{\substack{c\in\mathbb{R}\\ u\in\mathcal{X}}} \left\{ -c, \text{ s.t.: } H(x, \nabla u, D^2u) - c \le 0, \ a.e. \text{ in } \mathbb{R}^m \right\}. \tag{$\mathfrak{D}^\vee$}$$

In the proof of Theorem 4.4.1, *Step 1* remains unchanged, while in *Step 2* we will get a different representation for the Lagrange multipliers $y_\circ^*$ (as we did in the proof of Lemma 4.5), that is, $y_\circ^* := (-c_\circ, \mathcal{L}_{\boldsymbol{\alpha}_\circ} u_\circ)$ which satisfies (4.4.14). Then we proceed exactly as in *Step 3* of the proof of Theorem 4.4.1, recalling that we have $-f$ (and not f) in the objective function of $(\mathfrak{P}^\vee)$ and we have $-c$ (and not c) in the objective function of $(\mathfrak{D}^\vee)$. The no-*duality gap* still writes as $c_\circ = \langle f(\cdot\,,\boldsymbol{\alpha}_\circ(\cdot)), q_\circ\rangle$ and $\boldsymbol{\alpha}_\circ(\cdot)$ is now characterized by

$$\boldsymbol{\alpha}_\circ(x) \in \operatorname*{argmax}_{\alpha\in A} \left\{ -\mathcal{L}_\alpha u_\circ(x) + f(x,\alpha) \right\}, \quad q_\circ\text{-a.e. } x \in \mathbb{R}^m,$$

since we have $-f$ in the objective function of the problem $(\mathfrak{P}^\vee)$, which again yields

$$H(x, \nabla u_\circ(x), D^2u_\circ(x)) = -\mathcal{L}_{\boldsymbol{\alpha}_\circ} u_\circ(x) + f(x, \boldsymbol{\alpha}_\circ(x)), \quad q_\circ\text{-a.e. } x \in \mathbb{R}^m.$$

We conclude as in *Step 3* of the proof of Theorem 4.4.1.

Finally, to prove uniqueness, we consider again $(c_\circ, u_1(\cdot)), (c_\circ, u_2(\cdot))$ two solutions with a polynomial growth of order at most d, and the only difference with *Step 4* in the proof of Theorem 4.4.1 is that we need to check that there cannot exist non-constant super-solutions to the static HJB equation $\max_{\alpha\in A}\{-\mathcal{L}_\alpha u\} = 0$, since we have

$$\max_{\alpha\in A}\{\, -\mathcal{L}_\alpha(u_1 - u_2)\,\} \geq \max_{\alpha\in A}\{-\mathcal{L}_\alpha u_1 + f(\cdot\,,\alpha)\} - \max_{\alpha\in A}\{-\mathcal{L}_\alpha u_2 + f(\cdot\,,\alpha)\} = 0.$$

And Liouville property in [22, Theorem 2.2] again holds for the latter static HJB, using the Lyapunov function ψ as in *Step 3* in the proof of Theorem 4.3.2.

□

Remark 4.4.4. *In stochastic control problems, the Hamiltonian usually writes* $H(x,p,P) = \max_{\alpha\in A}\{\, -b(x,\alpha)\cdot p - \mathit{trace}(a(x,\alpha)P) + f(x,\alpha)\,\}$. *So recalling the definition of the corresponding dual problem* $(\mathfrak{D}^\vee)$*, one can see that the ergodic constant* c *that is given by Theorem 4.4.2 is the smallest one, in the sense that: if there exists an other solution* $(\widetilde{c},\widetilde{u}(\cdot))$*, then necessary* $c \leq \widetilde{c}$*. This is in line with the classical results on viscous ergodic Bellman equations for which one usually expects infinitely many possible ergodic constant (and solutions) but all larger than the critical (smallest) one; see [97, 104].*

4.4.4 The manifold setting

A similar result holds in the case when we consider, instead of $\mathbb{R}^m$, a non-compact complete and connected smooth Riemannian manifold M of dimension m (see [17]). Indeed most of the results in §4.2.3.2 and that we borrowed from [42] are still valid in the case of a Riemannian manifold, following the results in [43]. It is however more convenient (following [43]) to deal with second-order elliptic operators in the divergence form

$$\mathfrak{L}_{a,b}\varphi(x) \coloneqq \operatorname{div}(a(x)\nabla\varphi(x)) + b(x)\cdot\nabla\varphi(x), \quad \varphi \in C_0^\infty(M)$$

where $b(x)$ is a Borel-measurable vector field on M, $a(x)$ is non-negative operator on T_xM that is Borel-measurable in x, " $\cdot$ " denotes the inner product in T_xM, and div is divergence with respect to the Riemannian volume dx, that is, for each function $\varphi \in C_0^\infty(M)$, one has the equality

$$\int_M \varphi \operatorname{div} v\, dx = -\int_M \nabla\varphi\cdot v\, dx.$$

We recall the operator $\mathcal{L}$ defined earlier in (4.3.2) and we denote it by

$$\mathcal{L}_{a,b}\varphi(x) = \operatorname{trace}(a(x)D^2\varphi(x)) + b(x)\cdot\nabla\varphi(x),$$

and the set $\mathcal{M}_{\text{ell}}^{a,b}$ analogous to (4.2.24) but that we define now on the manifold M by

$$\mathcal{M}_{\text{ell}}^{a,b} := \big\{\mu \mid \mu \text{ a probability measure on } M \text{ satisfying } \mathfrak{L}_{a,b}^*\mu = 0\big\}. \tag{4.4.16}$$

where $\mathfrak{L}_{a,b}^*\mu = 0$ is understood in the distribution sense, that is,

$$\int_M \mathfrak{L}_{a,b}\varphi \,\mathrm{d}\mu = 0, \quad \forall\, \varphi \in C_0^\infty(M),$$

provided μ is a locally finite Borel measure on M and $\mathfrak{L}_{a,b}\varphi \in L^1(M,\mu)$, $\forall\,\varphi \in C_0^\infty(M)$.

When the coefficients a^{ij} and b^i satisfy assumption (A2), we have

$$\begin{aligned} \mathcal{L}_{a,b} &= \mathfrak{L}_{a,b_0}, \quad \text{where } b_0^i = b^i - \partial_j a^{ij},\\ \mathfrak{L}_{a,b} &= \mathcal{L}_{a,b_1}, \quad \text{where } b_1^i = b^i + \partial_j a^{ij}, \end{aligned} \tag{4.4.17}$$

that is, both representations are equivalent and their coefficients satisfy the same local conditions. Note however that when dealing with stochastic differential equations, the Ito form leads to the non-divergent elliptic operator $\mathcal{L}_{a,b}$, whereas the Stratonovich form (see [101]) leads to the elliptic operator in divergence form $\mathfrak{L}_{a,b}$.

Remark 4.4.5. *It is immediate to see that, using* (4.4.17)*, our previous results in* $\mathbb{R}^m$ *are still valid when we consider an operator in divergence form.*

Let us consider the operator $\mathfrak{L}_{a,b}$ where we drop the dependency on the coefficients a and b, but which depends now on the parameters $\alpha \in A$ as previously defined. We then write $\mathfrak{L}_\alpha$ and that we define by

$$\mathfrak{L}_\alpha\varphi(x) := \operatorname{div}(a(x,\alpha)\nabla\varphi(x)) + b(x,\alpha)\cdot\nabla\varphi(x), \quad \varphi \in C_0^\infty(M). \tag{4.4.18}$$

Similarly, we use the notation $\mathcal{M}_{\text{ell}}^\alpha$ to refer to (4.4.16) where measures satisfy $\mathfrak{L}_\alpha^*\mu = 0$.

We also recall $\mathcal{A}$ being the set of measurable functions $\boldsymbol{\alpha}(\cdot) : M \to A$, and $\mathfrak{L}_{\boldsymbol{\alpha}}$ is defined as in (4.4.18) where instead of α we have $\boldsymbol{\alpha}(x)$.

The following lemma is a collection of results from [43] which are analogous to those in §4.2.3.2. It allows us then to justify the results in §4.4 when in the setting of manifolds and using the operator $\mathfrak{L}_\alpha$ as in (4.4.18).

Lemma 4.6. *Let M be a non-compact complete and connected smooth Riemannian manifold of dimension m. Assume a, b satisfy conditions (A1), (A2) and (A3) in local coordinates. The following statements hold true.*

1. *(Existence and uniqueness) If in addition, for each $\alpha(\cdot) \in \mathcal{A}$, there exists a non-negative compact*[8] *function $V[\alpha] \in C^2(M)$ (a 'Lyapunov function') such that*

$$\lim_{|x|\to\infty} V[\alpha](x) = +\infty, \quad \lim_{|x|\to\infty} \mathfrak{L}_\alpha V[\alpha](x) = \infty \tag{4.4.19}$$

   in the following sense: for each $R > 0$, there exists a compact set K_R such that $\mathfrak{L}_\alpha V[\alpha](x) \leq -R$ for $x \notin K_{R,\alpha}$. Then the set $\mathcal{M}^{\alpha}_{ell}$ is a non-empty singleton, i.e., there exists a unique invariant probability measure satisfying $\mathfrak{L}^*_\alpha \mu_\alpha = 0$.

2. *(Regularity) The measure μ_α given in (i) enjoys the same properties as those in Theorem 4.2.5.*

3. *(Closed extension) There exists a closed extension $(\mathfrak{L}^{\mu_\alpha}_\alpha, D(\mathfrak{L}^{\mu_\alpha}_\alpha))$ of $(\mathfrak{L}_\alpha, C^\infty_0(M))$ generating a sub-Markovian contractive C_0-semigroup on $L^1(M, \mu)$. Moreover, Theorem 4.2.8 is still valid when replacing $\mathbb{R}^m$ with M.*

Proof *of Lemma 4.6.* Statement *(i)* is a consequence of [43, Theorem 5.7 & Example 5.1], and was obtained in [41, 44]. Statement *(ii)* is [43, Theorem 2.1]. And statement *(iii)* is [43, Theorem 2.3 & Theorem 2.8]. □

Remark 4.4.6. *As noted in [41, Remark 2.3], if M has Ricci curvature bounded from bellow, a Lyapunov function can be of the form $r(x) = d(x, o)^k$ defined outside the set of its singularities, and where o is a fixed point in M and $d(\cdot\,, \cdot)$ is the distance in M. We refer to [75, Chap. IX, §6] for further details. See also [41] and references therein.*

Armed with this Lemma, we can then perform the same duality procedure using the material of §4.2.1 and recover analogous results to Theorems 4.4.1 and 4.4.2.

However, we are lacking information on the moments of the invariant measure as in Lemma 2.3.1 and also Liouville type results [22] when in the setting of manifolds. Therefore in what follows, we only provide an existence result for a solution to an ergodic HJB equation on a manifold M, and hope we can tackle the remaining questions in a future work.

[8] A function V on M is said to be *compact* if the sets $\{V \leq c\}$, $c \geq 0$, are compact. When M is a non-compact manifold and denoting $|x| = \mathrm{dist}(x, o)$, where $o \in M$ is a fixed point, then a continuous function V is compact if and only if $V(x) \to +\infty$ when $|x| \to +\infty$.

Let us recall assumption (A*) which states that the domain of the closed extension is nonempty and does not depend on the parameters α, i.e. there exists $\overline{\alpha}$ such that $D(\mathfrak{L}_{\alpha}^{\mu_\alpha}) = D(\mathfrak{L}_{\overline{\alpha}}^{\mu_{\overline{\alpha}}})$, for all $\alpha \in A$. We also recall $\langle g(\cdot), \mu\rangle := \int_M g(x)\mathrm{d}\mu(x)$.

The following is the manifold case analogue of Theorems 4.4.1 and 4.4.2.

Theorem 4.4.3. *Let M be a non-compact complete and connected smooth Riemannian manifold of dimension m. Assume a, b satisfy conditions (A1), (A2) and (A3) in local coordinates, (A*) holds and there exists a Lyapunov function* (4.4.19) *as in Lemma 4.6(i). If the primal problem*

$$\min_{q\in\mathcal{M}^+(M)} \left\{ \min_{\alpha(\cdot)\in\mathcal{A}} \quad \langle f(\cdot\,,\alpha(\cdot)), q\rangle, \quad s.t.\ 1 - \langle 1, q\rangle = 0 \text{ and } q \in Ker(\mathcal{L}_{\alpha}^{*}) \right\}$$

admits a solution $(\mu_{\alpha_\circ}, \alpha_\circ(\cdot))$, then for any $r \in [1,\infty)$ such that $f(\cdot\,,\alpha_\circ(\cdot)) \in L^r(M, \mu_{\alpha_\circ})$, there exists $(c, u(\cdot)) \in \mathbb{R} \times W^{r,2}_{loc}(M)$ solution to

$$\min_{\alpha\in A}\{-div(a(x,\alpha)\nabla u(x)) - b(x,\alpha)\cdot\nabla u(x) + f(x,\alpha)\} = c, \quad a.e.\ in\ M.$$

Moreover, the ergodic constant is defined by $c = \langle f(\cdot\,,\alpha_\circ(\cdot)), \mu_{\alpha_\circ}\rangle$ and $\alpha_\circ$ is such that

$$\alpha_\circ(x) \in \operatorname*{argmin}_{\alpha\in A}\{-div(a(x,\alpha)\nabla u(x)) - b(x,\alpha)\cdot\nabla u(x) + f(x,\alpha)\}, \quad a.e.\ in\ M.$$

Finally, the same holds when we have a max *(respec.* argmax*) instead of a* min *(respec.* argmin*).*

Proof. The duality procedure described earlier remains valid in the manifold setting. We conclude using Theorem 4.2.2 (i), the calmness property proved in Lemma 4.2, together with the optimality conditions as we did in the proof of Theorem 4.4.1. □

Obviously, with the same proof as in Lemma 4.2, one can show that the *primal* problem in the above theorem admits a solution provided f is integrable w.r.t μ. This integrability condition can be handled if one has an information on which are the moments of the invariant measure μ that are finite, and hence can make an assumption on the growth of f as we did in (A5). This is also the reason why the existence result in this setting is restricted to $W^{r,2}$ with r such that $f \in L^r(M, \mu)$. Indeed, this latter condition comes from Theorem 4.2.8 and was not needed in our previous results in §4.4, since we have in hand Lemma 2.3.1.

4.4.5 On the ergodic constant

In this subsection, we consider the particular case where the diffusion matrix (a^{ij}) is the identity I_m and we shall be interested in comparing the ergodic constants of two HJB equations when the vector field b and the function f vary. More precisely, we are given the two ergodic HJB equations:

$$\begin{aligned} -\Delta u(x) + \min_{\alpha\in A}\{ -b_1(x,\alpha)\cdot\nabla u(x) + f(x,\alpha) \} = U, \quad a.e.\ x\in\mathbb{R}^m \\ -\Delta v(x) + \min_{\alpha\in A}\{ -b_2(x,\alpha)\cdot\nabla v(x) + g(x,\alpha) \} = V, \quad a.e.\ x\in\mathbb{R}^m \end{aligned} \tag{4.4.20}$$

whose respective solutions (insured by Theorem 4.4.1) are $(U,u(\cdot))$ and $(V,v(\cdot))$. We have in addition (again using Theorem 4.4.1) the existence of $\alpha_1(\cdot)$ and $\alpha_2(\cdot)$ such that the ergodic constants are represented by

$$U = \langle f(\cdot\,,\alpha_1(\cdot))\,,\,\mu\rangle \quad \text{and} \quad V = \langle g(\cdot\,,\alpha_2(\cdot))\,,\,\nu\rangle. \tag{4.4.21}$$

where μ and ν are the invariant probability measures corresponding to the adjoint operator $\mathcal{L}^*_{\alpha_1}$ and $\mathcal{L}^*_{\alpha_2}$, respectively. And our aim is to provide an estimate on $|U-V|$ in terms of the data. Before we do so, we will need the following result borrowed from [40, p. 171].

Lemma 4.7. *Let μ and ν be two probability solutions to* (4.2.22) *with a diffusion matrix $A = I_m$ the identity, and with locally bounded Borel coefficients b_1 and b_2, respectively, i.e. $L^*_{I_m,b_1}\mu = 0$ and $L^*_{I_m,b_2}\nu = 0$. Suppose that*

(H.1) $|b_1 - b_2| \in L^2(\mathbb{R}^m;\nu)$,

(H.2) at least one of the following two conditions is fulfilled:

- *(a)* $(1+|x|)^{-1}|b_1(x)| \in L^1(\mathbb{R}^m;\nu)$
- *(b) there exists a function $V\in C^2(\mathbb{R}^m)$ such that $L_{I_m,b_1}V(x) \le MV(x)$ for all x and some $M>0$ and*

$$\lim_{|x|\to+\infty} V(x) = +\infty, \qquad \frac{(b_1-b_2)\cdot\nabla V}{1+V} \in L^1(\mathbb{R}^m;\nu),$$

(H.3) μ and ν have second moments,

(H.4) the measure μ satisfies the logarithmic Sobolev inequality with constant C.

Then

$$\|\mu-\nu\|^2 \le \frac{C}{2}\int_{\mathbb{R}^m} |b_1-b_2|^2\, d\nu. \tag{4.4.22}$$

In case (ii) *in* $(H.2)$, *the estimate* (4.4.22) *holds on a smooth Riemannian manifold (instead of* $\mathbb{R}^m$*) provided that the condition* $\lim_{|x|\to+\infty} V(x) = +\infty$ *is replaced by the requirement that the sets* $\{V \leq R\}$ *are compact.*

Proof *of Lemma 4.7.* The estimate (4.4.22) is [40, Corollary 4.5.9, p. 171] and is based on an estimate in [40, Theorem 4.5.8, p. 171]. □

Remark 4.4.7. *Note that our standing assumptions already guarantee the fulfillment of (H.1, H.2(i), H.3) above. If moreover we strengthen assumption (A4) and consider instead*

$\mathbf{A}_\gamma$**.** $\quad (b(x) - b(y)) \cdot (x - y) \leq -\gamma|x - y|^2$ *for some* $\gamma > 0$ *and all* $x, y \in \mathbb{R}^m$,

where b *is again Borel locally bounded vector field on* $\mathbb{R}^m$, *and* (A_γ) *is satisfied uniformly in* α, *then (H.2(ii)) is satisfied with* $V(x) = |x|^2$ *and (H4) is satisfied with* $C = 2/\gamma$ *(see [40, Theorem 5.6.36, p. 225]).*

Proposition 4.4.2. *Assume the two ergodic HJB equations* (4.4.20) *fall in the framework of Theorem 4.4.1 where assumption (A4) is now replaced with* (A_γ). *Then there exists a constant* $M > 0$ *such that we have the following*

$$|U - V| \leq \|f - g\|_{L^1(\mathbb{R}^m;\nu)} + M\,\|b_1 - b_2\|^{1/2}_{L^2(\mathbb{R}^m;\nu)} \tag{4.4.23}$$

Such an estimate is important for applications to problems in singular perturbations and homogenization, and it is a refinement of [26, Proposition 4.4].

We recall the weighted norm, for $1 \leq p < +\infty$, defined (when exists) by

$$\|f\|_{L^p(\mathbb{R}^m;\nu)} := \left(\int_{\mathbb{R}^m} |f(x)|^p \,\mathrm{d}\nu(x)\right)^{\frac{1}{p}}$$

Proof *of Proposition 4.4.2.* For simplicity of notation, we shall drop in the sequel the dependency on α.
Let us consider the representation formula (4.4.21) of U and V, we have

$$\begin{aligned} U - V &= \langle f, \mu\rangle - \langle g, \nu\rangle \\ &= \langle f, \mu - \nu\rangle + \langle f - g, \nu\rangle. \end{aligned}$$

And observe that

$$\langle f - g, \nu\rangle \leq \int_{\mathbb{R}^m} |f(x) - g(x)| \,\mathrm{d}\nu(x) = \|f - g\|_{L^1(\mathbb{R}^m;\nu)}.$$

For the other term, using Cauchy-Schwarz inequality, and denoting by $\mathrm{d}\mu = \rho_\mu \mathrm{d}x$ and $\mathrm{d}\nu = \rho_\nu \mathrm{d}x$ the respective densities w.r.t. Lebesgue measure, we have

$$\begin{aligned}\langle f, \mu - \nu \rangle &= \int_{\mathbb{R}^m} f(x)\big(\rho_\mu(x) - \rho_\nu(x)\big)\,\mathrm{d}x \\ &\leq \left(\int_{\mathbb{R}^m} |f(x)|^2 \,|\rho_\mu(x) - \rho_\nu(x)|\,\mathrm{d}x\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^m} |\rho_\mu(x) - \rho_\nu(x)|\,\mathrm{d}x\right)^{\frac{1}{2}} \\ &\leq K_f \left(\int_{\mathbb{R}^m} \left(1 + |x|^d\right)^2 |\rho_\mu(x) - \rho_\nu(x)|\,\mathrm{d}x\right)^{\frac{1}{2}} \|\mu - \nu\|^{1/2}\end{aligned}$$

where in the last inequality, we have used assumption (A5). We can now use the estimate (4.4.22) and get

$$\langle f, \mu - \nu \rangle \leq M \,\|b_1 - b_1\|^{1/2}_{L^2(\mathbb{R}^m;\nu)}$$

where $M := \frac{K_f}{\gamma} \left(\int_{\mathbb{R}^m} \left(1 + |x|^d\right)^2 |\rho_\mu(x) - \rho_\nu(x)|\,\mathrm{d}x\right)^{\frac{1}{2}}$ and γ is as in assumption (A_γ) (see Remark 4.4.7). $\square$

4.5 Ergodic Mean-Field Games

4.5.1 Introduction

This section is devoted to the problem of existence and uniqueness of solutions to ergodic mean-field games (MFG) in the whole space $\mathbb{R}^m$ with unbounded data satisfying subexponential growth. Such a problem writes as

$$\begin{gathered}\textit{Find } (c, u, \mu) \in \mathbb{R} \times \mathcal{X}(\mathbb{R}^m) \times \mathcal{P}(\mathbb{R}^m), \textit{ s.t.:} \\ H(x, \nabla u(x), D^2 u(x), \mu) = c \quad \text{and} \quad -\mathcal{L}^*_{\alpha[u,\mu]}\mu = 0\end{gathered} \tag{4.5.1}$$

where $\mathcal{X}$ is a functional space (part of the unknowns), $\mathcal{P}$ is the set of probability measures and the Hamiltonian is of one of the two forms

$$H := \min_{\alpha \in A}\{\, -\mathcal{L}_\alpha u(x) + f(x, \alpha, \mu)\,\}, \quad \text{or} \quad H := \max_{\alpha \in A}\{\, -\mathcal{L}_\alpha u(x) + f(x, \alpha, \mu)\,\},$$

the diffusion operator $\mathcal{L}_\alpha$ is a linear operator given by

$$\mathcal{L}_\alpha \varphi(x) := \mathrm{trace}\big(a(x, \alpha) D^2 \varphi(x)\big) + b(x, \alpha) \cdot \nabla \varphi(x)$$

and its adjoint $\mathcal{L}_\alpha^*$ is then

$$\mathcal{L}_\alpha^*\mu(x) = \mathrm{trace}\big(D^2(a(x,\alpha)\mu(x))\big) - \mathrm{div}\big(b(x,\alpha)\mu(x)\big).$$

The second equation in (4.5.1) is nothing but $-\mathcal{L}_\alpha^*\mu = 0$ where we anticipate the dependence of α on (u,μ). The (*control*) parameters α are in a compact set A of $\mathbb{R}^k$ for some $k > 0$. The first equation is a Hamilton-Jacobi-Bellman equation (HJB for short) and the second one is a Fokker-Planck-Kolmogorov equation (FPK for short).

We denote by $\mathcal{M}(\mathbb{R}^m)$ (respec. $\mathcal{M}^+(\mathbb{R}^m)$) the space of totally finite signed (respec. non-negative) Borel measures on $\mathbb{R}^m$. With slight abuse of notation, an element $\mu \in \mathcal{M}(\mathbb{R}^m)$ will denote either a measure or a density (when exists). When μ is absolutely continuous with respect to (w.r.t) Lebesgue measure dx; we write $\mu \ll dx$. For some $d \geq 1$, we denote by $\mathcal{M}_d(\mathbb{R}^m)$ the subset of measures with finite d-moment. We equip $\mathcal{M}(\mathbb{R}^m)$ with the *Total-Variation* (TV) norm. And we denote by $\mathcal{P}(\mathbb{R}^m)$ the subset of probability measures. We write shortly for any measurable function $g : \mathbb{R}^m \to \mathbb{R}$ and $\mu \in \mathcal{M}(\mathbb{R}^m)$

$$\langle g(\cdot)\,,\mu\rangle = \int_{\mathbb{R}^m} g(x)\mathrm{d}\mu(x).$$

We recall that the differential operator $\mathcal{L}_\alpha$ can be interpreted as the infinitesimal generator of the controlled stochastic process

$$dX_t = b(X_t,\alpha_t)dt + \sqrt{2}\varrho(X_t,\alpha_t)dB_t \tag{4.5.2}$$

where B_t is a Wiener process while f is the running cost of a stochastic control problem. Note that (4.5.2) should be understood in its weak sense (see e.g. [114, 115]).

Throughout this section, we will make the following assumptions and refer to them wherever it is needed.

A0. The dimension $m \geq 2$.

A1. (i) $a = (a_\alpha^{ij})$ is a continuous mapping (uniformly in α) on $\mathbb{R}^m$ such that $a(x,\alpha) = \varrho(x,\alpha)\varrho(x,\alpha)^\top$ where ϱ is a continuous in x (uniformly in α) $m \times m_1$ matrix function (for some $m_1 \geq m$) and Borel-measurable in α,

(ii) $b = (b_\alpha^i) : \mathbb{R}^m \times A \to \mathbb{R}^m$ is a locally bounded Borel-measurable vector field.

A2. For $p > m$, $a^{ij}(\cdot,\alpha) \in W^{p,1}_{\mathrm{loc}}(\mathbb{R}^m)$ and $b^i(\cdot,\alpha) \in L^p_{\mathrm{loc}}(\mathbb{R}^m)$, uniformly in $\alpha \in A$.

A3. There exist $\overline{\Lambda} \geq \underline{\Lambda} > 0$ such that $\forall\, x,\xi \in \mathbb{R}^m,\ \underline{\Lambda}\|\xi\|^2 \leq \xi a(x,\alpha)\cdot\xi \leq \overline{\Lambda}\|\xi\|^2$, uniformly in $\alpha \in A$, i.e. (a^{ij}) is positive, uniformly bounded and nondegenerate.

A4. $\lim\limits_{|x|\to\infty} \sup\limits_{\alpha\in A} b(x,\alpha)\cdot x = -\infty$ (Recurrence condition).

A5. $f : \mathbb{R}^m \times A \times \mathcal{M}(\mathbb{R}^m) \to \mathbb{R}$ is such that

(i) $x \mapsto f(x,\alpha,\mu)$ is Borel-measurable on $\mathbb{R}^m$,

(ii) $\alpha \mapsto f(x,\alpha,\mu)$ is continuous on $A \subset \mathbb{R}^k$,

(iii) $f(\cdot\,,\alpha,\mu) \in L^1(\mathbb{R}^m; \mathrm{d}\mu)$, , uniformly in α, for $d \geq 1$ and for every $\mu \in \mathcal{M}_d^+(\mathbb{R}^m)$ such that $\mu \ll dx$,

(iv) $\mu \mapsto f(x,\alpha,\mu)$ has a *Fréchet* (or strong) directional derivative at every $\mu \in \mathcal{M}_d^+(\mathbb{R}^m)$ such that $\mu \ll dx$, i.e.

$$f(x,\alpha,\mu+h) = f(x,\alpha,\mu) + D_\mu f(x,\alpha,\mu)[h] + o(\|h\|), \quad \forall\, h \in \mathcal{M}_d(\mathbb{R}^m)$$

where $D_\mu f(x,\alpha,\mu)[h]$ is a bounded linear continuous functional of h and $\|h\|$ is its TV-norm.

(v) The Fréchet directional derivative of f in μ satisfies on the subset $\mathcal{M}_d^+(\mathbb{R}^m)$, uniformly in α, and for every $\mu \in \mathcal{M}_d^+(\mathbb{R}^m)$ such that $\mu \ll dx$

$$\langle\, D_\mu f(\cdot\,,\alpha,\mu)[h]\,,\,\mu\,\rangle \leq 0, \quad \forall\, h \in \mathcal{M}_d^+(\mathbb{R}^m).$$

A6. $\exists\, K_b > 0$ and $\beta \in [0,d]$ such that $|b(x,\alpha)| \leq K_b(1+|x|^\beta)$ for all $x \in \mathbb{R}^m, \alpha \in A$.

Whenever μ is an absolutely continuous measure on $\mathbb{R}^m$, we shall identify it with its Lebesgue density.

Notation. We shall keep the same notation $f(x,\alpha,\mu)$ whether f depends on μ in a *local* way, i.e. when we have $f(x,\alpha,\mu(x))$ defined on $\mathbb{R}^m \times A \times \mathbb{R}$, or f depends on μ in a *non-local* way, i.e. when we have $f(x,\alpha,\mu)$ defined on $\mathbb{R}^m \times A \times \mathcal{M}(\mathbb{R}^m)$, having in mind that one can represent (in the *local* case) $\mu(x)$ as a convolution with a Dirac measure with unit mass concentrated at zero, i.e. $\delta_0 * \mu(x)$.

Assumption (A4) is reminiscent of the existence of a Lyapunov function $w \in C^2(\mathbb{R}^m)$ s.t. $\lim\limits_{|x|\to\infty} w(x) = +\infty$ and $\lim\limits_{|x|\to\infty} \mathcal{L}w(x) = -\infty$; see Corollary 4.2.2.

In assumption (A5-(iii)), what we are asking is a polynomial growth of f in x of order at most d, since μ here is taken among $\mathcal{M}_d(\mathbb{R}^m)$. This is in fact a subexponential growth since $d \geq 1$ can be arbitrarily chosen. One can still handle an exponential growth provided assumption (A4) is strengthened (see Remark 4.2.2).

With assumption (A5-(iv)), $\mu \mapsto f(x,\alpha,\mu)$ is *Fréchet*-differentiable and hence there also exists *Gateaux* (directional) derivative and we have

$$\lim_{t\downarrow 0} t^{-1}(f(x,\alpha,\mu+th) - f(x,\alpha,\mu)) = D_\mu f(x,\alpha,\mu)[h], \quad \forall\, h \in \mathcal{M}_d(\mathbb{R}^m).$$

In particular, $\mu \mapsto f(x,\alpha,\mu)$ is continuous in the TV-norm.

Note that assumption (A5-(v)) is reminiscent of (but not exactly) the monotonicity assumption discovered by Lasry and Lions [122], usually present in the MFG literature [54, 55] and which writes (with our notations) as

$$\int_{\mathbb{R}^m} \big(f(x,\alpha,\mu_1) - f(x,\alpha,\mu_2)\big)\,\mathrm{d}(\mu_1-\mu_2)(x) \le 0, \quad \forall\, \mu_1,\mu_2 \in \mathcal{M}_d(\mathbb{R}^m). \tag{M}$$

Setting $\mu_1 = \mu + h$ and $\mu_2 = \mu$, and assuming f is Fréchet differentiable in the μ-variable, then one gets

$$\int_{\mathbb{R}^m} \big(f(x,\alpha,\mu+h) - f(x,\alpha,\mu)\big)\,\mathrm{d}h(x) = \langle\, D_\mu f(\cdot\,,\alpha,\mu)[h]\,,\, h\,\rangle + o(\|h\|^2)$$

and condition (M) hence implies

$$\langle\, D_\mu f(\cdot\,,\alpha,\mu)[h]\,,\, h\,\rangle \le 0, \quad \forall\, \mu, h \in \mathcal{M}_d(\mathbb{R}^m). \tag{M'}$$

We shall discuss this later in Remark 4.5.1.

Finally, it is worth mentioning that by requiring in (A5-(i)) the function f to be measurable only in x, we bypass regularity requirements of the measure μ on x in the case f locally depends on μ.

The **main result** (see Theorem 4.5.1 & Theorem 4.5.2) can be informally stated as: *Under assumptions including (A0-A6), the following are equivalent*

(I) There exists a pair $(q_\circ, \alpha_\circ)$ *such that*

$$(q_\circ,\alpha_\circ) \in \operatorname*{argmin}_{\substack{q\in\mathcal{M}_d^+(\mathbb{R}^m)\\ \alpha(\cdot)\in\mathcal{A}}} \Big\{\, \langle\, f(\cdot\,,\alpha(\cdot),q)\,,\, q\,\rangle\,, \quad \textit{s.t.:}\;\; 1 - \langle 1, q\rangle = 0 \;\textit{and}\;\; q \in Ker(\mathcal{L}^*_\alpha)\Big\},$$

where $\mathcal{A}$ *is the set of measurable functions from* $\mathbb{R}^m$ *to* A*.*

(II) There exists $(c_\circ, u_\circ, q_\circ) \in \mathbb{R} \times W^{r,2}_{loc}(\mathbb{R}^m) \times W^{s,1}_{loc}(\mathbb{R}^m)$ *for any* $r \geq 1$, $s > m$, *and there exists a measurable function* $\alpha_\circ(\cdot) : \mathbb{R}^m \to A$ *that solve the coupled system*

$$\begin{cases} -\, trace\big(a(x, \alpha_\circ(x)) D^2 u_\circ(x)\big) - b(x, \alpha_\circ(x)) \cdot \nabla u_\circ(x) + f(x, \alpha_\circ(x), q_\circ) = c_\circ \\ -\, trace\big(D^2(a(x, \alpha_\circ(x)) q_\circ(x))\big) + div\big(b(x, \alpha_\circ) q_\circ(x)\big) = 0, \qquad a.e.\ in\ \mathbb{R}^m \end{cases}$$

and such that

(i) the constant $c_\circ$ *is defined by* $c_\circ = \langle f(\cdot\,, \alpha_\circ(\cdot), q_\circ), q_\circ \rangle$,

(ii) the function $\alpha_\circ(\cdot)$ *satisfies*

$$\alpha_\circ(x) \in \underset{\alpha \in A}{\operatorname{argmin}} \{-\mathcal{L}_\alpha u_\circ(x) + f(x, \alpha, q_\circ)\}, \quad a.e.\ \ x \in \mathbb{R}^m.$$

We will show in particular (see Corollary 4.5.1) that in the case where f is *separable*, i.e. $f(x, \alpha, q) = g(x, \alpha) + k(x, q)$, statement (I) above is satisfied, and hence there exists a solution to the MFG system as in (II). In the non-*separable* case, the same result holds (see Corollary 4.5.2) under additional assumptions (mainly smoothness of the coefficients) that we will later make precise.

The theory of Mean-Field Games started with the seminal work of [95, 120, 121, 122]. Since then there is a huge literature on MFGs in general and those of ergodic type in particular, with mainly two approaches: PDEs or probability, but also with many connections with control theory, differential games and optimal transport. For ergodic MFGs, we would like to refer to [61] and the many references therein. However many of the existing results consider bounded domains (mainly the torus), and very few treat the problem in the whole space. We refer to [24, 25] for the linear-quadratic setting where the solvability of the MFG system is reduced to the solvability of an algebraic Riccati equation and a Sylvester equation which also allow to get (at least in some examples) explicit solutions. In [60], existence of classical solutions is proved in the whole space $\mathbb{R}^m$ for ergodic (stationary) MFGs of the form

$$\begin{cases} -\, \varepsilon \Delta u + H(Du) + c = g(m) + V(x) \\ -\, \varepsilon \Delta q - div(q DH(Du)) = 0, \text{ in } \mathbb{R}^m \end{cases}$$

where the potential V is assumed to be coercive and g is a local coupling term. The Hamiltonian H also satisfies some growth assumptions. Their approach is variational based on the analysis of the non-convex energy associated to the system. They have also studied the vanishing viscosity limit, i.e. when $\varepsilon \to 0$. Another work in this same vein is the one in [66] where the coupling term is local, decreasing and unbounded satisfying some growth conditions. In this case, existence and non-existence results are shown using Sobolev regularity of the invariant measure and a blow-up procedure, and additional results in the case where the coupling term is local and increasing are also proven. In the latter references, the setting is (with our notations) $a = I$ identity matrix, $b(x, \alpha) = \alpha$, $f(x, \alpha, q) = H^*(\alpha) - V(x) - g(q)$ where H^* is Legendre transform of H which is usually assumed to behave as a power $H(p) = \frac{1}{\gamma}|p|^\gamma, \gamma > 1$ (and hence also H^*). Another difference is that we are interested in *weak solutions* whereas they are concerned with *classical solutions*. Another recent result with a setting that is closer to ours is the one in [6]. Their setting is the one of (ergodic) stochastic control: the drift $b = b(x, \alpha)$ and the diffusion term $\varrho = \varrho(x)$ in (4.1.2) are locally Lipschitz with an affine growth and local non-degeneracy, and the running cost f satisfies some growth conditions. They proved existence of MFG solutions defined as:

$\eta \in C([0, +\infty), \mathcal{P}(\mathbb{R}^m))$ for which there exists v. such that

$$\begin{gathered} \mathrm{d}X_t = b(X_t, v_t) + \varrho(X_t)\mathrm{d}W_t, \\ \text{with } \mathrm{Law}(X_t) = \eta_t, \quad X_0 = x \end{gathered}$$

and $J_x(U, \eta) \geq J_x(v, \eta)$ for all admissible controls U where

$$J_x(U, \eta) = \limsup_{T \to +\infty} \frac{1}{T} \mathbb{E}_x \left[\int_0^T f(X_t, U_t, \eta_t) \mathrm{d}t \right].$$

is the objective function of the ergodic stochastic control problem.

When v above takes values in $\mathcal{P}(A)$, the MFG solution is said to be *relaxed*, and when it takes values in A, the solution is said *strict*. Their approach is based on the ergodic control formulation and relies on regularity of set-values maps corresponding to ergodic occupation measures and invariant measures together with an application of Kakutani-Fan-Glicksberg fixed point theorem and convex analytic tools.

Our method seems to be new in this regard. It relies on optimization on abstract Banach spaces, taking advantage of existing results in the theory of Dirichlet forms (and diffusion operators). We shall also work with the Total-Variation norm (and not the Wasserstein metric as it is customary); see Remark 4.5.3. Finally, let us mention that our assumptions concern the coefficients of the diffusion operator (or the underlying stochastic differential equation) rather than the structure of the Hamiltonian.

The following is organized as follows. In **Section 4.5.2** we prove preliminary results that will be needed throughout this section. In particular, we will define the *primal* and *dual* optimization problems and also prove *calmness* property which plays a key role in the sequel. We will then be ready in **Section 4.5.3** to state and prove the main existence and uniqueness results for the ergodic MFG system.

4.5.2 Preliminary results

Let us denote again by $\mathcal{M}(\mathbb{R}^m)$ the space of totally finite signed Borel measures on $\mathbb{R}^m$ and equipped with the *Total Variation* norm[9] $\|\mu\| = \mu^+(\mathbb{R}^m) + \mu^-(\mathbb{R}^m)$, where $\mu = \mu^+ - \mu^-$ is the Jordan decomposition of μ. It is known that $(\mathcal{M}(\mathbb{R}^m), \|\cdot\|)$ is a Banach space (see e.g. [74, §IV.2.16]), and hence also a locally convex topological vector space when equipped with its norm topology.

We also denote by $\mathcal{M}_d(\mathbb{R}^m)$ (respectively, $\mathcal{M}_d^+(\mathbb{R}^m)$) the subset of signed (resp. non-negative) totally finite Borel measures with finite moments of order d, where we recall d is the growth order of f as in assumption (A5). And since the latter two subsets are closed, they are also Banach spaces.

Let us define the duality product in $\mathcal{M}_d(\mathbb{R}^m)$ by

$$\langle h(\cdot), \mu \rangle = \int_{\mathbb{R}^m} h(x) d\mu(x), \quad \text{for all } \ \mu \in \mathcal{M}_d(\mathbb{R}^m)$$

where h is a Borel measurable function with at most a polynomial growth of order d.

Recall that a linear functional h on the normed space $(\mathcal{M}(\mathbb{R}^m), \|\cdot\|)$ is continuous if and only if it is bounded on the unit ball, i.e. if

$$\|h\|_* := \sup_{\|\mu\| \leq 1} \langle h(\cdot), \mu \rangle \ < \infty$$

And so, the topological dual space $(\mathcal{M}(\mathbb{R}^m))^*$ (i.e. set of continuous linear functionals, equipped with the dual norm $\|\cdot\|_*$) is again a Banach space. It is easy to see that Borel-measurable functions with at most a polynomial growth of order d are in $(\mathcal{M}_d(\mathbb{R}^m))^*$.

It can be quite hard to deal with $(\mathcal{M}(\mathbb{R}^m))^*$ which can indeed be seen as the bidual of the space of continuous and bounded functions. But we will see that we can avoid these difficulties provided we find a subset of the latter, which will turn out to be more convenient to work with. We refer the interested reader to the work of S. Kaplan on the bidual of the space of continuous functions [106, 107].

[9] To check it is a norm, the only technical step is in the triangle inequality; to prove that $\|\mu + \nu\| \leq \|\mu\| + \|\nu\|$ we need to consider a Hahn decomposition $\mathbb{R}^m = A \uplus B$ for $\mu + \nu$.

We denote by $C_0^\infty(\mathbb{R}^m)$ the set of all real-valued, infinitely differentiable functions on $\mathbb{R}^m$ with compact support. And we define as usual $W^{p,k}(\mathbb{R}^m)$, for $p \geq 1, k > 0$, the Sobolev space of all functions on $\mathbb{R}^m$ with generalized derivatives up to order k in $L^p(dx)$, where dx denotes Lebesgue measure on $\mathbb{R}^m$. $W^{p,k}_{\mathrm{loc}}(\mathbb{R}^m)$ denotes the corresponding local Sobolev space, i.e. $f \in W^{p,k}_{\mathrm{loc}}(\mathbb{R}^m)$ if $\zeta f \in W^{p,k}(\mathbb{R}^m)$ for all $\zeta \in C_0^\infty(\mathbb{R}^m)$.

We shall also need an assumption that will play a crucial role in the validity of our method: besides the standing assumptions (A0-A6), we denote again by the operator $(\mathcal{L}_\alpha, D(\mathcal{L}_\alpha))$ its closed extension $(\overline{L}^\mu_{A,b}, D(\overline{L}^\mu_{A,b}))$ as given by Theorem 4.2.7 and Theorem 4.2.8 and we assume the following holds true

(A*) The domain $D(\mathcal{L}_\alpha)$ of the closed extension is nonempty and is independent of α.

Such an assumption can be encountered for example in [85, §III.6, p. 130]. It means that there exists $\widetilde{\alpha}(\cdot) \in \mathcal{A}$ such that for all $\alpha(\cdot) \in \mathcal{A}$, one has $D(\mathcal{L}_\alpha) = D(\mathcal{L}_{\widetilde{\alpha}})$, and $\mathcal{L}_{\widetilde{\alpha}}$ falls in the framework of the previous sections, in particular it satisfies Theorem 4.2.8. The nonemptiness assumption is trivial otherwise the PDE problem (4.5.1) does not make sense. We will hereafter denote by $D(\mathcal{L}_0)$ the later domain. We will see that the functional space $\mathcal{X}$ is a subset of $D(\mathcal{L}_0)$ and we will shortly after make it precise.

Before we go any further, let us comment in the following remark on our assumption (A5-(v)).

Remark 4.5.1. *There is a twofold difference between* (M') *and our assumption (A5-(v)): firstly, the choice of measures in* (M') *is the whole space $\mathcal{M}_d(\mathbb{R}^m)$, whereas in our case we require the assumption to hold only in the positive cone*[10] *$\mathcal{M}_d^+(\mathbb{R}^m)$; secondly, the averaging $\langle\, \cdot\,,\, h\,\rangle$ in* (M') *is taken with respect to the same measure h as in the Fréchet derivative $D_\mu f(x, \alpha, \mu)[h]$, whereas in our case, the averaging $\langle\, \cdot\,,\, \mu\,\rangle$ is taking w.r.t. the measure μ where the derivative has been computed. This difference makes it difficult to compare the two conditions. However, in the case f depends linearly on the measure μ, e.g.*

$$f(x, \alpha, \mu) = \int_{\mathbb{R}^m} K(x - y, \alpha)\, d\mu(y),$$

then the Fréchet derivative $D_\mu f(x, \alpha, \mu)[h] = f(x, \alpha, h)$ is independent of μ. Hence, our condition (A5-(v)) requires $\langle f(\,\cdot\,, \alpha, h)\,,\, \mu\rangle \leq 0$ for all $h, \mu \in \mathcal{M}_d^+(\mathbb{R}^m)$, while condition (M') *writes as $\langle f(\,\cdot\,, \alpha, h)\,,\, h\rangle \leq 0$ for all $h \in \mathcal{M}_d(\mathbb{R}^m)$. Therefore, in the case of a linear dependency on the measure, (A5-(v)) is stronger than* (M') *when restricted to the*

[10] In (A5-(v)), we ask $\langle\, D_\mu f(\cdot\,, \alpha, \mu)[h]\,, \mu\,\rangle \leq 0$ to hold $\forall\, h, \mu \in \mathcal{M}_d^+(\mathbb{R}^m)$ s.t. $\mu \ll dx$. But in this ongoing discussion, we forget deliberately about absolute continuity of μ w.r.t. Lebesgue measure in order to focus rather on the *structure* of the assumption when compared to (M').

positive cone $\mathcal{M}_d^+(\mathbb{R}^m)$*. If we assume in addition that the kernel* $K(\cdot\,,\alpha)$ *is odd, then condition (A5-(v)) implies* (M')*. Indeed, writing* $h = h^+ - h^-$ *the Jordan decomposition of* h*, one has as (we drop the dependency on* α*)*

$$\begin{aligned}\iint K(x-y)\,dh(y)\,dh(x) = &\iint K(x-y)\,dh^+(y)\,dh^+(x) + \iint K(x-y)\,dh^-(y)\,dh^-(x)\\ &- \iint K(x-y)\,dh^+(y)\,dh^-(x) - \iint K(x-y)\,dh^-(y)\,dh^+(x).\end{aligned}$$

Assuming K *to be odd, the last term can be written as*

$$\begin{aligned}\iint K(x-y)\,dh^-(y)\,dh^+(x) &= -\iint K(y-x)\,dh^-(y)\,dh^+(x)\\ &= -\iint K(x-y)\,dh^-(x)\,dh^+(y), \qquad (i)\\ &= -\iint K(x-y)\,dh^+(y)\,dh^-(x), \qquad (ii)\end{aligned}$$

where in line (i) *we exchanged the notations of the mute variables* x *and* y*, and then in line* (ii) *we exchanged the order of the two integrals. Substituting the latter term in the previous equality, it cancels out and condition* (M') *writes as*

$$\begin{aligned}\iint K(x-y)\,dh(y)\,dh(x) = &\iint K(x-y)\,dh^+(y)\,dh^+(x) + \iint K(x-y)\,dh^-(y)\,dh^-(x)\\ &\leq 0.\end{aligned}$$

Therefore, noting that $h^+, h^- \in \mathcal{M}_d^+(\mathbb{R}^m)$*, one gets: (A5-(v)) implies* (M')*.*
A nonlinear version of this example can be

$$f(x,\alpha,\mu) = F\left(x,\alpha,\int_{\mathbb{R}^m} K(x-y,\alpha)\,d\mu(y)\right).$$

Let us denote by D_3 *the derivative in the third variable of* $F : \mathbb{R}^m \times A \times \mathbb{R} \to \mathbb{R}$*. Then one gets*

$$\begin{aligned}D_\mu f(x,\alpha,\mu)[h] &= \int_{\mathbb{R}^m} D_3F\left(x,\alpha,\int_{\mathbb{R}^m} K(x-z,\alpha)\,d\mu(z)\right) K(x-y,\alpha)\,dh(y)\\ &= \int_{\mathbb{R}^m} \phi(x,\alpha,\mu) K(x-y,\alpha)\,dh(y)\end{aligned}$$

where ϕ is the term coming from D_3F in the previous line. In this case, assumption (M') *writes as:*

$$\iint_{\mathbb{R}^{2m}} [\phi(x,\alpha,\mu)K(x-y,\alpha)] \; dh(y)\, dh(x) \le 0, \quad \forall\, h \in \mathcal{M}_d(\mathbb{R}^{2m}),$$

and assumption (A5-(v)) is now:

$$\iint_{\mathbb{R}^{m}} [\phi(x,\alpha,\mu)K(x-y,\alpha)] \; dh(y)\, d\mu(x) \le 0, \quad \forall\, h,\mu \in \mathcal{M}_d^+(\mathbb{R}^{m}).$$

In this example, it is sufficient to have the term between brackets non-positive almost everywhere to satisfy assumption (A5-(v)) since h,μ are non-negative measures. But this is not sufficient to guarantee assumption (M') *since h can be any (signed) measure. We refer to [20, §2.3] for various other examples with different interpretations on the convolution kernel K considered above.*

4.5.2.1 An exchange property

The following proposition allows us to exchange the order of the minimization (or maximization) with the integration with respect to a measure $q \in \mathcal{M}_d^+(\mathbb{R}^m)$, i.e. non-negative totally finite Borel measure with finite moment of order d.

Proposition 4.5.1. *Let f satisfies (A5). The following holds for any $q \in \mathcal{M}_d^+(\mathbb{R}^m)$*

$$\int_{\mathbb{R}^m} \min_{\alpha\in A} f(x,\alpha,q)\, dq(x) = \min_{\alpha(\cdot)\in\mathcal{A}} \int_{\mathbb{R}^m} f(x,\alpha(x),q)\, dq(x) \tag{4.5.3}$$

where A is a compact subset of $\mathbb{R}^k$, for some $k>0$, and $\mathcal{A}$ is the set of measurable functions $\alpha(\cdot) : \mathbb{R}^m \to A$. And the same holds true with max *instead of* min*.*

Remark 4.5.2. *In the context of stochastic control, the set $\mathcal{A}$ needs to be the one of progressively measurable functions. In fact, these are the admissible controls.*

Proof. We refer to Proposition 4.4.1 where the same proof holds in the present setting, provided we let $q \in \mathcal{M}_d^+(\mathbb{R}^m)$ be arbitrarily fixed and $f : \mathbb{R}^m \times A \times \mathcal{M}_d^+(\mathbb{R}^m) \to \mathbb{R}$ satisfies (A5-(i,ii,iii)). □

The *exchange property* proved in Proposition 4.4.1 will be much needed in the sequel. It insures that we can exchange the minimization over the parameters α and the duality product in $\mathcal{M}_d(\mathbb{R}^m)$ provided we define the second argument in f as measurable functions $\alpha(\cdot) \in \mathcal{A}$ instead of vectors $\alpha \in A$, that is

$$\min_{\alpha(\cdot)\in\mathcal{A}} \langle\, f(\cdot\,,\alpha(\cdot),q)\;,\; q\,\rangle = \langle\, \min_{\alpha\in A} f(\cdot\,,\alpha,q)\;,\; q\,\rangle$$

4.5.2.2 The primal problem

In what follows, we will first deal with the case where the Hamiltonian is given by

$$H(x,\nabla u(x),D^2u(x),q) = \min_{\alpha\in A}\{\,-\mathcal{L}_\alpha u(x) + f(x,\alpha,q)\,\}.$$

Using the notation $a\wedge b = \min(a,b)$, $a \vee b = \max(a,b)$, we will refer to the *primal* problem by $(\mathfrak{P}^\wedge)$ and its *dual* by $(\mathfrak{D}^\wedge)$. We will then recover the case where we have in the Hamiltonian a max instead of a min, and use the notation $(\mathfrak{P}^\vee)$ and $(\mathfrak{D}^\vee)$.

We state our *primal* problem as follows

$$\min_{q\in\mathcal{M}_d^+(\mathbb{R}^m)}\left\{\min_{\alpha(\cdot)\in\mathcal{A}}\quad \langle f(\cdot\,,\alpha(\cdot),q),q\rangle,\quad \text{s.t.: } 1-\langle 1,q\rangle = 0 \text{ and } q\in\mathrm{Ker}(\mathcal{L}_\alpha^*)\right\} \qquad (\mathfrak{P}^\wedge)$$

where we recall $\langle f(\cdot\,,\alpha(\cdot),q),q\rangle = \int_{\mathbb{R}^m} f(x,\alpha(x),q)\mathrm{d}q(x)$. For the convenience of the reader, we will use the same notation as in §4.2.1, that is,

$$X = \mathcal{M}_d(\mathbb{R}^m) \quad\text{and}\quad Q = \mathcal{M}_d^+(\mathbb{R}^m)$$

$$G_1 : X \to \mathbb{R},\quad \text{s.t.}\quad G_1(q) = 1-\langle 1,q\rangle$$

$$G_2 : X\to X,\quad \text{s.t.}\quad G_2(q) = q$$

$$G = (G_1,G_2)\quad\text{and}\quad Y = \mathbb{R}\times X$$

$$K_1 = \{0\},\; K_2(\alpha) = \mathrm{Ker}(\mathcal{L}_\alpha^*)\quad\text{and}\quad K_\alpha = K_1\times K_2(\alpha)\subset Y$$

The *primal* problem then writes

$$\min_{q\in Q}\left\{\min_{\alpha(\cdot)\in\mathcal{A}}\langle f(\cdot\,,\alpha(\cdot),q),q\rangle,\quad\text{s.t.: } G(q)\in K_\alpha\right\} \qquad (\mathfrak{P}^\wedge)$$

Setting $F(G(q),\alpha) := I_{K_\alpha}(G(q))$ the indicator function which is 0 if $G(q)\in K_\alpha$ and $+\infty$ otherwise, we can finally write the *primal* problem as

$$\min_{q\in Q}\left\{\min_{\alpha(\cdot)\in\mathcal{A}}\langle f(\cdot\,,\alpha(\cdot),q),q\rangle + F(G(q),\alpha)\right\}. \qquad (\mathfrak{P}^\wedge)$$

In the case where the Hamiltonian is given by

$$H(x, \nabla u(x), D^2u(x), q) = \max_{\alpha \in A}\{ -\mathcal{L}_\alpha u(x) + f(x, \alpha, q) \}.$$

we write the *primal* problem in the form

$$\min_{q \in \mathcal{M}_d^+(\mathbb{R}^m)} \left\{ \min_{\alpha(\cdot) \in \mathcal{A}} \langle -f(\cdot\,, \alpha(\cdot), q), q \rangle, \quad \text{s.t.: } 1 - \langle 1, q \rangle = 0 \text{ and } q \in \operatorname{Ker}(\mathcal{L}_\alpha^*) \right\}, \qquad (\mathfrak{P}^\vee)$$

that is,

$$\min_{q \in Q} \left\{ \min_{\alpha(\cdot) \in \mathcal{A}} \langle -f(\cdot\,, \alpha(\cdot), q), q \rangle, \quad \text{s.t.: } G(q) \in K_\alpha \right\}. \qquad (\mathfrak{P}^\vee)$$

Remark 4.5.3. *The Total-Variation (TV) norm, although it is somehow dictated by the results in §4.2.2, seems to be natural in regards to our primal problem* $(\mathfrak{P}^\vee)$ *where the constraint* $q \in Ker(\mathcal{L}_\alpha^*)$ *is nothing but* (4.2.22) *in §4.2.3.2, that is requiring* q *to be an invariant (stationary) measure. Therefore, there is no idea of "transportation" which the Wasserstein metric seems to capture the best. Roughly speaking, in optimal transport, one seeks a transport plan (unknown) such that for a given initial measure, its image with the transport plan matches a given target measure. Whereas in our case, one seeks measures that remain invariant (in the sense* (4.2.26)*) w.r.t. to a given analogue of the transport plan (known), that is, the* C_0*-semigroup* $(T_t)_{t\geq 0}$ *on* $L^1(\mathbb{R}^m, \mu)$ *which has* $\mathcal{L}_\alpha^*$ *as a generator. And the latter invariance needs to hold for every* $t \geq 0$*. In this sense, one needs a stronger distance than Wasserstein and TV seems to be well suited. Note also that the space of totally finite Borel measures on* $\mathbb{R}^m$ *is a Banach space when equipped with TV norm (see e.g. [74, §IV.2.16]) which is the right setting for §4.2.1.*

4.5.2.3 Calmness property

In this subsection we assume the function $f(x, \alpha, \mu)$ to be *separable*, that is,

(B0) There exist $g(\cdot\,,\cdot)$ and $k(\cdot\,,\cdot)$ such that (A5) is satisfied and

$$f(x, \alpha, q) = g(x, \alpha) + k(x, q)$$

The next result shows that the *primal* problem enjoys *calmness* (see Definition 4.2.1) and moreover admits a solution in the case (B0) holds. We shall later come back to the more general case, that is when f is not *separable*.

Lemma 4.8. *Under the standing assumptions, the primal problem* $(\mathfrak{P}^\wedge)$ *is calm. If we assume in addition (B0) to hold, then the problem admits an optimal solution* $(q_\circ, \alpha_\circ)$*. The same also holds when the primal problem is* $(\mathfrak{P}^\vee)$*.*

Proof. We need to check that the value of the primal problem $(\mathfrak{P}^\wedge)$ is finite and that the value function $v(y)$ is subdifferentiable in 0. Then we prove existence of an optimal solution $(q_\circ, \alpha_\circ)$ assuming (B0) to hold.

Step 1. (val$(\mathfrak{P}^\wedge) < +\infty$*)*
This is true since the constraints sets and the domain of the objective function are nonempty. Indeed, using Theorem 4.2.6 (and Corollary 4.2.2), for every $\alpha \in \mathcal{A}$, there exists an invariant probability measure q and hence (q, α) is an admissible solution.

Step 2. $(\partial v(0) \neq \emptyset)$
Using the above notations, let $y \in Y$ such that $y := (\lambda, z)$ where $\lambda \in \mathbb{R}$ and $z \in X$. We define the value function $v(y)$ as in §4.2.1, and we have

$$
\begin{aligned}
v(y) &= \inf_{q\in Q} \min_{\alpha(\cdot)\in\mathcal{A}} \langle f(\cdot\,, \alpha(\cdot)), q\rangle, \quad \text{s.t. } G(q) + y \in K_\alpha \\
&= \inf_{q\in Q} \min_{\alpha(\cdot)\in\mathcal{A}} \langle f(\cdot\,, \alpha(\cdot)), q\rangle + I_{K_\alpha}(G(q) + y) \\
&\geq \inf_{q\in Q} \min_{\alpha(\cdot)\in\mathcal{A}} \{\, \langle f(\cdot\,, \alpha(\cdot)), q\rangle + I_{K_\alpha}(G(q))\,\} + \inf_{q\in Q} \min_{\alpha(\cdot)\in\mathcal{A}} \{I_{K_\alpha}(G(q) + y) - I_{K_\alpha}(G(q))\}
\end{aligned}
$$

where in the last inequality we used "$\min(A + B) \geq \min A + \min B$". Note that the first term in the right hand-side is $v(0)$ and hence, one gets, for any $y \in Y$

$$
v(y) - v(0) \geq \inf_{q\in Q} \min_{\alpha(\cdot)\in\mathcal{A}} \{I_{K_\alpha}(G(q) + y) - I_{K_\alpha}(G(q))\} \tag{4.5.4}
$$

Recalling the definition of the subdifferential (see §4.2.1), one has

$$
I_{K_\alpha}(G(q) + y) - I_{K_\alpha}(G(q)) \geq \langle y^*, y\rangle_{Y^*,Y} \quad \text{for all} \quad y^* \in \partial I_{K_\alpha}(G(q)). \tag{4.5.5}
$$

It suffices then to have $\partial I_{K_\alpha}(G(q))$ nonempty for any $\alpha(\cdot) \in \mathcal{A}$, in order to show that $\partial v(0)$ is nonempty. Hence, letting $\alpha(\cdot) \in \mathcal{A}$ be arbitrarily fixed, we first need to have $G(q) \in K_\alpha$, and noting that (recalling $y := (\lambda, z) \in \mathbb{R} \times X$)

$$
\begin{aligned}
I_{K_\alpha}(G(q) + y) - I_{K_\alpha}(G(q)) = I_{\{0\}}(G_1(q) + \lambda) - I_{\{0\}}(G_1(q)) \\
+ I_{\mathrm{Ker}(\mathcal{L}^*_\alpha)}(G_2(q) + z) - I_{\mathrm{Ker}(\mathcal{L}^*_\alpha)}(G_2(q)),
\end{aligned}
$$

it suffices that the polar (negative dual) cones $\{0\}^-$ and $\big(\mathrm{Ker}(\mathcal{L}_\alpha^*)\big)^-$ are nonempty, since $\{0\}$ and $\mathrm{Ker}(\mathcal{L}_\alpha^*)$ are nonempty, closed and convex cones (the same argument is used when deriving the equivalent optimality conditions (4.2.7), (4.2.8) and (4.2.9) in §4.2.1). This holds true, since $\{0\}^- = \mathbb{R}$ and $\big(\mathrm{Ker}(\mathcal{L}_\alpha^*)\big)^- = \mathrm{cl}(\mathrm{range}(\mathcal{L}_\alpha))$ are nonempty. Indeed, for $z^* \in X^*$ to be in $\mathrm{cl}(\mathrm{range}(\mathcal{L}_\alpha))$ it suffices that there exists $u \in D(\mathcal{L}_\alpha)$ such that $z^* = \mathcal{L}_\alpha u$. But $D(\mathcal{L}_\alpha) = D(\mathcal{L}_0)$ is nonempty (thanks to (A*)), and hence there exists $z^* = \mathcal{L}_\alpha u$ for $u \in D(\mathcal{L}_0)$. So there exists $y^* = (\lambda^*, z^*) \in \{0\}^- \times \big(\mathrm{Ker}(\mathcal{L}_\alpha^*)\big)^- \subset Y^*$ and y^* depends on $\alpha(\cdot)$ (in fact only z^* depends on $\alpha(\cdot)$), satisfying (4.5.5).

To sum up, for any $\alpha(\cdot) \in \mathcal{A}$, there exists $q \in Q$ satisfying $G(q) \in K_\alpha$ (indeed $\{q \in Q \,:\, G(q) \in K_\alpha\} = \{\mu_\alpha\}$ a singleton, as shown in §4.2.3.2), and moreover there exists $y^* \in \{0\}^- \times \big(\mathrm{Ker}(\mathcal{L}_\alpha^*)\big)^-$ satisfying (4.5.5). The set $\mathcal{A}$ being closed and recalling (4.5.4), we conclude that there exists $\overline{y}^* \in Y^*$ such that $v(y) - v(0) \geq \langle \overline{y}^*, y \rangle$, i.e. $\partial v(0) \neq \emptyset$.

Step 3. (There exists an optimal solution)
Recall that the feasible set of our *primal* problem $(\mathfrak{P}^\wedge)$ is $\{q \in Q \,:\, G(q) \in K_\alpha\} = \{\mu_\alpha\}$ a singleton, where $\mu_\alpha \in \mathcal{P}_d(\mathbb{R}^m)$. Hence, $(\mathfrak{P}^\wedge)$ equivalently writes as

$$\min_{\alpha(\cdot)\in\mathcal{A}} \langle f(\cdot\,, \alpha(\cdot), \mu_\alpha), \mu_\alpha \rangle.$$

We proceed using a fixed-point approach: we first fix $\alpha_1(\cdot) \in \mathcal{A}$, hence also μ_{α_1}, and then show that $\alpha_2(\cdot) \in \underset{\alpha(\cdot)\in\mathcal{A}}{\mathrm{argmin}} \langle f(\cdot\,, \alpha(\cdot), \mu_{\alpha_1}), \mu_{\alpha_1} \rangle$ exists. The next step is then to consider the corresponding unique invariant probability measure μ_{α_2} in the objective function, and repeat the process. We get a fixed point when $\alpha_\circ(\cdot) \in \underset{\alpha(\cdot)\in\mathcal{A}}{\mathrm{argmin}} \langle f(\cdot\,, \alpha(\cdot), \mu_{\alpha_\circ}), \mu_{\alpha_\circ} \rangle$.

Let $\alpha_1(\cdot) \in \mathcal{A}$ be arbitrarily fixed, and let μ_{α_1} be its corresponding unique invariant probability measure. Using (4.5.3) from Proposition 4.4.1, and assuming (B0), i.e. f is separable, one has

$$\begin{aligned}
&\min_{\alpha(\cdot)\in\mathcal{A}} \langle f(\cdot\,, \alpha(\cdot), \mu_{\alpha_1}), \mu_{\alpha_1} \rangle \\
&\qquad = \big\langle \min_{\alpha\in A} f(\cdot\,, \alpha, \mu_{\alpha_1})\,, \mu_{\alpha_1} \big\rangle \\
&\qquad = \big\langle \min_{\alpha\in A} g(\cdot\,, \alpha) + k(\cdot\,, \mu_{\alpha_1})\,, \mu_{\alpha_1} \big\rangle \\
&\qquad = \big\langle k(\cdot\,, \mu_{\alpha_1})\,, \mu_{\alpha_1} \big\rangle + \int_{\mathbb{R}^m} \min_{\alpha\in A} g(x, \alpha)\, \mathrm{d}\mu_{\alpha_1}(x)
\end{aligned}$$

The minimization problem is then reduced to a finite dimensional optimization problem that is, to minimize $g(x, \alpha)$ over $\alpha \in A \subset \mathbb{R}^k$, for each $x \in \mathbb{R}^m$. The function $\alpha \mapsto g(x, \alpha)$

being continuous over a compact set A, a minimizer α_x to the latter finite dimensional optimization problem exists. We then define $\alpha_\circ : \mathbb{R}^m \ni x \mapsto \alpha_x \in A$ a measurable function, and we have $\alpha_\circ(\cdot) \in \underset{\alpha(\cdot)\in\mathcal{A}}{\operatorname{argmin}} \langle f(\cdot\,, \alpha(\cdot), \mu_{\alpha_1}), \mu_{\alpha_1}\rangle$. But $\alpha_\circ(\cdot)$ is independent of μ_{α_1} since it is obtained from the minimization of $g(x,\alpha)$ for $\alpha \in A$. Hence, one gets the desired fixed point by considering $\mu_{\alpha_\circ}$ which is the corresponding unique invariant probability measure, and $(\mu_{\alpha_\circ}, \alpha_\circ)$ is an optimal solution for $(\mathfrak{P}^\wedge)$.

Step 4. (On the problem $(\mathfrak{P}^\vee)$*)*
The same argumentation as in the previous steps remains valid when we deal with the problem $(\mathfrak{P}^\vee)$ since the only difference is in the sign in front of f, while the constraints set is unchanged. □

4.5.2.4 The dual problem

In order to deduce the corresponding *dual* problem, we follow a parametric (conjugate) duality scheme as in [46, §2.5.3, p. 107]. Therefore we embed the problem $(\mathfrak{P}^\wedge)$ in a family of parameterized problems, where $y \in Y$ is the parameter vector and consider the function

$$\phi(q,y) = \min_{\alpha(\cdot)\in\mathcal{A}} \{ \langle f(\cdot\,, \alpha(\cdot), q), q\rangle + F(G(q)+y, \alpha) \}\,.$$

It is clear that when setting $y=0$, we recover the objective function in $(\mathfrak{P}^\wedge)$.

Lemma 4.9. *Under the standing assumptions, ϕ is lower semi-continuous.*

Proof. We have $q \mapsto \langle f(\cdot\,, \alpha(\cdot), q), q\rangle$ and $y \mapsto F(y, \alpha)$ are lower semi-continuous (l.s.c), and $q \mapsto G(q)$ is continuous. And $y \mapsto F(y,\alpha)$ is l.s.c. if and only if K_α is closed, and this holds in our setting. □

We also consider the following (Lagrangian) function, $L : X \times Y^* \times \mathcal{A} \to \mathbb{R}$, analogue to (4.2.2) and such that

$$L(q, y^*, \alpha) := \langle f(\cdot\,, \alpha(\cdot), q), q\rangle + \langle y^*, G(q)\rangle_{Y^*,Y}. \tag{4.5.6}$$

Using the Legendre-Fenchel transform, we have

$$\begin{aligned}
\phi^*(q^*, y^*) &= \sup_{q\in Q, y\in Y} \{ \langle q^*, q\rangle + \langle y^*, y\rangle - \phi(q,y) \} \\
&= \sup_{q\in Q, y\in Y} \left\{ \langle q^*, q\rangle + \langle y^*, y\rangle - \min_{\alpha(\cdot)\in\mathcal{A}} \{ \langle f(\cdot\,, \alpha(\cdot), q), q\rangle + F(G(q)+y, \alpha)\} \right\} \\
&= \sup_{q\in Q, y\in Y} \left\{ \max_{\alpha(\cdot)\in\mathcal{A}} \{\langle q^*, q\rangle + \langle y^*, y\rangle - \big(\langle f(\cdot\,, \alpha(\cdot), q), q\rangle + F(G(q)+y, \alpha)\big) \} \right\}
\end{aligned}$$

$$
\begin{aligned}
&= \max_{\alpha(\cdot)\in\mathcal{A}} \left\{ \sup_{q\in Q, y\in Y} \left\{ \langle q^*, q\rangle + \langle y^*, y\rangle - \big(\langle f(\cdot\,,\alpha(\cdot),q),q\rangle + F(G(q)+y,\alpha)\big) \right\} \right\} \\
&= \max_{\alpha(\cdot)\in\mathcal{A}} \left\{ \sup_{q\in Q} \left\{ \langle q^*, q\rangle - \langle f(\cdot\,,\alpha(\cdot),q),q\rangle - \langle y^*, G(q)\rangle \right\} + \right. \\
&\qquad\qquad\qquad\qquad \left. + \sup_{y\in Y}\left\{ \langle y^*, G(q)+y\rangle - F(G(q)+y,\alpha) \right\} \right\} \\
&= \max_{\alpha(\cdot)\in\mathcal{A}} \left\{ \sup_{q\in Q} \left\{ \langle q^*, q\rangle - L(q,y^*,\alpha) + F^*(y^*,\alpha) \right\} \right\} \\
&= \sup_{q\in Q} \left\{ \langle q^*, q\rangle + \max_{\alpha(\cdot)\in\mathcal{A}} \left\{ -L(q,y^*,\alpha) + F^*(y^*,\alpha) \right\} \right\} \\
&= \sup_{q\in Q} \left\{ \langle q^*, q\rangle - \min_{\alpha(\cdot)\in\mathcal{A}} \left\{ L(q,y^*,\alpha) - F^*(y^*,\alpha) \right\} \right\}
\end{aligned}
$$

The *dual* of the parameterized *primal* problem is then obtained as

$$
\max_{y^*\in Y^*} \left\{ \langle y^*, y\rangle - \phi^*(0,y^*) \right\}
$$

which writes

$$
\max_{y^*\in Y^*} \left\{ \langle y^*, y\rangle + \inf_{q\in Q} \min_{\alpha(\cdot)\in\mathcal{A}} \left\{ L(q,y^*,\alpha) - F^*(y^*,\alpha) \right\} \right\}
$$

Finally, the *dual* problem associated to $(\mathfrak{P}^\wedge)$ is obtained by setting $y = 0$, and writes as

$$
\max_{y^*\in Y^*} \left\{ \inf_{q\in Q} \min_{\alpha(\cdot)\in\mathcal{A}} \left\{ L(q,y^*,\alpha) - F^*(y^*,\alpha) \right\} \right\}. \tag{$\mathfrak{D}^\wedge$}
$$

We will now make $(\mathfrak{D}^\wedge)$ more explicit. We denote the support of a non-negative measure q by $spt(q) := \{x \in \mathbb{R}^m \,:\, q(x) > 0\}$.

Lemma 4.10. *The problem $(\mathfrak{D}^\wedge)$ is equivalent to*

$$
\max_{\substack{c\in\mathbb{R}\\ u\in\mathcal{X}}} \left\{ c + \inf_{q\in Q} \left\{ \langle H(x,\nabla u, D^2u, q) - c, q\rangle \right\} \right\}, \tag{$\mathfrak{D}^\wedge$}
$$

where $H(x,\nabla u(x), D^2u(x), q) = \min_{\alpha\in A}\{ -\mathcal{L}_\alpha u(x) + f(x,\alpha,q) \}$ and $\mathcal{X}$ is such that

$$
\mathcal{X} = D(\mathcal{L}_0) \cap \{u : \mathbb{R}^m \to \mathbb{R}, \textit{Borel-meas.} \mid \exists\, C > 0,\ |u(x)| \le C(1+|x|^\kappa)\} \tag{4.5.7}
$$

with $\kappa = d + 1 - \beta$, that is, the two optimization problems have the same set of optimal solutions and the same optimal value.

Remark 4.5.4. *Assumption **(A*)** together with Theorem 4.2.8 insure that $D(\mathcal{L}_0) \subset W^{r,2}_{loc}(\mathbb{R}^m)$.*

Proof *of Lemma 4.10.* Recalling that F is an indicator function, its conjugate is the support function as defined in (4.2.4), that is

$$\begin{aligned} F^*(y^*, \alpha) &= I^*_{K_\alpha}(y^*) = \sigma(y^*; K_\alpha) \\ &= \begin{cases} 0, & \text{if } \; y^* \in (K_\alpha)^- \\ +\infty, & \text{otherwise} \end{cases} \end{aligned}$$

And recalling the definition $K_\alpha = \{0\} \times \mathrm{Ker}(\mathcal{L}_\alpha)$, we have

$$\begin{aligned} y^* \in (K_\alpha)^- &\Leftrightarrow \; (c, \omega) \in \Big(\{0\} \times \mathrm{Ker}(\mathcal{L}_\alpha)\Big)^- \\ &\Leftrightarrow \; (c, \omega) \in \mathbb{R} \times (\mathrm{Ker}(\mathcal{L}_\alpha))^\perp \\ &\Leftrightarrow \; (c, \omega) \in \mathbb{R} \times \mathrm{cl}(\mathrm{range}(\mathcal{L}_\alpha)) \end{aligned}$$

Since we are working with $\mathcal{L}_\alpha$ in its closed extension, we have

$$\begin{aligned} \omega \in \mathrm{cl}(\mathrm{range}(\mathcal{L}_\alpha)) &\Leftrightarrow \exists \; u \in D(\mathcal{L}_\alpha), \text{ s.t. } \; \omega = -\mathcal{L}_\alpha u \\ &\Leftrightarrow \exists \; u \in D(\mathcal{L}_0), \text{ s.t. } \; \omega = -\mathcal{L}_\alpha u \end{aligned}$$

where the last equivalence is obtained thanks to the assumption (A*) which guarantees that $D(\mathcal{L}_\alpha) = D(\mathcal{L}_0)$ for all $\alpha(\cdot) \in \mathcal{A}$. Note however that ω still depends on α through its definition as $\omega = -\mathcal{L}_\alpha u$. The fact that u belongs to a domain which is independent of α is important in this scheme, since the maximization over y^* is not in the same order as the minimization over α. Indeed, our *dual* problem now writes

$$\max_{y^* \in Y^*} \inf_{q \in Q} \min_{\alpha(\cdot) \in \mathcal{A}} \{ L(q, y^*, \alpha) \,, \quad \text{s.t.: } \; y^* = (c, -\mathcal{L}_\alpha u) \text{ and } (c, u) \in \mathbb{R} \times D(\mathcal{L}_0) \}, \qquad (\mathfrak{D}\hat{}\,)$$

and the new variables on which we perform the maximization are now (c, u) and they belong to a domain $\mathbb{R} \times D(\mathcal{L}_0)$. The latter being independent of $\alpha(\cdot)$, we can write the *dual* problem as

$$\max_{\substack{c \in \mathbb{R} \\ u \in D(\mathcal{L}_0)}} \inf_{q \in Q} \min_{\alpha(\cdot) \in \mathcal{A}} \{ L(q, y^*, \alpha), \quad \text{s.t.: } \; y^* = (c, -\mathcal{L}_\alpha u) \, \}. \qquad (\mathfrak{D}\hat{}\,)$$

Recalling the definition (4.5.6) of L and the notations introduced earlier, we have

$$\begin{aligned} L(q, y^*, \alpha) &= \langle f(\cdot\,, \alpha(\cdot), q), q \rangle + \langle y^*, G(q) \rangle_{Y^*, Y} \\ &= \langle f(\cdot\,, \alpha(\cdot), q), q \rangle + c(1 - \langle 1, q \rangle) + \langle -\mathcal{L}_\alpha u(\cdot), q \rangle \\ &= c + \langle f(\cdot\,, \alpha(\cdot), q) - \mathcal{L}_\alpha u(\cdot) - c, q \rangle \end{aligned}$$

hence we have, using the *exchange property* in Proposition 4.4.1,

$$\begin{aligned}
\min_{\alpha(\cdot)\in\mathcal{A}}\{L(q,y^*,\alpha), \quad \text{s.t.: } \ y^*=(c,-\mathcal{L}_\alpha u)\} = \\
&= c+\min_{\alpha(\cdot)\in\mathcal{A}}\{\langle f(\cdot\,,\alpha(\cdot),q)-\mathcal{L}_\alpha u(\cdot)-c,q\rangle\} \\
&= c+\langle\min_{\alpha\in A}\{f(\cdot\,,\alpha,q)-\mathcal{L}_\alpha u(\cdot)\}-c,q\rangle \\
&= c+\langle H(x,\nabla u,D^2u,q)-c,q\rangle.
\end{aligned}$$

But since Q is made of non-negative measures with finite moment of order d, we need u to have a polynomial growth of order at most $\kappa=d+1-\beta$ (see (A3), (A5-(iii)) and (A6)). The *dual* problem finally writes as

$$\max_{\substack{c\in\mathbb{R}\\ u\in\mathcal{X}}}\left\{c+\inf_{q\in Q}\langle H(x,\nabla u,D^2u,q)-c,q\rangle\right\}.$$

where the functional space $\mathcal{X}$ is defined as

$$\mathcal{X}=D(\mathcal{L}_0)\cap\{u:\mathbb{R}^m\to\mathbb{R},\textit{Borel-meas.}\mid\exists\, C>0,\ |u(x)|\leq C(1+|x|^\kappa)\}$$

and $\kappa=d+1-\beta$. This concludes the proof. $\square$

In the case where the Hamiltonian is given by

$$H(x,\nabla u(x),D^2u(x),q)=\max_{\alpha\in A}\{-\mathcal{L}_\alpha u(x)+f(x,\alpha,q)\},$$

and the corresponding *primal* problem is $(\mathfrak{P}^\vee)$, we proceed in the same way as before, noting that the only difference with $(\mathfrak{P}^\wedge)$ is that instead of f we now consider $-f$. Then we define

$$\phi(q,y)=\min_{\alpha(\cdot)\in\mathcal{A}}\ \{\langle -f(\cdot\,,\alpha(\cdot),q),q\rangle+F(G(q)+y,\alpha)\}.$$

and in this case the Lagrangian (compared with (4.2.2) or (4.5.6)) writes as

$$L(q,y^*,\alpha)\coloneqq\langle -f(\cdot\,,\alpha(\cdot),q),q\rangle+\langle y^*,G(q)\rangle_{Y^*,Y}. \tag{4.5.8}$$

Then we compute the Legendre-Fenchel transform $\phi^*(q^*,y^*)$ and recover the *dual* problem similar to $(\mathfrak{D}^\wedge)$ which is given by

$$\max_{y^*\in Y^*}\inf_{q\in Q}\min_{\alpha(\cdot)\in\mathcal{A}}\{L(q,y^*,\alpha)-F^*(y^*,\alpha)\}. \tag{$\mathfrak{D}^\vee$}$$

The following is an analogue of Lemma 4.10.

Lemma 4.11. *The problem* $(\mathfrak{D}^\vee)$ *is equivalent to*

$$\max_{\substack{c\in\mathbb{R}\\ u\in\mathcal{X}}} \left\{ -c + \inf_{q\in Q} \left\{ -\langle H(x,\nabla u, D^2u, q) - c, q\rangle \right\} \right\} \qquad (\mathfrak{D}^\vee)$$

where $H(x,\nabla u(x), D^2u(x), q) = \max_{\alpha\in A}\{ -\mathcal{L}_\alpha u(x) + f(x,\alpha,q) \}$ *and* $\mathcal{X}$ *is such that*

$$\mathcal{X} = D(\mathcal{L}_0) \cap \{u : \mathbb{R}^m \to \mathbb{R}, \textit{Borel-meas.} \mid \exists\, C>0,\ |u(x)| \le C(1+|x|^\kappa)\} \qquad (4.5.9)$$

with $\kappa = d+1-\beta$*, that is, the two optimization problems have the same set of optimal solutions and the same optimal value.*

We keep the *primal* problem $(\mathfrak{P}^\vee)$ and the *dual* problem $(\mathfrak{D}^\vee)$ written in this formulation because it will be needed when we will set the optimality conditions in the next section.

Proof *of Lemma 4.11.* The proof follows the one of Lemma 4.10. The main difference is in the choice of the representation of the dual variable $y^* \in (K_\alpha)^-$ which we now write as

$$\begin{aligned} y^* \in (K_\alpha)^- &\Leftrightarrow (-c,-\omega) \in \Big(\{0\}\times \operatorname{Ker}(\mathcal{L}_\alpha)\Big)^- \\ &\Leftrightarrow (-c,-\omega) \in \mathbb{R}\times (\operatorname{Ker}(\mathcal{L}_\alpha))^\perp \\ &\Leftrightarrow (-c,-\omega) \in \mathbb{R}\times \operatorname{cl}(\operatorname{range}(\mathcal{L}_\alpha)) \end{aligned}$$

We set again as in Lemma 4.10,

$$\omega \in \operatorname{cl}(\operatorname{range}(\mathcal{L}_\alpha)) \Leftrightarrow \exists\, u \in D(\mathcal{L}_0),\ \text{s.t.}\ \ \omega = -\mathcal{L}_\alpha u.$$

And the *dual* problem $(\mathfrak{D}^\vee)$ writes as

$$\max_{y^*\in Y^*} \inf_{q\in Q} \min_{\alpha(\cdot)\in\mathcal{A}} \{L(q,y^*,\alpha) \quad \text{s.t.}\ \ y^* = (-c,\mathcal{L}_\alpha u) \text{ and } (c,u)\in\mathbb{R}\times D(\mathcal{L}_0)\}, \qquad (\mathfrak{D}^\vee)$$

Recalling the definition (4.5.6) of L and the notations introduced earlier, we have

$$\begin{aligned} L(q,y^*,\alpha) &= \langle -f(\cdot\,,\alpha(\cdot),q),q\rangle + \langle y^*, G(q)\rangle_{Y^*,Y} \\ &= \langle -f(\cdot\,,\alpha(\cdot),q),q\rangle - c(1-\langle 1,q\rangle) + \langle \mathcal{L}_\alpha u(\cdot),q\rangle \\ &= -c - \langle f(\cdot\,,\alpha(\cdot),q) - \mathcal{L}_\alpha u(\cdot) - c, q\rangle \end{aligned}$$

hence we have, using the *exchange property* in Proposition 4.4.1,

$$\begin{aligned}
\min_{\alpha(\cdot)\in\mathcal{A}} \{ L(q, y^*, \alpha), \quad \text{s.t. } \; y^* = (-c, \mathcal{L}_\alpha u) \} = \\
&= -c + \min_{\alpha(\cdot)\in\mathcal{A}} \{ -\langle f(\cdot\,, \alpha(\cdot), q) - \mathcal{L}_\alpha u(\cdot) - c, q\rangle \} \\
&= -c - \max_{\alpha(\cdot)\in\mathcal{A}} \{ \langle f(\cdot\,, \alpha(\cdot), q) - \mathcal{L}_\alpha u(\cdot) - c, q\rangle \} \\
&= -c - \langle \max_{\alpha\in A}\{ f(\cdot\,, \alpha, q) - \mathcal{L}_\alpha u(\cdot)\} - c, q\rangle \\
&= -c - \langle H(x, \nabla u, D^2 u, q) - c, q\rangle.
\end{aligned}$$

The *dual* problem $(\mathfrak{D}^\vee)$ then writes as

$$\max_{\substack{c\in\mathbb{R}\\ u\in\mathcal{X}}} \left\{ -c + \inf_{q\in Q} \{ -\langle H(x, \nabla u, D^2 u, q) - c, q\rangle \} \right\}.$$

And we conclude in the same way as in the proof of Lemma 4.10. □

4.5.3 Main results III: ergodic MFG system

The PDE problem

We address the problem of existence and uniqueness of solutions to an ergodic mean-field games (MFG) system, that is

$$\begin{gathered}
\textit{Find } (c, u, \mu) \in \mathbb{R}\times\mathcal{X}(\mathbb{R}^m)\times\mathcal{P}(\mathbb{R}^m), \textit{ s.t.:} \\
H(x, \nabla u(x), D^2 u(x), \mu) = c \quad \text{and} \quad -\mathcal{L}^*_{\alpha[u,\mu]}\mu = 0
\end{gathered} \tag{4.5.10}$$

where $\mathcal{X}$ is a functional space (part of the unknowns), $\mathcal{P}$ is the set of probability measures and the Hamiltonian is of one of the two forms

$$H \coloneqq \min_{\alpha\in A}\{ -\mathcal{L}_\alpha u(x) + f(x, \alpha, \mu) \}, \quad \text{or} \quad H \coloneqq \max_{\alpha\in A}\{ -\mathcal{L}_\alpha u(x) + f(x, \alpha, \mu) \},$$

the diffusion operator $\mathcal{L}_\alpha$ is a linear operator given by

$$\mathcal{L}_\alpha\varphi(x) \coloneqq \operatorname{trace}\big(a(x,\alpha)D^2\varphi(x)\big) + b(x,\alpha)\cdot\nabla\varphi(x)$$

and its adjoint $\mathcal{L}^*_\alpha$ is then

$$\mathcal{L}^*_\alpha\rho(x) = \operatorname{trace}\big(D^2(a(x,\alpha)\rho(x))\big) - \operatorname{div}\big(b(x,\alpha)\rho(x)\big).$$

The second equation in (4.5.10) is nothing but $-\mathcal{L}^*_\alpha\mu = 0$ where we anticipate the dependence of α on (u, μ).

Optimality conditions

We first consider the case where the Hamiltonian is given by

$$H(x, \nabla u(x), D^2 u(x), q) = \min_{\alpha \in A} \{ -\mathcal{L}_\alpha u(x) + f(x, \alpha, q) \}.$$

We check that the optimality conditions as stated in §4.2.1, in particular (4.2.8) and (4.2.9), still hold in our framework. In order to do so, we start from the *duality gap* (or *duality inequality*) which states that the value of the *dual* problem ($\mathfrak{D}^\wedge$) is less or equal than the value of the *primal* problem ($\mathfrak{P}^\wedge$). Recalling the definition (4.5.6) of the Lagrangian function L

$$L(q, y^*, \alpha) = \langle f(\cdot\,, \alpha(\cdot), q), q\rangle + \langle y^*, G(q)\rangle_{Y^*,Y}$$

and the value of the *dual* problem being less or equal the value of the *primal* problem (see §4.2.1), we have

$$\begin{aligned}
&\max_{y^* \in Y^*} \min_{q \in Q} \min_{\alpha(\cdot) \in \mathcal{A}} \{L(q, y^*, \alpha) - F^*(y^*, \alpha)\} \\
&\qquad \le \min_{q \in Q} \min_{\alpha(\cdot) \in \mathcal{A}} \{\langle f(\cdot\,, \alpha(\cdot)), q\rangle + F(G(q), \alpha)\} \\
&\qquad \le \min_{q \in Q} \min_{\alpha(\cdot) \in \mathcal{A}} \{L(q, y^*, \alpha) + F(G(q), \alpha) - \langle y^*, G(q)\rangle_{Y^*,Y}\}, \ \forall\, y^* \in Y^*.
\end{aligned}$$

Let us denote by $(q_\circ, \alpha_\circ)$ an optimal solution in the *primal* problem ($\mathfrak{P}^\wedge$) and by $y^*_\circ$ an optimal solution in the *dual* problem ($\mathfrak{D}^\wedge$). We then have

$$\begin{aligned}
\min_{q \in Q} \min_{\alpha(\cdot) \in \mathcal{A}} \{L(q, y^*_\circ, \alpha) - F^*(y^*_\circ, \alpha)\} &\le L(q_\circ, y^*_\circ, \alpha_\circ) + F(G(q_\circ), \alpha_\circ) - \langle y^*_\circ, G(q_\circ)\rangle_{Y^*,Y} \\
&= \langle f(\cdot\,, \alpha_\circ(\cdot), q_\circ), q_\circ\rangle + F(G(q_\circ), \alpha_\circ)
\end{aligned}$$

Optimality conditions are then obtained when we reach equality in the above inequality. We can then characterize the optimal *primal* and *dual* solutions and provide a no-*duality gap* condition. Suppose the left hand side minimization in the above inequality is reached in the pair of optimal solutions $(q_\circ, \alpha_\circ)$. Therefore, we firstly need to have $F^*(y^*_\circ, \alpha_\circ) = 0$ i.e.

$$y^*_\circ \in (K_{\alpha_\circ})^- \tag{4.5.11}$$

since F is an indicator function and hence F^* is a support function which is either 0 if $y^*_\circ \in (K_{\alpha_\circ})^-$ or $+\infty$ otherwise. Then, and secondly, since

$$L(q_\circ, y^*_\circ, \alpha_\circ) = \langle f(\cdot\,, \alpha_\circ(\cdot), q_\circ), q_\circ\rangle + \langle y^*_\circ, G(q_\circ)\rangle_{Y^*,Y}$$

then from the optimality of $(q_\circ, \alpha_\circ)$ we have

$$\langle y_\circ^*, G(q_\circ)\rangle_{Y^*,Y} = 0. \tag{4.5.12}$$

And finally the inequality is reduced to

$$-F^*(y_\circ^*, \alpha_\circ) \leq F(G(q_\circ), \alpha_\circ) - \langle y_\circ^*, G(q_\circ)\rangle_{Y^*,Y}$$

which is the Fenchel-Young inequality. The latter is an equality if and only if we have

$$y_\circ^* \in \partial F(G(q_\circ), \alpha_\circ) = \partial I_{K_{\alpha_\circ}}(G(q_\circ)) = N_{K_{\alpha_\circ}}(G(q_\circ)) \tag{4.5.13}$$

And since K_{α_0} is a convex cone, then $y_\circ^* \in N_{K_{\alpha_\circ}}(G(q_\circ))$ is equivalent to

$$G(q_\circ) \in K_{\alpha_\circ}, \quad y_\circ^* \in (K_{\alpha_\circ})^- \text{ and } \langle y_\circ^*, G(q_\circ)\rangle_{Y^*,Y} = 0. \tag{4.5.14}$$

To sum up, we have the following sufficient optimality conditions which also guarantee the absence of the *duality gap*

$$\begin{cases} (q_\circ, \alpha_\circ) \in \underset{q\in Q, \alpha(\cdot)\in\mathcal{A}}{\operatorname{argmin}} \; L(q, y_\circ^*, \alpha) \\ G(q_\circ) \in K_{\alpha_\circ}, \quad y_\circ^* \in (K_{\alpha_\circ})^- \text{ and } \langle y_\circ^*, G(q_\circ)\rangle_{Y^*,Y} = 0. \end{cases} \tag{4.5.15}$$

They are indeed analogue to (4.2.8) and (4.2.9).

And the same optimality conditions (4.5.15) hold when the Hamiltonian is given by

$$H(x, \nabla u(x), D^2u(x), q) = \max_{\alpha\in A}\{\, -\mathcal{L}_\alpha u(x) + f(x, \alpha, q)\,\},$$

provided we write $-f$ (instead of f) in the above computations and make use of definition (4.5.8) for the Lagrangian function L

$$L(q, y^*, \alpha) = \langle -f(\cdot\,, \alpha(\cdot), q), q\rangle + \langle y^*, G(q)\rangle_{Y^*,Y}.$$

4.5.4 Existence and uniqueness

Our first main result is a necessary and sufficient theorem for existence and uniqueness of a solution to the ergodic MFG system (4.5.10).

Theorem 4.5.1. *Assuming (A0-A6) and (A*) hold true, the following are equivalent*

(I) The primal problem $(\mathfrak{P}^\wedge)$ *admits a solution* $(q_\circ, \alpha_\circ)$*, that is,*

$$(q_\circ, \alpha_\circ) \in \operatorname*{argmin}_{\substack{q \in \mathcal{M}_d^+(\mathbb{R}^m) \\ \alpha(\cdot) \in \mathcal{A}}} \Big\{ \langle f(\,\cdot\,, \alpha(\cdot), q),\, q \,\rangle\,,\ \textit{s.t.: } 1 - \langle 1, q \rangle = 0 \textit{ and } q \in Ker(\mathcal{L}_\alpha^*) \Big\}.$$

(II) There exists $(c_\circ, u_\circ, q_\circ) \in \mathbb{R} \times W^{r,2}_{loc}(\mathbb{R}^m) \times W^{s,1}_{loc}(\mathbb{R}^m)$ *for any* $r \geq 1$*,* $s > m$*, and there exists a measurable function* $\alpha_\circ(\cdot) : \mathbb{R}^m \to A$ *that solve the MFG system*

$$\begin{cases} -\,trace\big(a(x, \alpha_\circ(x)) D^2 u_\circ(x)\big) - b(x, \alpha_\circ(x)) \cdot \nabla u_\circ(x) + f(x, \alpha_\circ(x), q_\circ) = c_\circ \\ -\,trace\big(D^2(a(x, \alpha_\circ(x)) q_\circ(x))\big) + div\big(b(x, \alpha_\circ) q_\circ(x)\big) = 0, \qquad a.e.\ in\ \mathbb{R}^m \end{cases} \tag{4.5.16}$$

and moreover

(a) the constant $c_\circ$ *is defined by* $c_\circ = \langle f(\,\cdot\,, \alpha_\circ(\cdot), q_\circ), q_\circ \rangle$*,*

(b) the function $u_\circ(\cdot)$ *satisfies:* $|u_\circ(x)| \leq K(1 + |x|^\kappa)$*, with* $\kappa = d + 1 - \beta$ *and* $K > 0$ *a constant,*

(c) $q_\circ(\cdot)$ *is the density of a probability measure, absolutely continuous w.r.t. Lebesgue,*

(d) the function $\alpha_\circ(\cdot)$ *satisfies*

$$\alpha_\circ(x) \in \operatorname*{argmin}_{\alpha \in A} \{ -\mathcal{L}_\alpha u_\circ(x) + f(x, \alpha, q_\circ) \}, \quad a.e.\ \ x \in \mathbb{R}^m.$$

If moreover $(q_\circ, \alpha_\circ)$ *in (I) is unique and the vector field* b *is locally Lipschitz continuous in* x *with* $\beta = 1$ *in (A6), then* $u_\circ(\cdot)$ *is unique in* $W^{r,2}_{loc}(\mathbb{R}^m)$ *with* $r > \frac{m}{2}$*, that is, if* $(c_\circ, u_1(\cdot))$ *and* $(c_\circ, u_2(\cdot))$ *are two solutions in the sense of (II), then* $u_1(\cdot) - u_2(\cdot)$ *is a constant.*

Theorem 4.5.2. *The statement in Theorem 4.5.1 remains valid when the Hamiltonian is given by* max *(instead of* min*), provided we consider in (I) the problem* $(\mathfrak{P}^\vee)$ *(instead of* $(\mathfrak{P}^\wedge)$*) and define* $(q_\circ, \alpha_\circ)$ *as an element of the* argmax *(instead of* argmin*), and in (II-d) we define* $\alpha_\circ$ *as an element of the* argmax *(instead of* argmin*).*

Remark 4.5.5. *Some observations regarding regularity and notion of the solution:*

- *We recall that functions in* $W^{r,2}_{loc}(\mathbb{R}^m)$*, with* $2r > m > 1$*, are continuous and pointwise twice differentiable almost everywhere (see e.g. [50, Appendix C]). And for* $s > m$*, one has* $W^{s,1}_{loc}(\mathbb{R}^m) \subset C^{1-\frac{m}{s}}(\mathbb{R}^m)$ *(see e.g. [137, p. 28] or [1, p. 97]).*

• *In fact $u_\circ(\cdot)$ is an L-viscosity solution (see e.g. [50, 70]), which is as expected as when we consider C-viscosity solutions for the continuous (and bounded) case. Recall that in our setting, the vector field b and the function f are assumed to be measurable (and unbounded) in x.*

Remark 4.5.6. *Some observations on uniqueness of the solution:*
• *Uniqueness of $(q_\circ, \alpha_\circ)$ in statement (I) requires the (primal) optimization problem to be jointly convex in (q, α). This is hardly satisfied because of the constraint $q \in Ker(\mathcal{L}^*_\alpha)$. Therefore, one does not expect uniqueness for the MFG system, at least not in our setting.*
• *The ergodic constant $c_\circ$ is in general not unique. In fact, there might be infinitely many constants for which there exists a solution (u, q). We refer to Remark 4.4.4*

Note that by the latter theorems, we reduced the problem of existence of a solution (c, u, q) for the MFG system (4.5.10) to the solvability of an (infinite dimensional) optimization problem where the unknown is (q, α). The next main result relies on the particular case where $f(x, \alpha, q)$ is *separable* and for which we can solve the optimization problem (hence prove existence of a solution to the MFG system). The non-*separable* case requires additional assumptions and shall be discussed afterwards.

Corollary 4.5.1. *In the situation of Theorem 4.5.1, if we assume in addition that (B0) holds, i.e. $f(x, \alpha, q) = g(x, \alpha) + k(x, q)$, then the ergodic MFG system admits a solution satisfying the properties (II) in Theorem 4.5.1.*
The same also holds true when in the situation of Theorem 4.5.2.

4.5.5 Proofs of the main results

Proof *of Theorem 4.5.1.* The proof is a consequence of Theorem 4.2.2 (i) and Lemma 4.2, provided we express the optimality conditions (4.5.15) in terms of a PDE system as in the statement (II). And to do so, we rely on Lemma 4.10 and the results in §4.2.2.

Step 1. (An application of Theorem 4.2.2 (i))
Thanks to Lemma 4.2, the *primal* problem $(\mathfrak{P}^\wedge)$ is *calm*. Therefore, the statement (i) of Theorem 4.2.2 insures that there is no duality gap, and $(q_\circ, \alpha_\circ)$ is an optimal solution of $(\mathfrak{P}^\wedge)$ if and only if there exists $y^*_\circ \in Y^* = \mathbb{R} \times (\mathcal{M}(\mathbb{R}^m))^*$ such that conditions (4.5.15) are satisfied.

Step 2. (On the conditions (4.5.15)*)*
Let us first note that, using Proposition 4.2.1, $y^*_\circ$ is an optimal solution of the *dual*

problem ($\mathfrak{D}^\wedge$). And following Lemma 4.10 (see also its proof), one can substitute the dual variables y^* with the pairs of variables $(c, u) \in \mathbb{R} \times \mathcal{X}$ where $\mathcal{X}$ is as defined in (4.5.7). And the optimal dual variables are given by $y^*_\circ = (c_\circ, -\mathcal{L}_{\alpha_\circ} u_\circ)$.

Now, the no-*duality gap* yields

$$c_\circ + \inf_{q \in Q} \{ \langle H(x, \nabla u_\circ, D^2 u_\circ, q) - c_\circ, q \rangle \} = \langle f(\cdot\,, \alpha_\circ(\cdot), q_\circ), q_\circ \rangle \tag{4.5.17}$$

and the last condition in (4.5.15) that is $\langle y^*_\circ, G(q_\circ)\rangle_{Y^*,Y} = 0$, writes as

$$c_\circ(1 - \langle 1, q_\circ\rangle) + \langle -\mathcal{L}_{\alpha_\circ} u_\circ(\cdot), q_\circ\rangle = 0,$$

i.e. $c_\circ = \langle c_\circ, q_\circ\rangle - \langle -\mathcal{L}_{\alpha_\circ} u_\circ(\cdot), q_\circ\rangle$. Substituting $c_\circ$ in (4.5.17) yields

$$\inf_{q \in Q} \{ \langle H(x, \nabla u_\circ, D^2 u_\circ, q) - c_\circ, q \rangle \} = \langle -\mathcal{L}_{\alpha_\circ} u_\circ(\cdot) + f(\cdot\,, \alpha_\circ(\cdot), q_\circ) - c_\circ, q_\circ \rangle. \tag{4.5.18}$$

Thanks to the *exchange property* (4.5.3), the latter equality writes as

$$\inf_{q \in Q} \min_{\alpha(\cdot) \in \mathcal{A}} \{ \langle -\mathcal{L}_{\alpha} u_\circ(\cdot) + f(\cdot\,, \alpha(\cdot), q) - c_\circ, q \rangle \} = \langle -\mathcal{L}_{\alpha_\circ} u_\circ(\cdot) + f(\cdot\,, \alpha_\circ(\cdot), q_\circ) - c_\circ, q_\circ \rangle, \tag{4.5.19}$$

that is

$$(q_\circ, \alpha_\circ) \in \operatorname*{argmin}_{\substack{q \in Q \\ \alpha(\cdot) \in \mathcal{A}}} \{ \langle -\mathcal{L}_{\alpha} u_\circ(\cdot) + f(\cdot\,, \alpha(\cdot), q) - c_\circ, q \rangle \}. \tag{4.5.20}$$

In particular, when setting q to its optimal value $q_\circ$, one has

$$\alpha_\circ(\cdot) \in \operatorname*{argmin}_{\alpha(\cdot) \in \mathcal{A}} \{ \langle -\mathcal{L}_{\alpha} u_\circ(\cdot) + f(\cdot\,, \alpha(\cdot), q_\circ) - c_\circ, q_\circ \rangle \} \tag{4.5.21}$$

which yields thanks to the *exchange property* (4.5.3)

$$\alpha_\circ(x) \in \operatorname*{argmin}_{\alpha \in A} \{ -\mathcal{L}_{\alpha} u_\circ(x) + f(x, \alpha, q_\circ) \}, \quad q_\circ - \text{a.e. } x \in \mathbb{R}^m, \tag{4.5.22}$$

i.e. $H(x, \nabla u_\circ(x), D^2 u_\circ(x), q_\circ) = -\mathcal{L}_{\alpha_\circ} u_\circ(x) + f(x, \alpha_\circ(x), q_\circ)$, $q_\circ$-almost everywhere. And thanks to Theorem 4.2.5, $q_\circ$ is absolutely continuous with respect to Lebesgue measure and hence the result almost everywhere in $\mathbb{R}^m$.

Analogously, when setting $\alpha(\cdot)$ to its optimal value $\alpha_\circ(\cdot)$, one has

$$q_\circ \in \operatorname*{argmin}_{q \in Q} \{ \langle -\mathcal{L}_{\alpha_\circ} u_\circ(\cdot) + f(\cdot\,, \alpha_\circ(\cdot), q) - c_\circ, q \rangle \}. \tag{4.5.23}$$

And recalling the definition of the *primal* problem ($\mathfrak{P}^\wedge$), the condition $G(q_\circ) \in K_{\alpha_\circ}$ in

(4.5.15) means in particular that $\langle 1, q_\circ \rangle = 1$, and since $q \in Q = \mathcal{M}_d^+(\mathbb{R}^m)$, then $q_\circ$ is a probability measure.

We will now show (using the results in §4.2.2) that an optimality condition for the optimization problem (4.5.23) allows to prove that $(c_\circ, u_\circ)$ solves the PDE $-\mathcal{L}_{\alpha_\circ} u_\circ + f(\cdot\,, \alpha_\circ(\cdot), q_\circ) = c_\circ$ a.e. in $\mathbb{R}^m$, i.e. $H(x, \nabla u_\circ(x), D^2 u_\circ(x), q_\circ) = c_\circ$ a.e. in $\mathbb{R}^m$.

Step 2.1. (On the problem (4.5.23)*)*
We define $\widetilde{f} : \mathbb{R}^m \times \mathcal{M}(\mathbb{R}^m) \to \mathbb{R}$ and $g : \mathbb{R}^m \to \mathbb{R}$ respectively by

$$\widetilde{f}(x, q) := f(x, \alpha_\circ(x), q), \qquad g(x) := -\mathcal{L}_{\alpha_\circ} u_\circ(x) - c_\circ$$

and we set

$$\Psi(q) := \langle\, \widetilde{f}(\,\cdot\,, q) + g(\cdot)\,,\, q\,\rangle.$$

The optimization problem (4.5.23) writes equivalently as

$$\min\ \big\{\ \Psi(q)\,, \quad \text{s.t.:}\ \ q \in \mathcal{M}_d^+(\mathbb{R}^m)\ \big\}. \tag{4.5.24}$$

With this formulation, it is easy to see that any measure q satisfying the constraint in (4.5.24) is *regular* in the sense of Definition 4.2.2. Indeed, it suffices to set, in the notation of (4.2.17), $Q = \mathcal{M}_d^+(\mathbb{R}^m)$, $G(q) = q$ and $K = \mathcal{M}_d(\mathbb{R}^m)$. Thanks to assumption (A5-(iv)), the function Ψ is Fréchet differentiable and we can apply Theorem 4.2.3 together with Theorem 4.2.4 and Corollary 4.2.1 to obtain the following first-order necessary condition for $q_\circ$ to be a minimum of (4.5.24) (or equivalently of problem (4.5.23)):

$$D\Psi(q_\circ)[h] \geq 0, \quad \forall\, h \in \{\, h \in \mathcal{M}_d(\mathbb{R}^m)\ :\ h^- \ll q_\circ\}, \tag{4.5.25}$$

where, using the definition of Ψ, one has

$$D\Psi(q_\circ)[h] = \langle \widetilde{f}(\cdot\,, q_\circ) + g(\cdot)\,,\, h\rangle + \langle D_\mu \widetilde{f}(\cdot\,,\, q_\circ)[h], q_\circ\rangle.$$

Step 2.2. (We show that $\widetilde{f}(\cdot\,, q_\circ) + g(\cdot) \geq 0$ *in* $\mathbb{R}^m$*)*
We proceed by contradiction. Suppose $\exists\, \overline{x} \in \mathbb{R}^m$ such that $\widetilde{f}(\overline{x}, q_\circ) + g(\overline{x}) < 0$.

We choose $h = \delta_{\overline{x}}$, the Dirac measure with unit mass concentrated at $\overline{x}$. It is a positive measure and is clearly in $T_{\mathcal{M}_d^+(\mathbb{R}^m)}$. When used in (4.5.25), one gets

$$\begin{aligned} 0 &\leq \langle \widetilde{f}(\cdot\,, q_\circ) + g(\cdot)\,,\, \delta_{\overline{x}}\rangle + \langle D_\mu \widetilde{f}(\cdot\,,\, q_\circ)[\delta_{\overline{x}}], q_\circ\rangle \\ &\leq \widetilde{f}(\overline{x}, q_\circ) + g(\overline{x}) + \langle D_\mu \widetilde{f}(\cdot\,,\, q_\circ)[\delta_{\overline{x}}], q_\circ\rangle \end{aligned}$$

But using assumption (A5-(v)), we have $\langle D_\mu \widetilde{f}(\cdot\,,\, q_\circ)[\delta_{\overline{x}}], q_\circ\rangle \leq 0$ and this yields a contradiction with $\widetilde{f}(\overline{x}, q_\circ) + g(\overline{x}) < 0$. Hence, the function $\widetilde{f}(\cdot\,, q_\circ) + g(\cdot)$ is non-negative for all $x \in \mathbb{R}^m$.

Step 2.3. (We show that $\widetilde{f}(x, q_\circ) + g(x) = 0$ almost everywhere in $\mathbb{R}^m$)
We proceed by contradiction. Suppose there exists a Borel subset B (open set in $\mathbb{R}^m$) such that $q_\circ(B) \neq 0$ and a constant $\Gamma > 0$, such that

$$\Gamma := q_\circ - \operatorname*{ess\,sup}_{x\in B}\{\ \widetilde{f}(x, q_\circ) + g(x)\ \} = \inf\{\gamma \in \mathbb{R}\ :\ \widetilde{f}(x, q_\circ) + g(x) \leq \gamma,\ q_\circ - \text{a.e. in } B\}.$$

We will first show that the pair $(q_\circ, \alpha_\circ)$ in the problem (4.5.20) remains the same when we subtract to $f(\cdot\,, \alpha(\cdot), q)$ a positive constant. Then we will show that Γ cannot be positive, which together with the previous *Step 2.2* yields the desired result.

Observe that $(q_\circ, \alpha_\circ)$ besides being a minimizer for the problem (4.5.20), it is determined by the optimality conditions (4.5.15). In particular, it is a minimizer for the *primal* problem $(\mathfrak{P}^\wedge)$. Therefore, we start from the latter problem $(\mathfrak{P}^\wedge)$ where we will subtract to f a constant $n\Gamma$ where $n \geq 1$ (although the choice of the constant here is not important, we keep considering Γ as defined above to avoid introducing new constants). Recall the *primal* problem $(\mathfrak{P}^\wedge)$

$$\min_{q\in\mathcal{M}_d^+(\mathbb{R}^m)} \left\{ \min_{\alpha(\cdot)\in\mathcal{A}} \quad \langle f(\cdot\,, \alpha(\cdot), q), q\rangle, \quad \text{s.t.: } 1 - \langle 1, q\rangle = 0 \text{ and } q \in \operatorname{Ker}(\mathcal{L}_\alpha^*) \right\}. \qquad (\mathfrak{P}^\wedge)$$

Thanks to existence and uniqueness theorems in §4.2.3.2 (Theorem 4.2.6 and Theorem 4.2.7) together with Lemma 2.3.1, the admissible (feasible) set for q in $(\mathfrak{P}^\wedge)$ writes as a singleton that depends on $\alpha(\cdot)$

$$q \in \mathcal{P}_d(\mathbb{R}^m) \cap \operatorname{Ker}(\mathcal{L}_\alpha^*) = \{\mu_\alpha\}$$

where μ_α is the unique invariant probability measure associated to $\mathcal{L}_\alpha^*$. Hence the problem $(\mathfrak{P}^\wedge)$ equivalently writes as

$$\min_{\alpha(\cdot)\in\mathcal{A}} \langle f(\cdot\,, \alpha(\cdot), \mu_\alpha)\,,\, \mu_\alpha\rangle.$$

Now subtracting a constant $n\Gamma$ to f in $(\mathfrak{P}^\wedge)$, i.e. considering as objective function

$$(q, \alpha) \mapsto \langle f(\cdot\,, \alpha(\cdot), q) - n\Gamma\,,\, q\rangle$$

yields the optimization problem

$$\min_{\alpha(\cdot)\in\mathcal{A}} \langle f(\cdot\,, \alpha(\cdot), \mu_\alpha) - n\Gamma\,,\, \mu_\alpha\rangle.$$

But μ_α being a probability measure, the latter writes as

$$-n\Gamma + \min_{\alpha(\cdot)\in\mathcal{A}} \langle f(\cdot\,, \alpha(\cdot), \mu_\alpha)\,,\, \mu_\alpha\rangle.$$

And $(q_\circ, \alpha_\circ)$ is again a minimizer for the latter problem. In other words, subtracting a constant to f in the objective function in $(\mathfrak{P}^\wedge)$ does not alter the optimality of the pair $(q_\circ, \alpha_\circ)$. And ultimately the optimality conditions (4.5.15) also remain the same.

Therefore, one can still consider $(c_\circ, u_\circ, q_\circ, \alpha_\circ)$ as in (4.5.19) even if we subtract to f a constant $n\Gamma$, i.e.

$$\begin{aligned}\inf_{q\in Q}\min_{\alpha(\cdot)\in\mathcal{A}} \{\, &\langle -\mathcal{L}_\alpha u_\circ + f(\cdot\,, \alpha(\cdot), q) - n\Gamma - c_\circ, q\rangle \,\}\\ &= \langle -\mathcal{L}_{\alpha_\circ} u_\circ + f(\cdot\,, \alpha_\circ(\cdot), q_\circ) - n\Gamma - c_\circ, q_\circ\rangle.\end{aligned}$$

In particular, $q_\circ$ is again a minimizer as it is for the problem (4.5.23) but when we subtract to f a constant, i.e.

$$q_\circ \in \operatorname*{argmin}_{q\in Q} \big\{\, \langle -\mathcal{L}_{\alpha_\circ} u_\circ + f(\cdot\,, \alpha_\circ(\cdot), q) - n\Gamma - c_\circ, q\rangle \,\big\}.$$

The latter writes in the notations of *Step 2.1*

$$\min\ \big\{\, \langle\, \widetilde{f}(\cdot\,, q) - n\Gamma + g(\cdot)\,,\, q\,\rangle\,, \quad \text{s.t.:}\ \ q \in \mathcal{M}_d^+(\mathbb{R}^m)\,\big\}. \tag{4.5.26}$$

The first-order necessary optimality conditions (4.5.25) written for the latter problem (4.5.26) now yields

$$\langle \widetilde{f}(\cdot\,, q_\circ) - n\Gamma + g(\cdot)\,,\, h\rangle + \langle D_\mu \widetilde{f}(\cdot\,, q_\circ)[h], q_\circ\rangle \geq 0, \quad \forall\, h \in \{\, h \in \mathcal{M}_d(\mathbb{R}^m)\ :\ h^- \ll q_\circ\}$$

Thanks to assumption (A5-(v)), the second term in the above inequality is non-positive when h is non-negative. So it suffices to choose h as a positive measure supported on the Borel subset B that we have fixed in our hypothesis, and recalling the definition of

Γ, one has $\widetilde{f}(\cdot\,,q_\circ)+g(\cdot)-n\Gamma<0$ for n sufficiently large ($n>1$ is indeed enough) which yields a contradiction. Hence there cannot be any Borel subset of non-zero measure in which $\widetilde{f}(\cdot\,,q_\circ)+g(\cdot)$ is positive, i.e. $\widetilde{f}(x,q_\circ)+g(x)\leq 0$ $q_\circ$-almost everywhere in $\mathbb{R}^m$, and together with the conclusion of *Step 2.2* we finally have $\widetilde{f}(x,q_\circ)+g(x)=0$ $q_\circ$-almost everywhere in $\mathbb{R}^m$. We conclude with Theorem 4.2.5 which insures that $q_\circ$ is absolutely continuous with respect to Lebesgue measure, and hence the desired result:

$$-\mathcal{L}_{\alpha_\circ}u_\circ(x)+f(x,\alpha_\circ(x),q_\circ)=c_\circ,\quad \text{almost everywhere in } \mathbb{R}^m \tag{4.5.27}$$

which writes, thanks to (4.5.22), as $H(x,\nabla u_\circ(x),D^2u_\circ(x),q_\circ)=c_\circ$ a.e. in $\mathbb{R}^m$.

Step 2.4. (Conclusion)
At this stage of the proof, we have shown that $(q_\circ,\alpha_\circ)$ is an optimal solution of $(\mathfrak{P}^\wedge)$ if and only if there exists a pair $(c_\circ,u_\circ)\in\mathbb{R}^m\times\mathcal{X}$ satisfying the optimality conditions (4.5.15). And the latter conditions yield the no-*duality gap*, also the growth condition of the function $u_\circ$ is given by the definition of $\mathcal{X}$ as in (4.5.7) (i.e. the statement (II-b)), the properties of the measure $q_\circ$ are insured by Theorem 4.2.5 (i.e. the statement (II-c)) and we have the characterization (4.5.22) of $\alpha_\circ$ (i.e. the statement (II-d)) noting that $q_\circ$ is equivalent to Lebesgue measure. Finally, the equation (4.5.27) together with $q_\circ\in\mathrm{Ker}(\mathcal{L}^*_{\alpha_\circ})$ and (4.5.22) yield the PDE system (4.5.16), and $(u_\circ,q_\circ)$ being in $W^{r,2}_{\mathrm{loc}}(\mathbb{R}^m)\times W^{s,1}_{\mathrm{loc}}(\mathbb{R}^m)$, for $r>\frac{m}{2}$ and $s>m$, is a direct consequence of Theorem 4.2.5 and Theorem 4.2.8 (see Remark 4.5.4). Substituting (4.5.27) in the equation (4.5.17) yields the characterization of the constant $c_\circ=\langle f(\cdot\,,\alpha_\circ(\cdot),q_\circ),q_\circ\rangle$, hence the statement (II-a).

We are therefore left with the proof of the last statement.

Step 3. (Uniqueness of $u_\circ$)
Assume here the primal problem (statement (I) of the theorem) enjoys uniqueness.
To prove that $u_\circ(\cdot)$ is unique, we need to assume in addition that the vector field $b(x,\alpha)$ is locally Lipschitz continuous with at most a linear growth in x, uniformly in α, i.e. $\beta=1$ in (A6) and hence $\kappa=d$. We also need $r>\frac{m}{2}$ in order to ensure continuity of $u_\circ(\cdot)$ following Remark 4.5.5. This setting will allow us to apply the Liouville type result in [22].

Suppose $(c_\circ, u_1(\cdot)), (c_\circ, u_2(\cdot))$ are two solutions with a polynomial growth of order at most d. Then we have, using the inequality "$\min(A - B) \leq \min(A) - \min(B)$"

$$\min_{\alpha\in A}\{\, -\mathcal{L}_\alpha(u_1 - u_2)\,\} \leq \min_{\alpha\in A}\{-\mathcal{L}_\alpha u_1 + f(\cdot\,, \alpha, q_\circ)\} - \min_{\alpha\in A}\{-\mathcal{L}_\alpha u_2 + f(\cdot\,, \alpha, q_\circ)\} = 0$$

Therefore uniqueness of a solution $(c_\circ, u_\circ(\cdot))$ is reduced to proving that there cannot exist non-constant sub-solutions to the static HJB equation $\min_{\alpha\in A}\{-\mathcal{L}_\alpha v\} = 0$, where $v := u_1 - u_2$ i.e. whether Liouville property holds for the latter static HJB equation. This is answered positively in [22] using the following
<u>*claim:*</u> *there exist a function* $\psi \in C^\infty(\mathbb{R}^m)$ *and* $R_o > 0$ *such that*

$$\min_{\alpha\in A}\{-\mathcal{L}_\alpha \psi(x)\} \geq 0 \quad \text{in } \overline{B(0, R_o)}^C, \quad \psi(x) \to +\infty \text{ when } |x| \to +\infty \tag{4.5.28}$$

and such that

$$\lim_{|x|\to+\infty} \frac{v(x)}{\psi(x)} = 0 \tag{4.5.29}$$

Hence, a Liouville type result [22, Theorem 2.1] insures that v is constant, i.e. $u_1(\cdot) - u_2(\cdot) \equiv$ constant.

Proof of the claim:
We check that $\psi(x) := |x|^d \log(|x|)$ satisfies (4.5.28) and (4.5.29): Using the polynomial growth of u_1 and u_2, (4.5.29) is immediate. To check the validity of (4.5.28), we compute $-\mathcal{L}_\alpha\psi(x)$ and make use of assumptions (A3, A4, A6). This has been done in the claim in step 3 of the proof of Theorem 4.3.2. □

Proof *of Theorem 4.5.2.* The proof follows exactly the one of Theorem 4.5.1 with minor modifications. We refer to the proof of Theorem 4.4.2 for further details. In particular, the Liouville type result we will need is [22, Theorem 2.2]. □

Proof *of Corollary 4.5.1.* This is a consequence of Theorem 4.5.1 together with Lemma 4.2 (see also Theorem 4.2.2 (ii)). □

4.5.5.1 The non-*separable* case

In this subsection, we shall prove a result analogue to Corollary 4.5.1 but where we drop the assumption (B0). To do so, we need to solve the (infinite dimensional) optimization problem $(\mathfrak{P}^\wedge)$ in the more general case of f being non-*separable.* The conclusion then follows using Theorem 4.5.1. The case of $(\mathfrak{P}^\vee)$ is analogous.

From uniqueness of the invariant probability measure (see Theorem 4.2.7), the problem ($\mathfrak{P}^\wedge$) can be equivalently expressed by

$$\min_{\alpha(\cdot)\in\mathcal{A}} \langle f(\cdot\,,\alpha(\cdot),\mu_\alpha),\mu_\alpha\rangle$$

since for each α, we have a unique measure μ_α in the constraints set. In the light of Theorem 4.2.5, μ_α has a density ρ_α and we can therefore write

$$\langle f(\cdot\,,\alpha(\cdot),\mu_\alpha),\mu_\alpha\rangle = \int_{\mathbb{R}^m} f(x,\alpha(x),\rho_\alpha)\,\rho_\alpha(x)\,\mathrm{d}x.$$

Recall that in the third argument of f, we can have either a local dependence on μ_α (hence on ρ_α) or a non-local dependence as discussed earlier in the introduction. We would like now to *exchange* the minimization and the integral. But to do so, we need a result on the map $\alpha \mapsto \rho_\alpha$.

In the sequel, we shall make the following additional assumptions (which remedy the absence of (B0))

(B1) $A = [0,1]$,

(B2) The restrictions of a^{ij}_α and b^i_α to every ball $U \subset \mathbb{R}^m$ are continuous in α in the space $L^1(U)$,

(B3) The family of measures μ_α solution to $\mathcal{L}^*_\alpha\mu_\alpha = 0$ is uniformly tight[11].

We have the following result in [45] (see also [40, Proposition 3.7.4, p. 123]).

Proposition 4.5.2. *([45, Proposition 1.1]) Assume* $(A1, A2, A3)$ *and* $(B1, B2, B3)$ *hold. Then one can choose densities* ρ_α *of* μ_α *such that the function* $\rho_\alpha(x)$ *is jointly continuous. In addition, the mapping* $\alpha \mapsto \rho_\alpha$ *with values in* $L^1(\mathbb{R}^m)$ *is continuous, i.e., the mapping* $\alpha \mapsto \mu_\alpha$ *is continuous in the variation norm.*

Remark 4.5.7. *([45] or [40, p.124]) A sufficient condition for (B3) to hold is the existence of a single Lyapunov function* V *such that* $V(x) \to +\infty$ *and* $\sup_{\alpha\in A}\mathcal{L}_\alpha V(x) \to -\infty$ *as* $|x| \to +\infty$*. And this is satisfied under the conditions* $(A3, A4)$ *in view of Corollary 4.2.2.*

As a direct consequence of Proposition 4.5.2, we obtain the desired *exchange property.*

[11] A family $\mathfrak{S}$ of probability measures is uniformly tight if, for each $r > 0$, there is a compact set K such that $\mu(\mathbb{R}^m \setminus K) \le r$ for all $\mu \in \mathfrak{S}$.

Lemma 4.12. *Assuming* $(A1{-}A5(i,ii,iii,iv))$ *and* $(B1,B2)$ *hold, we have the following*

$$\min_{\alpha(\cdot)\in\mathcal{A}} \langle f(\cdot\,,\alpha(\cdot),\mu_\alpha),\mu_\alpha\rangle = \int_{\mathbb{R}^m} \min_{\alpha\in A} f(x,\alpha,\rho_\alpha)\rho_\alpha\, dx \tag{4.5.30}$$

Proof *of Lemma 4.12.* The continuity of $[0,1]\ni\alpha\mapsto f(x,\alpha,\rho_\alpha)\rho_\alpha$ is a direct consequence of Proposition 4.5.2 together with the assumptions $(A1,A2,A3,A5(ii,iv))$ and $(B1,B2)$, noting in addition that $(B3)$ is satisfied with $(A3,A4)$ using Remark 4.5.7. We are therefore in the situation of Proposition 4.4.1, since $x\mapsto f(x,\alpha(x),\rho_\alpha)\rho_\alpha(x)$ is in $L^1(\mathbb{R}^m;\mathrm{d}x)$ where we recall $\mathrm{d}x$ is Lebesgue measure, thanks to assumption $(A5(iii))$. The proof then follows using the same arguments as for Proposition 4.4.1. □

Finally, we have the existence (and uniqueness) result for the system MFG in the non-*separable* case.

Corollary 4.5.2. *In the situation of Theorem 4.5.1, if we assume in addition that (B1,B2) hold, then the ergodic MFG system admits a solution satisfying the properties (II) in Theorem 4.5.1.*
The same also holds true when in the situation of Theorem 4.5.2.

Proof *of Corollary 4.5.2.* It suffices to show existence of a solution for (I) in Theorem 4.5.1. This is true and is in the same spirit as in *Step 3* in the proof of Lemma 4.2, thanks to the exchange property in Lemma 4.12 which reduces the infinite dimensional optimization problem to a finite dimensional optimization problem where we minimize a continuous functions over a compact set.

The case of max (instead of min) as in Theorem 4.5.2 follows analogously. □

4.6 Conclusion and future perspectives

We addressed the solvability of the ergodic Bellman problem in the whole space with unbounded and measurable data using a new method based on abstract optimization techniques together with results from Dirichlet forms theory. We also discussed uniqueness of the solution under additional assumptions and characterized the ergodic constant as being the critical one which then allows us to provide an estimate that measures its dependency on the data of the problem. Moreover, we showed that the method can be extend to the case of non-compact Riemannian manifolds with no boundaries.

Our strategy allows us also to tackle the problem of ergodic Mean-Field games. We provided necessary and sufficient conditions for existence of a solution to the ergodic

MFG system. Uniqueness is also discussed. And we showed moreover that these conditions are satisfied both in the *separable* and in the non-*separable* cases (under additional smoothness assumptions for the latter).

Further extensions

Besides the questions on generalization (see our Remark 4.2.2), e.g. using [40, Chapter 5], several challenging problems are still not clear.

A first interesting and natural open problem would be to prove in our setting the convergence of the approximating *δ-cell problem* and *t-cell problem* towards the *true cell problem* which we called here *ergodic PDE*; see [4, Chapter 2] for the convergence results under different assumptions and using different techniques.

A challenging second question would be whether one can adapt such techniques in the case of *ergodic stochastic games*, i.e. for Hamilton-Jacobi-Isaacs equations (see [37] and also [4]). The difficulty indeed appears when we perform Legendre-Fenchel duality to construct the dual problem as in §4.4.2.

One could also try to use these results to solve fully nonlinear elliptic PDEs more general than those of Bellman type as we have considered, using for example Pucci's extremal operator; see, e.g., [51, §2.2].

Another work which made use of convex duality and invariant measures to address uniqueness problem for viscous HJB is [5], unlike in our case where we used duality for the existence problem. It could be interesting to see how much these methods can complement each other to address both existence and uniqueness.

Finally, it would be interesting to highlight the link of the method in this chapter with weak KAM theory. It is not difficult to see that what we called *primal* problem is strongly related to Mather's variational problem in the stochastic setting [91]. And indeed, the method presented here is very reminiscent of [77], and to some extent to [92, 102, 103]. Also the definition of the ergodic constant as the value of the dual problem insures that it is indeed the the *critical value* or *Mañé critical value* (see Remark 4.4.4).

We hope we can tackle these problems in a future work.

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