Advertising and exogenous interference in a segmented market

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Abstract   We propose a model of a monopolist firm which advertises a product in a segmented market where a constant exogenous interference is present. Using the framework of Nerlove-Arrow advertising model, we describe the interference using a constant negative addendum in the goodwill ODE. This effect may vary over the different market segments. Hence, we admit that the firm goodwill concerning any segment may become negative and we associate 0 demand with negative goodwill values. By using the simplest demand model, i.e. a piecewise linear function, we formulate a nonsmooth infinite horizon optimal control problem. The features of an optimal advertising policy depend on the different effects in each segment of the advertising effort and of the exogenous interference. We describe an optimal solution for a single wide-spectrum medium, which requires a constant advertising effort after some finite time. We characterize the segment sets which may be a possible target for the firm using a given advertising medium.

Key words   Advertising; Market Segmentation; Optimal Control.
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1 Introduction

Nerlove and Arrow's paper [12] is a cornerstone of dynamic advertising modelling and its discretization in Little's *Brandaid* model [11] is one of the most important references for marketing practitioners. Its importance in a theoretical and practical setting is well shown in [10] and in the references therein. Usually, the literature based on Nerlove-Arrow model assumes a homogeneous market, although heterogeneous markets are a common experience in practice. Marketing theory proposes segmentation strategies to

handle market heterogeneity, but the description of a segmented market in a dynamic setting is quite recent. To the best of our knowledge, the problem of introducing a new product in a segmented market has been studied for the first time in [5], where a standard way to describe segmentation in a dynamic advertising model is presented. That approach is exploited in [8] to analyze the effects of advertising on sales in the long run, when a segmented market is considered.

The extension of Nerlove-Arrow model to the dynamic game framework is an interesting and growing subject of research (see [10]). A still unexplored area of research concerns the connection between equilibrium advertising policies in a dynamic game setting and market segmentation. A first step in this direction is the paper [9]. Here we want to go further.

We assume that a firm perceives the action of all other players in the market arena as an interference hampering its own action. Actually, the interference phenomenon has been an object of attention by marketing scholars recently (see e.g. [3], [4], [6], [13]). For the sake of simplicity, we consider the overall interference for a firm as a constant negative effect on the goodwill evolution of its brand. The basic model is the same as that described in [9], but enriched to represent a segmented market. Actually, the right framework to describe competition is the dynamic games setting; however in this paper we are still assuming a single decision maker and we are summarizing the actions of the other players in an exogenus term. Hence, this paper is only a first step towards the competitive scenario.

In Section 2 we use the results obtained in [9] to present and analyze the extention of the Nerlove-Arrow model, which includes an exogenus interference in the goodwill ODE. If the firm has a set of *segment specific advertising media*, two kinds of strategies are possibly optimal for each segment: either stay in the segment in the long run with a constant advertising effort, or exit from the segment in the short run. Hence only some segments are relevant for the firm.

In Section 3 we assume that the firm can advertise to the whole market with a *single medium* and we ask which segments are relevant for the firm in this case. This answer is crucial from a practical point of view, because it permits to associate an advertising medium with a set of segments which can be supported by that medium. In Section 4 we suggest some future research directions.

2 The model: segmented market, advertising, interference

Let the consumer population be partitioned into groups (segments), each one specified by the value $a \in A$ of a suitable parameter *(segmentation attribute)*. Here A is a finite set. Typically, a small number of segments – two or three – is considered in practice. We refer to the definition of *goodwill* given by Nerlove and Arrow [12] to describe the variable which summarizes the effects of present and past advertising on the demand. The goodwill

needs an advertising effort to increase, while it is subject to a spontaneous decay. As done in [5], here we assume that there exists a different state variable (goodwill) for each segment in the market.

Let $G_a(t)$ represent the stock of goodwill of the product at time t, for the (consumers in the) a segment. We assume that the product demand rate in each segment a is a piecewise linear function of its goodwill,

$$D_a(G_a) \;=\; \beta\sigma_a \cdot \max\{0, G_a\} \;=\; \beta\sigma_a \cdot [G_a]^+ \,, \quad a \in A, \qquad (1)$$

with $\sigma_a > 0$ and $\sum_{a\in A}\sigma_a = 1$. Then $\beta\sigma_a > 0$ is the marginal demand of goodwill from segment a, when $G_a > 0$, and σ_a represents the percentage of sales in market segment a, provided that the goodwill value is the same for all segments. This is the most elementary model of demand dependence on goodwill and in fact it is rather common in the marketing literature: see for example [7, p. 303].

The goodwill values $G_a(t)$, $a \in A$, are the outcome of an advertising process, which is activated by the decision maker, and of an exogenous interference (done for example by other players in the market arena). The latter may have a different weight among the segments. The firm action is a set of advertising intensities $w_a(t) \in [0, w_a^{\max}]$, $a \in A$, directed to the different segments and we assume that the goodwill of each segment evolves in time according to the differential equations

$$\dot{G}_a(t) \;=\; w_a(t) - \zeta_a - \delta\,G_a(t)\,, \quad a \in A \qquad (2)$$

where

- $\delta > 0$ represents the goodwill depreciation rate, the same in all segments;
- $\zeta_a \geq 0$ is the exogenous *interference* toward segment a goodwill.

Moreover, we assume the goodwill levels at the initial time to be known,

$$G_a(0) = \alpha_a \; > 0\,, \quad a \in A. \qquad (3)$$

The dynamics of the goodwill given by the linear equation (2) is essentially the same as the one proposed by [12]. Here, we have further assumed that in each segment there is a negative interference which contrasts the advertising effort. It is not only a theoretical point: for instance, if a retailer sells both a national and a store brand, then the advertising effort for the store brand may interfere negatively with the goodwill evolution of the national brand (see [1,2]).

We assume that the firm wants to maximize the discounted profit

$$J\left[(w_a)_{a\in A}\right] = \sum_{a\in A} \int_0^\infty \left\{ \pi\beta\sigma_a\, [G_a(t)]^+ - \frac{\kappa_a}{2} w_a^2(t) \right\} e^{-\rho t} dt, \qquad (4)$$

where $\pi, \beta, \rho, \kappa_a > 0$. In particular, ρ is the discount factor; π is the marginal profit of the decision maker with respect to demand; $\beta\sigma_a$ is the marginal

demand of goodwill; $\kappa_a w_a^2 / 2$ is the cost intensity associated with the advertising intensity w_a for segment a. This is a nonsmooth optimal control problem.

In view of the additive form of the objective functional and of the fact that the advertising intensities are subject only to the non-negativity constraints, we observe that maximizing $J\left[(w_a)_{a \in A}\right]$ is equivalent to maximizing independently each term

$$J_a\left[w_a\right] = \int_0^\infty \left\{ \pi \beta \sigma_a \left[G_a(t)\right]^+ - \frac{\kappa_a}{2} w_a^2(t) \right\} e^{-\rho t} dt\,, \tag{5}$$

of the sum (4), with conditions (2), (3) and the non-negativity constraint $w_a(t) \geq 0$. The segment-related functional (5) represents the profit coming from segment a.

The solution of the segment-related problem is the same as the one described in [9]. For each segment there exists an optimal solution $(\bar{w}_a, \bar{G}_a)$ and the advertising strategy $\bar{w}_a$ can take one of the following two forms:

– either the constant

$$\bar{w}_a(t) \equiv w_a^* = \frac{\pi \beta \sigma_a}{(\delta + \rho) \kappa_a} \tag{6}$$

and in this case $\bar{G}_a(t) > 0$ for all $t \in [0, +\infty)$;
– or the monotonically decreasing function

$$\bar{w}_a(t) = w_a^{\tau_a}(t) = \frac{\pi \beta \sigma_a}{(\delta + \rho) \kappa_a} \left[1 - e^{(\delta + \rho)(t - \tau_a)} \right]^+\,, \tag{7}$$

where, $\tau_a \in (0, +\infty)$ is the *exit time*, the first time at which the goodwill $\bar{G}_a(t)$ for segment a vanishes. The exit time τ_a is the solution of the equation

$$\frac{\zeta_a - w_a^*}{\delta}(e^{\delta \tau} - 1) + \frac{w_a^*(e^{\delta \tau} - e^{-(\delta + \rho)\tau})}{2\delta + \rho} \;=\; \alpha_a\,. \tag{8}$$

Sometimes, only one of these two strategies is a candidate to optimality: e.g. if equation (8) has no solutions, then the constant advertising strategy (6) is optimal. In other cases both necessary conditions are satisfied and to decide which strategy is the best one we have only to calculate and compare the two profits $J_a\left[w_a^*\right]$ and $J_a\left[w_a^{\tau_a}\right]$.

In [9] we have proved a sufficient condition for the firm to stay in a market segment a in the long run. If the exogenous disturbance in segment a is sufficiently small

$$\zeta_a < \frac{\delta + \rho}{2\delta + \rho} w_a^*, \tag{9}$$

and the initial goodwill value $G_a(0) = \alpha_a$ is sufficiently high

$$\alpha_a \geq \frac{w_a^* - \zeta_a}{\delta} - \frac{w_a^*}{\delta} \left(1 - \frac{\zeta_a}{w_a^*} \Big/ \frac{\delta + \rho}{2\delta + \rho} \right)^{\frac{\delta + \rho}{2\delta + \rho}}\,, \tag{10}$$

then the optimal advertising effort in the segment a is (6) and the goodwill of this segment is always positive.

This result has also an interesting and practical interpretation: the relevant segments for the firm are the ones where the interference produced by the other competitors is small and where the firm has a good position at the beginning of the programming interval. Two are the ingredients for staying in the market in the long run in a given segment: a light competition and a good (initial) position of the firm. Both these characteristics are held by the firms which are successful in a niche market.

3 A single wide-spectrum advertising medium

In the real world it may be difficult and expensive to plan an advertising campaign using a set of media which hit independently each segment, as it is described in the previous section. In practice, often the firm has to use a medium which reaches several segments with variable segment-effectiveness. To represent this situation we assume that the goodwill evolution in the different segments is driven by a single advertising effort $u(t)$ according to the motion equations (2), where the advertising intensities are proportional to the effort,

$$w_a(t) = \gamma_a u(t), \qquad a \in A, \tag{11}$$

and $\gamma_a \geq 0$, $\sum_{a \in A} \gamma_a > 0$. We call $(\gamma_a)_{a \in A}$ the medium *(segment-)spectrum*; its components provide the different effectiveness of the advertising medium on the market segments. The decision maker wants to maximize the discounted profit functional

$$J[u] = \int_0^\infty \left\{ \pi \beta \sum_{a \in A} \sigma_a \left[G_a(t) \right]^+ - \frac{\kappa}{2} u^2(t) \right\} e^{-\rho t} dt, \tag{12}$$

with initial conditions (3) and control bound $u(t) \leq u^{\max}$. Here $\kappa u^2/2$ is the cost intensity associated with the advertising effort u. In the previous section assumptions, the firm can drive the evolution of the goodwill of each segment independently (control and state have the same number of components). Here, on the contrary, the firm has only one control (the effort using the single medium), and drives the evolution of all the state variables (the goodwill of all the segments) by means of it.

The following theorem provides a characterization of the optimal control in an ideal situation for the firm: when it is optimal to be present with a positive goodwill in all segments at all times.

Theorem 1 *Let $(\bar{u}, \bar{G})$ be an optimal control-state pair and let*

$$\bar{G}_a(t) > 0, \quad t \geq 0, \quad \text{for all } a \in A; \tag{13}$$

then

$$\bar{u}(t) \equiv u_A = \frac{\pi \beta}{(\delta + \rho)\kappa} \sum_{a \in A} \gamma_a \sigma_a, \tag{14}$$

and

$$\gamma_a u_A \geq \zeta_a \,, \qquad a \in A \,. \tag{15}$$

Proof The inequalities (13) imply that the value of the objective functional at the solution $(\bar{u}, \bar{G})$ coincides with the value of the functional

$$\Phi(u) \;=\; \int_0^{+\infty} e^{-\rho t} \left[\pi\beta \sum_{a \in A} \sigma_a G_a(t) - \frac{\kappa}{2} u^2(t) \right] dt \,, \tag{16}$$

which has a smooth integrand function. Then the solution $(\bar{u}, \bar{G})$ must satisfy the optimality necessary conditions for the problem of maximizing the objective functional (16), subject to the motion equations (2) and the initial conditions (3). We rely on the necessary conditions for such infinite horizon optimal control problem by [14]. The Hamiltonian is

$$H(G, u, p, t) = \sum_{a \in A} \left[(\pi\beta\sigma_a - \delta p_a) G_a - p_a \zeta_a \right] + \sum_{a \in A} \gamma_a p_a u - \frac{\kappa}{2} u^2 \,, \tag{17}$$

which is a strictly concave function of u. Hence there exists the unique maximum point

$$u = \frac{1}{\kappa} \left[\sum_{a \in A} \gamma_a p_a \right]^{+} \,. \tag{18}$$

The adjoint functions $p_a(t)$ must be solutions to the differential equations

$$\dot{p}_a(t) = -\pi\beta\sigma_a + (\delta + \rho)\, p_a(t) \,, \qquad a \in A \,, \tag{19}$$

and therefore

$$p_a(t) = \frac{\beta\pi\sigma_a}{\delta + \rho} + e^{(\delta + \rho)t} \left(p_a(0) - \frac{\beta\,\pi\,\sigma_a}{\delta + \rho} \right) \,. \tag{20}$$

This function is bounded if and only if

$$p_a(t) \equiv \frac{\beta\pi\sigma_a}{\delta + \rho} \,. \tag{21}$$

Using (18), we obtain the control (14). In view of the motion equations (2), such constant control is compatible with the positive goodwill path hypothesis (13) if and only if the inequalities (15) hold.   □

   Both in this discussion and in the analysis in the previous section we neglect to consider the upper bounds to the controls. This approach is correct if and only if $u^{\max}$, or $u_a^{\max}$, are greater than the control values (14), or (6) respectively. We assume that the inequality holds, because we want to consider the trade off between revenue and advertising costs (the same is done in [8], [9]).
   In practice, the ideal situation depicted by the hypothesis of the above theorem is seldom encountered. Therefore, a firm wants to ask the following question: "Using optimally a single medium, which are the segments where the goodwill remains positive in the long run?" We provide an answer.

Corollary 1 *Let $(u(t),\, G(t))$ be an optimal control-state pair and let there exist a time τ and a subset $\bar{A}$ of the segment set A, $\bar{A} \subseteq A$, such that*

$$\begin{cases} G_a\,(t) > 0\,, & t \geq \tau\,,\ a \in \bar{A}\,, \\ G_a\,(t) \leq 0\,, & t \geq \tau\,,\ a \notin \bar{A}\,, \end{cases} \tag{22}$$

then

$$u(t) = u_{\bar{A}} = \frac{\pi\beta}{(\delta + \rho)\kappa} \sum_{a \in \bar{A}} \gamma_a \sigma_a, \quad \text{for all } t \geq \tau\,, \tag{23}$$

and

$$\bar{A} = \{\, a \in A \mid \gamma_a u_{\bar{A}} \geq \zeta_a \,\}\,. \tag{24}$$

Proof Let $(u(t),\, G(t))$ be an optimal control-state pair such that (22) holds for some $\tau \geq 0$ and some $\bar{A} \subseteq A$. Hence the control function $\bar{u}|_{[\tau, +\infty)}$, i.e. the restriction of $\bar{u}$ to the interval $[\tau, +\infty)$, is an optimal solution of the problem of maximizing

$$\Phi_\tau(u) = \int_\tau^{+\infty} e^{-\rho t} \left[\pi\beta \sum_{a \in \bar{A}} \sigma_a G_a\,(t) - \frac{\kappa}{2} u^2(t) \right] dt\,, \tag{25}$$

subject to the motion equations (2) and the initial conditions

$$G_a\,(\tau) = \bar{G}_a\,(\tau) \; > \; 0\,, \quad a \in \bar{A}\,. \tag{26}$$

This is essentially the same problem as the one analyzed in Theorem 1, so that from (14) and (15) we obtain (23) and (24). $\qquad\square$

Corollary 1 provides a necessary condition for the firm to be optimally present in a submarket $\bar{A} \subseteq A$ at all times, while quitting business outside $\bar{A}$. Equation (24) has a practical interest: a subset of segments that satisfy this equation characterizes the optimal behavior of the firm in the long run. Hence this equation provides a very interesting information: given a medium, a subset of A that satisfies this equation is a submarket where the goodwill (and therefore the demand) may remain positive for all t. Such a submarket is a possible target for the firm.

We can associate a medium with the collection $\mathcal{A}$ of subsets of A which may be the target for the firm in the long run: a subset $\bar{A}$ of segments belongs to $\mathcal{A}$ if and only if it satisfies the two conditions (23) and (24). Even if we have neglected the first part of the programming interval, we can characterize the portions of the market which are relevant for the firm, given that a single medium is used. For each subset $\bar{A}$ belonging to $\mathcal{A}$, there exists a constant control which is a candidate to coincide with an optimal advertising effort after a suitable critical time.

In the following we want to apply the results of Corollary 1 to a numeric example. Let us consider a market with three segments, so that $A = \{1, 2, 3\}$; let the discount and decay rates and the cost parameter be such

Table 1 Segment related parameters

a	1	2	3
ζ_a	6	8	9

a	1	2	3
$\beta\pi\sigma_a$	5	2	6

a	1	2	3
γ_a	2	1,5	1

that $\delta + \rho = 0.05$, $\kappa = 20$; let the values of the segment related parameters be as in Table 1.

Then, in Table 2 the segment subsets which satisfy the necessary conditions to be target submarkets are checked by "$\sqrt{}$". We see, for instance,

Table 2 Submarkets

A^*	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
	$\sqrt{}$	no	no	$\sqrt{}$	$\sqrt{}$	no	$\sqrt{}$

that the segment 2 cannot be taken as an independent submarket, but must be associated with the segment 1; the subset $\{1,2\}$ satisfies the necessary conditions to be a target, whereas $\{2,3\}$ does not.

4 Conclusion

In this paper we present an extension of Nerlove-Arrow advertising model in order to deal with customer segmentation and with the presence of an exogenous interference in the market.

Here we characterize the segments which are relevant for the firm in the segment specific advertising setting and also in the single medium advertising setting. We have obtained a set of necessary conditions in order to keep a positive demand in the long run in some segments.

An interesting issue, which has not been tackled in this paper, is the comparison between two advertising media with different advertising spectra.

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