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Transferring L^p eigenfunction bounds from S^{2n+1} to h^n

by

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Abstract. By using the notion of contraction of Lie groups, we transfer L^p-L^2 estimates for joint spectral projectors from the unit complex sphere S^{2n+1} in $\mathbb{C}^{n+1}$ to the reduced Heisenberg group h^n. In particular, we deduce some estimates recently obtained by H. Koch and F. Ricci on h^n. As a consequence, we prove, in the spirit of Sogge's work, a discrete restriction theorem for the sub-Laplacian L on h^n.

1. Introduction. In the last twenty-five years the notion of *contraction* (or *continuous deformation*) of Lie algebras and Lie groups, introduced in 1953 in a physical context by E. Inönu and E. P. Wigner, was developed in a mathematical framework as well. The basic idea is that, given a Lie algebra $\mathfrak{g}_1$, from a family of non-degenerate transformations of its structure constants it is possible to obtain, in a limit sense, a non-isomorphic Lie algebra $\mathfrak{g}_2$.

It turns out that the deformed algebra $\mathfrak{g}_2$ inherits analytic and geometric properties from $\mathfrak{g}_1$ and that the same holds for the corresponding Lie groups. As a consequence, transference results have attracted considerable attention, in particular in the context of Fourier multipliers. In fact, contraction has been successfully used to transfer L^p multiplier theorems from one Lie group to another. There is an extensive literature on this topic, centered about deLeeuw's theorems; we only mention here the results by A. H. Dooley, G. Gaudry, J. W. Rice and R. L. Rubin ([D], [DGa], [DRi1], [DRi2], [Ru]), concerning, in particular, contraction of rotation groups and semisimple Lie groups.

The primary purpose of this paper is to show that contraction is an effective tool to transfer L^p eigenfunction bounds as well. In particular, we shall focus on a contraction from the complex unit sphere S^{2n+1} in $\mathbb{C}^{n+1}$ to the reduced Heisenberg group h^n.

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We recall that, if P is a second order self-adjoint elliptic differential operator on a compact manifold M and if P_λ denotes the spectral projection corresponding to the eigenvalue λ^2, a classical problem is to estimate the norm ν_p of P_λ as an operator from $L^p(M)$, $1 \le p \le 2$, to $L^2(M)$. Sharp estimates for ν_p have been obtained by C. Sogge ([So2]), who proved that

$$\|P_\lambda\|_{(p,2)} \le C\lambda^{\gamma(1/p,n)}, \quad 1 \le p \le 2, \tag{1.1}$$

where γ is the piecewise affine function on $[1/2, 1]$ defined by

$$\gamma\left(\frac{1}{p}, n\right) := \begin{cases} n\left(\frac{1}{p} - \frac{1}{2}\right) - \frac{1}{2} & \text{if } 1 \le p \le \tilde{p}, \\ \frac{n-1}{2}\left(\frac{1}{p} - \frac{1}{2}\right) & \text{if } \tilde{p} \le p \le 2, \end{cases}$$

with *critical point* $\tilde{p}$ given by $\tilde{p} := 2(n+1)/(n+3)$.

The starting point for our approach is a sharp two-parameter estimate for joint spectral projections on complex spheres, recently obtained by the first author ([Ca]). More precisely, we consider the Laplace–Beltrami operator $\varDelta_{S^{2n+1}}$ and the sub-Laplacian $\mathcal{L}$ on S^{2n+1} (they form a basis for the algebra of $U(n+1)$-invariant differential operators on S^{2n+1}). It is possible to work out a joint spectral theory. In particular, we denote by $\mathcal{H}^{l,l'}$, $l, l' \ge 0$, the joint eigenspace with eigenvalue $\mu_{l,l'}$ for $\varDelta_{S^{2n+1}}$, where $\mu_{l,l'} := -(l+l')(l+l' + 2n)$, and with eigenvalue $\lambda_{l,l'}$ for $\mathcal{L}$, where $\lambda_{l,l'} := -2ll' - n(l + l')$ ([Kl]). It is a classical fact ([VK, Ch. 11]) that

$$L^2(S^{2n+1}) = \sum_{l,l'=0}^{\infty} \oplus \mathcal{H}^{l,l'}. \tag{1.2}$$

We denote by $\pi_{l,l'}$ the joint spectral projector from $L^2(S^{2n+1})$ onto $\mathcal{H}^{l,l'}$. In [Ca] the first author proved the following two-parameter L^p eigenfunction bounds:

$$\|\pi_{l,l'}\|_{(p,2)} \lesssim C(2q_l + n)^{\alpha(1/p,n)}(1 + Q_l)^{\beta(1/p,n)} \quad \text{for all } l, l' \ge 0, \tag{1.3}$$

where $Q_l := \max\{l, l'\}$, $q_l := \min\{l, l'\}$ and α and β are the piecewise affine functions represented in Figure 1 at the end of Section 2. We remark that the critical exponent in our case is $2(2n+1)/(2n+3)$ and cannot be directly deduced from Sogge's results. Observe moreover that $2q_l + n$ and Q_l are related to the eigenvalues $\lambda_{l,l'}$ and $\mu_{l,l'}$, since they grow, respectively, as $|\lambda_{l,l'}|/(l + l')$ and $|\mu_{l,l'}|^{1/2}$.

On the other hand, on the reduced Heisenberg group h^n, defined as $h^n := \mathbb{C}^n \times \mathbb{T}$, with product

$$(\mathbf{z}, e^{it})(\mathbf{w}, e^{it'}) := (\mathbf{z} + \mathbf{w}, e^{i(t+t'+\operatorname{Im} \mathbf{z}\cdot\overline{\mathbf{w}})}),$$

with $\mathbf{z}, \mathbf{w} \in \mathbb{C}^n$, $t, s \in \mathbb{R}$, we consider the sub-Laplacian L and the operator $i^{-1}\partial_t$. The pairs $(2|m|(2k+1), m)$ with $m \in \mathbb{Z} \setminus \{0\}$ and $k \in \mathbb{N}$ give the discrete joint spectrum of these operators. Recently H. Koch and F. Ricci proved the following L^p-L^2 estimate for the orthogonal projector $P_{m,k}$ onto the joint eigenspace:

$$\|P_{m,k}\|_{(L^p(h^n),L^2(h^n))} \lesssim C(2k+n)^{\alpha(1/p,n)}|m|^{\beta(1/p,n)} \tag{1.4}$$

for $1 \le p \le 2$, where α and β are given by (1.3) ([KoR]).

We start by showing in Section 2 that $P_{m,k}$ may be obtained as the limit in the L^2-norm of a sequence of joint spectral projectors on S^{2n+1}. Then we give an alternative proof of (1.4) by a contraction argument.

A contraction from $SU(2)$ to the one-dimensional Heisenberg group H^1 was studied by F. Ricci and R. L. Rubin ([R], [RRu]). In [Ca] the first author used some ideas from [R] to transfer L^p-L^2 estimates for norms of harmonic projection operators from the unit sphere S^3 in $\mathbb{C}^2$ to the reduced Heisenberg group h^1. In this paper we discuss the higher-dimensional case.

A contraction from the unit sphere S^{2n+1} to the Heisenberg group H^n for $n > 1$ was analyzed by A. H. Dooley and S. K. Gupta; in a first paper they adapted the notion of Lie groups contraction to the homogeneous space $U(n+1)/U(n)$ and described the relationship between certain unitary irreducible representations of $U(n+1)$ and H^n ([DG1]), in a second paper they proved a deLeeuw type theorem on H^n by transferring results from S^{2n+1} ([DG2]). The contraction we use here is essentially that introduced by Dooley and Gupta; however, their approach is mainly algebraic, while our interest is directed to the analytic features of the problem.

As an application of (1.3) we prove in Section 3 a discrete restriction theorem for the sub-Laplacian L on h^n in the spirit of Sogge's work ([So1], see also (1.1)). More precisely, let Q_N be the spectral projection corresponding to the eigenvalue N associated to L on h^n, that is,

$$Q_N f := \sum_{(2k+n)|m|=N} P_{m,k} f.$$

The study of L^p-L^2 mapping properties of Q_N was suggested by D. Müller in his paper about the restriction theorem on the Heisenberg group ([M]). In [Th1] S. Thangavelu proved that

$$\|Q_N\|_{(L^p(h^n),L^2(h^n))} \le C(N^n d(N))^{1/p-1/2}, \qquad 1 \le p \le 2, \tag{1.5}$$

where $d(N)$ is the divisor-type function defined by

$$d(N) := \sum_{2k+n|N} \frac{1}{2k+n}, \tag{1.6}$$

and the estimate is sharp for $p = 1$. By writing $a \,|\, b$ we mean that a divides b.

Other types of restriction theorems on the Heisenberg group were discussed by Thangavelu in [Th2].

By using orthogonality, we add up the estimates in (1.3) and obtain L^p-L^2 bounds for the norm of Q_N, which in some cases improve (1.5). The exponent appearing in (1.5) is an affine function of $1/p$. In our estimate the exponent of $d(N)$ is, as in Sogge's results, a piecewise affine function of $1/p$. In other words, there is a critical point $\tilde{p}$ where the slope of the exponent changes. This critical point is the same as that found on complex spheres ([Ca]).

Our bounds are in general not sharp. The reason is that with our procedure we disregard the interferences between eigenfunctions. We show however that there are arithmetic progressions N_m in $\mathbb{N}$ for which our estimates for $\|Q_{N_m}\|_{(p,2)}$ are sharp and better than (1.5). Moreover, since the behaviour of $d(N)$ is highly irregular, we inquire about the average size of $\|Q_N\|_{(p,2)}$. We prove in this case that L^p-L^2 estimates do not involve divisor-type functions and that the critical point disappears.

2. Preliminaries. In this section we introduce some notation and recall a few results, that will be used in the following.

2.1. *Some notation.* For $n \geq 1$ let $\mathbb{C}^{n+1}$ denote the n-dimensional complex space endowed with the scalar product $\langle \mathbf{z}, \mathbf{w} \rangle := \mathbf{z} \cdot \overline{\mathbf{w}} := z_1\overline{w}_1 + \cdots + z_{n+1}\overline{w}_{n+1}$, $\mathbf{z}, \mathbf{w} \in \mathbb{C}^{n+1}$, and let S^{2n+1} denote the unit sphere in $\mathbb{C}^{n+1}$, that is,

$$S^{2n+1} := \{\mathbf{z} = (z_1, \ldots, z_{n+1}) \in \mathbb{C}^{n+1} : \langle \mathbf{z}, \mathbf{z} \rangle = 1\}.$$

The symbol $\mathbf{1}$ will denote the north pole of S^{2n+1}, that is, $\mathbf{1} := (0, \ldots, 0, 1)$.

For every $l, l' \in \mathbb{N}$ the symbol $\mathcal{H}^{l,l'}$ will denote the space of restrictions to S^{2n+1} of harmonic polynomials $p(\mathbf{z}, \bar{\mathbf{z}}) = p(z_1, \ldots, z_{n+1}, \overline{z}_1, \ldots, \overline{z}_{n+1})$, of homogeneity degree l in $z_1, \ldots, z_{n+1}$ and of homogeneity degree l' in $(\overline{z}_1, \ldots, \overline{z}_{n+1})$, i.e. such that

$$p(a\mathbf{z}, b\bar{\mathbf{z}}) = a^l b^{l'} p(\mathbf{z}, \bar{\mathbf{z}}), \qquad a, b \in \mathbb{R},\ \mathbf{z} \in \mathbb{C}^n.$$

For a detailed description of the spaces $\mathcal{H}^{l,l'}$ see Chapter 11 in [VK]. We only recall here that a polynomial p in $\mathbf{z}, \bar{\mathbf{z}}$ is said to be *harmonic* if

$$\Delta_{S^{2n+1}} p := \frac{1}{4}\left(\frac{\partial^2}{\partial z_1 \partial \overline{z}_1} + \cdots + \frac{\partial^2}{\partial z_{n+1} \partial \overline{z}_{n+1}}\right) p = 0, \tag{2.1}$$

where $\Delta_{S^{2n+1}}$ denotes the Laplace–Beltrami operator.

A *zonal function* of bidegree (l, l') on S^{2n+1} is a function in $\mathcal{H}^{l,l'}$ which is constant on the orbits of the stabilizer of $\mathbf{1}$ (which is isomorphic to $U(n)$). Given a zonal function f, we may associate to f a map ${}^b\!f$ on the unit disk by

$$f(\mathbf{z}) = {}^b\!f(\langle \mathbf{z}, \mathbf{1} \rangle), \qquad \mathbf{z} \in S^{2n+1}$$

(by using the notation in Section 11.1.5 of [VK] we have $\langle \mathbf{z}, \mathbf{1} \rangle = z_n = e^{i\varphi}\cos\theta$, where $\varphi \in [0, 2\pi]$ and $\theta \in [0, \pi/2]$).

By means of ${}^b f$ we may define a convolution of a zonal function f and an arbitrary function g on S^{2n+1}. More precisely, we set

$$(f * g)(\mathbf{z}) := \int_{S^{2n+1}} {}^b f(\langle \mathbf{z}, \mathbf{w} \rangle) g(\mathbf{w})\, d\sigma(\mathbf{w}),$$

where $d\sigma$ is the measure invariant under the action of the unitary group $U(n+1)$ (see (3.4) for an explicit formula). In the following we shall write $f(\theta, \varphi)$ instead of ${}^b f(e^{i\varphi}\cos\theta)$.

Let $L^2(S^{2n+1})$ be the Hilbert space of functions on S^{2n+1} endowed with the inner product $(f, g) := \int_{S^{2n+1}} f(\mathbf{z})\overline{g(\mathbf{z})}\, d\sigma(\mathbf{z})$.

It is a classical fact ([VK, Ch. 11]) that $L^2(S^{2n+1})$ is the direct sum of the pairwise orthogonal and $U(n+1)$-invariant subspaces $\mathcal{H}^{l,l'}$, $l, l' \geq 0$. In other words, every $f \in L^2(S^{2n+1})$ admits a unique expansion

$$f = \sum_{l,l'=0}^{\infty} Y^{l,l'},$$

where $Y^{l,l'} \in \mathcal{H}^{l,l'}$ for every $l, l' \geq 0$ and the series on the right converges to f in the $L^2(S^{2n+1})$-norm.

The orthogonal projector onto $\mathcal{H}^{l,l'}$,

$$\pi_{l,l'} : L^2(S^{2n+1}) \ni f \mapsto Y^{l,l'} \in \mathcal{H}^{l,l'}, \tag{2.2}$$

may be written as

$$\pi_{l,l'} f := {}^b\mathbb{Z}_{l,l'} * f,$$

where $\mathbb{Z}_{l,l'}$ is the zonal function from $\mathcal{H}^{l,l'}$, given by

$$\mathbb{Z}_{l,l'}(\theta, \varphi) := \frac{d_{l,l'}}{\omega_{2n+1}} \frac{q_l!(n-1)!}{(q_l+n-1)!} e^{i(l'-l)\varphi} (\cos\theta)^{|l-l'|} P_{q_l}^{(n-1,|l-l'|)}(\cos 2\theta) \tag{2.3}$$
$$l, l' \geq 1,\ \varphi \in [0, 2\pi],\ \theta \in [0, \pi/2],$$

where $q_l = \min(l, l')$, ω_{2n+1} denotes the surface area of S^{2n+1}, $P_{q_l}^{(n-1,|l-l'|)}$ is the Jacobi polynomial and

$$d_{l,l'} := \dim \mathcal{H}^{l,l'} = n\,\frac{l+l'+n}{ll'} \binom{l+n-1}{l-1} \binom{l'+n-1}{l'-1} \qquad \text{for all } l, l' \geq 1.$$

Recall finally that $\mathcal{H}^{l,0}$ consists of holomorphic polynomials and $\mathcal{H}^{0,l}$ consists of polynomials whose complex conjugates are holomorphic. In both cases, the dimension of the space is given by

$$\dim \mathcal{H}^{l,0} = \dim \mathcal{H}^{0,l} = \binom{l+n-1}{l}$$

and the zonal function is

$$\mathbb{Z}_{l,0}(\theta,\varphi) := \frac{1}{\omega_{2n-1}}\binom{l+n-1}{l} e^{-il\varphi}(\cos\theta)^l, \qquad \varphi\in[0,2\pi],\ \theta\in[0,\pi/2].$$

In this paper we shall adopt the convention that C denotes a constant which is not necessarily the same at each occurrence.

2.2. *Some useful results.* In order to transfer L^p bounds from S^{2n+1} to h^n we shall need both a pointwise estimate for the Jacobi polynomials, due to Darboux and Szegö ([Sz, pp. 169, 198]), and a Mehler–Heine type formula, relating Jacobi and Laguerre polynomials ([Sz], [R]).

LEMMA 2.1. *Let $\alpha,\beta > -1$. Fix $0 < c < \pi$. Then*

$$P_l^{(\alpha,\beta)}(\cos\theta) = \begin{cases} O(l^\alpha) & \text{if } 0\le\theta\le c/l,\\ l^{-1/2}k(\theta)(\cos(N_l\theta+\gamma)+(l\sin\theta)^{-1}O(1)) & \text{if } c/l\le\theta\le\pi-c/l,\\ O(l^\beta) & \text{if } \pi-c/l\le\theta\le\pi, \end{cases}$$

where

$k(\theta) := \pi^{1/2}\big(\sin\frac{\theta}{2}\big)^{-\alpha-1/2}\big(\cos\frac{\theta}{2}\big)^{-\beta-1/2}$, $N_l := l+\frac{\alpha+\beta+1}{2}$, $\gamma := -\big(\alpha+\frac12\big)\frac{\pi}{2}$.

PROPOSITION 2.2 ([R, p. 224]). *Let $n\ge 1$ and let x be a real number. Fix k and j in $\mathbb{N}$, $j\ge k$. Then*

$$\lim_{N\to\infty}\cos^{N-j-k}\Big(\frac{x}{\sqrt{N-j-k}}\Big)P_k^{(j-k,N-j-k)}\Big(\cos\frac{2x}{\sqrt{N-j-k}}\Big) = L_k^{j-k}(x^2)e^{-x^2/2}. \tag{2.4}$$

Our proof is based on the following two-parameter estimate for the L^p-L^2 norm of the complex harmonic projectors $\pi_{l,l'}$ defined by (2.2).

THEOREM 2.3 ([Ca]). *Let $n\ge 2$ and let l,l' be non-negative integers. Then*

$$\|\pi_{l,l'}\|_{(p,2)} \lesssim C\left(\frac{2ll'+n(l+l')}{l+l'}\right)^{\alpha(1/p,n)}(l+l')^{\beta(1/p,n)} \qquad \text{if } 1\le p\le 2, \tag{2.5}$$

where

$$\alpha\Big(\frac1p,n\Big) := \begin{cases} n\Big(\frac1p-\frac12\Big)-\frac12 & \text{if } 1\le p<\tilde p,\\ \frac14-\frac{1}{2p} & \text{if } \tilde p\le p\le 2, \end{cases} \tag{2.6}$$

with $\tilde p = 2(2n+1)/(2n+3)$, and

$$\beta\Big(\frac1p,n\Big) = n\Big(\frac1p-\frac12\Big) \qquad \text{for all } 1\le p\le 2, \tag{2.7}$$

The above estimates are sharp.

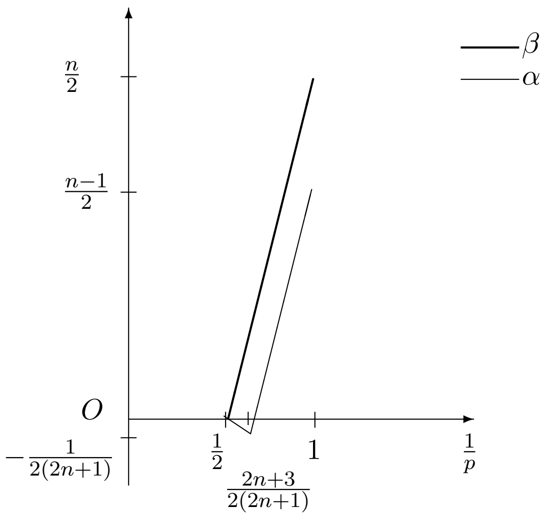

Fig. 1. The exponents α and β as functions of $1/p$

3. L^p eigenfunction bounds on H^n. The Heisenberg group H^n is a Lie group with underlying manifold $\mathbb{C}^n \times \mathbb{R}$, endowed with the product

$$(\mathbf{z}, t)(\mathbf{w}, s) := (\mathbf{z} + \mathbf{w}, t + s + \operatorname{Im} \mathbf{z} \cdot \overline{\mathbf{w}})$$

for $\mathbf{z}, \mathbf{w} \in \mathbb{C}^n$, $t, s \in \mathbb{R}$.

We denote an element in H^1 by $(\rho e^{i\varphi}, t)$, where $\rho \in [0, \infty)$, $\varphi \in [0, 2\pi]$, $t \in \mathbb{R}$, and an element in H^n by $(\rho\boldsymbol{\eta}, t)$, where $\rho \in [0, \infty)$, $t \in \mathbb{R}$ and $\boldsymbol{\eta} \in S^{2n-1}$ is given by

$$\boldsymbol{\eta} = \begin{cases} e^{i\varphi_1} \sin\theta_{n-1} \sin\theta_{n-2} \ldots \sin\theta_1, \\ e^{i\varphi_2} \sin\theta_{n-1} \sin\theta_{n-2} \ldots \cos\theta_1, \\ \vdots \\ e^{i\varphi_n} \cos\theta_{n-1}, \end{cases} \tag{3.1}$$

with $\varphi_k \in [0, 2\pi]$, $k = 1, \ldots, n$, and $\theta_j \in [0, \pi/2]$, $j = 1, \ldots, n-1$.

Observe that $\boldsymbol{\eta} = \boldsymbol{\eta}(\Theta_{n-1}, \Phi_n)$, where $\Theta_{n-1} := (\theta_1, \ldots, \theta_{n-1})$ and $\Phi_n := (\varphi_1, \ldots, \varphi_n)$.

Define now a map $\Psi : H^n \to S^{2n+1}$ by

$$\Psi : (\rho\boldsymbol{\eta}, t) \mapsto (\Theta_{n-1}, \rho, \Phi_n, t), \tag{3.2}$$

where $(\Theta_{n-1},\rho,\Phi_n,t)\in S^{2n+1}$ is given by

$$(3.3)\qquad (\Theta_{n-1},\rho,\Phi_n,t):=\begin{cases} e^{i\varphi_1}\sin\rho\,\sin\theta_{n-1}\sin\theta_{n-2}\ldots\sin\theta_1,\\ e^{i\varphi_2}\sin\rho\sin\theta_{n-1}\sin\theta_{n-2}\ldots\cos\theta_1,\\ \vdots\\ e^{i\varphi_n}\sin\rho\cos\theta_{n-1},\\ e^{it}\cos\rho.\end{cases}$$

We introduce in this way a coordinate system $(\Theta_{n-1},\rho,\Phi_n,t)$ on S^{2n+1}, if ρ and t are restricted, respectively, to $[0,\pi/2]$ and $[-\pi,\pi]$.

The invariant measure $d\sigma_{S^{2n+1}}$ on S^{2n+1} in the spherical coordinates (3.3) is

$$(3.4)\qquad \frac{n!}{2\pi^{n+1}}\prod_{k=1}^{n}d\varphi_k\,dt\sin^{2n-1}\rho\cos\rho\,d\rho\prod_{j=1}^{n-1}\sin^{2j-1}\theta_j\cos\theta_j\,d\theta_j.$$

The factor $n!/(2\pi^{n+1})$ is introduced in order to make the measure of the whole sphere equal to 1.

The Haar measure on H^n in these coordinates is

$$\frac{n!}{2\pi^{n+1}\sqrt{\omega_{2n+1}}}\,\rho^{2n-1}\,d\rho\,d\varphi_1\ldots d\varphi_n\prod_{j=1}^{n-1}\sin^{2j-1}\theta_j\cos\theta_j\,d\theta_j.$$

The *reduced Heisenberg group* h^n is defined as $h^n:=\mathbb{C}^n\times\mathbb{T}$, with product

$$(\mathbf{z},e^{it})(\mathbf{w},e^{it'}):=(\mathbf{z}+\mathbf{w},e^{i(t+t'+\operatorname{Im}\mathbf{z}\cdot\overline{\mathbf{w}})})$$

for $\mathbf{z},\mathbf{w}\in\mathbb{C}^n$, $t,s\in\mathbb{R}$.

Let now f be a function on h^n with compact support. Let $\tilde f$ be the function f extended by periodicity on $\mathbb{R}$ with respect to the variable t. Define the function f_ν on S^{2n+1} by

$$(3.5)\qquad f_\nu(\Theta_{n-1},\rho,\Phi_n,t):=\nu^n\tilde f(\rho\sqrt{\nu}\,\boldsymbol{\eta},t\nu),\qquad \nu\in\mathbb{N}.$$

LEMMA 3.1. *Let f be an integrable function on h^n with compact support. If $1\le p\le\infty$, then*

$$\nu^{-n/p'}\|f_\nu\|_{L^p(S^{2n+1})}<\|f\|_{L^p(h^n)}$$

and

$$\lim_{\nu\to\infty}\nu^{-n/p'}\|f_\nu\|_{L^p(S^{2n+1})}=\|f\|_{L^p(h^n)}.$$

Proof. The proof is similar to that of Lemma 2 in [RRu] and is omitted. Compare also with Lemma 4.3 in [DG2]. ∎

Throughout the paper we shall consider a pair of strongly commuting operators on h^n. The first is the left-invariant sub-Laplacian L, defined by

$$L := -\sum_{j:=1}^{n}(X_j^2 + Y_j^2),$$

where $X_j := \partial_{x_j} - y_j\partial_t$ and $Y_j := \partial_{y_j} + x_j\partial_t$. The second is the operator $T := i^{-1}\partial_t$. These operators generate the algebra of differential operators on h^n invariant under left translation and under the action of the unitary group. One can work out a joint spectral theory; the pairs $(2|m|(2k+n), m)$ with $m \in \mathbb{Z}\setminus\{0\}$ and $k \in \mathbb{N}$ give the discrete joint spectrum of L and $i^{-1}\partial_t$. We shall denote by $P_{m,k}$ the orthogonal projector onto the joint eigenspace.

By considering the Fourier decomposition of functions in $L^2(h^n)$ with respect to the central variable, we obtain an orthogonal decomposition of $L^2(h^n)$ as

$$L^2(h^n) = \mathcal{H}_0 \oplus \mathcal{H},$$

where

$$\mathcal{H}_0 := \Big\{f \in L^2(h^n) : \int_{\mathbb{T}} f(z,t)\,dt = 0\Big\}.$$

The projectors $P_{m,k}$ map $L^2(h^n)$ onto $\mathcal{H}$ and provide a spectral decomposition for $\mathcal{H}$. The importance of this decomposition is due to the fact that the spectral analysis of L on $\mathcal{H}_0$ essentially reduces to the analysis of the Laplacian on $\mathbb{C}^n$.

On the complex sphere S^{2n+1} the algebra of $U(n+1)$-invariant differential operators is commutative and generated by two elements; a basis is given by the Laplace–Beltrami operator $\Delta_{S^{2n+1}}$, defined by (2.1), and the Kohn Laplacian $\mathcal{L}$ on S^{2n+1}, defined by

$$\mathcal{L} := \sum_{j<k}(M_{jk}\overline{M}_{jk} + \overline{M}_{jk}M_{jk})$$

with

$$M_{jk} := \bar{z}_j\partial_{z_k} - \bar{z}_k\partial_{z_j} \quad \text{and} \quad \overline{M}_{jk} := z_j\partial_{\bar{z}_k} - z_k\partial_{\bar{z}_j}.$$

We shall denote by $\mathcal{H}^{l,l'}$ the joint eigenspace of $\Delta_{S^{2n+1}}$ and $\mathcal{L}$ with eigenvalues respectively $\mu_{l,l'} := -(l+l')(l+l'+2n)$ and $\lambda_{l,l'} = -2ll' - n(l+l')$ ([Kl]).

The next task is to prove that the joint spectral projection $P_{m,k}$ on h^n may be obtained as limit in the L^2-norm of an appropriate sequence of joint spectral projectors on S^{2n+1}.

Proposition 3.2. *Let f be a continuous function on h^n with compact support. Take $m \in \mathbb{N}\setminus\{0\}$ and $k \in \mathbb{N}$. For every $\nu \in \mathbb{N}$ let $N(\nu) \in \mathbb{N}$ be such* *that*

$$\lim_{\nu\to\infty}\frac{N(\nu)}{\nu}=m. \tag{3.6}$$

Then

$$\|P_{m,k}f\|_{L^2(h^n)}=\lim_{\nu\to\infty}\frac{1}{\nu^{n/2}}\,\|\pi_{k,N(\nu)-k}f_\nu\|_{L^2(S^{2n+1})}, \tag{3.7}$$

$$\|P_{-m,k}f\|_{L^2(h^n)}=\lim_{\nu\to\infty}\frac{1}{\nu^{n/2}}\,\|\pi_{N(\nu)-k,k}f_\nu\|_{L^2(S^{2n+1})}. \tag{3.8}$$

Proof. The scheme of the proof is similar to that of Proposition 4.4 in [Ca]. Since the higher dimensional case is more involved, we present the proof for more transparency.

Fix two integers $m>0$ and $k\in\mathbb{N}$.

First of all, if $\mathbf{z},\mathbf{w}\in\mathbb{C}^n$, by writing $\mathbf{z}:=\rho\boldsymbol{\eta}$ and $\mathbf{w}:=\rho'\boldsymbol{\eta}'$ with $\rho,\rho'\in[0,\infty)$ and $\boldsymbol{\eta},\boldsymbol{\eta}'\in S^{2n-1}$, a simple computation yields

$$\begin{aligned}\operatorname{Im}\mathbf{z}\cdot\overline{\mathbf{w}}=\rho\rho'(&\sin(\varphi_1-\varphi_1')\sin\theta_{n-1}\sin\theta_{n-1}'\cdot\ldots\ldots\sin\theta_1\sin\theta_1'\\&+\sin(\varphi_2-\varphi_2')\sin\theta_{n-1}\sin\theta_{n-1}'\cdot\ldots\ldots\cos\theta_1\cos\theta_1'\\&+\cdots+\sin(\varphi_n-\varphi_n')\cos\theta_{n-1}\cos\theta_{n-1}')\end{aligned} \tag{3.9}$$

and

$$\begin{aligned}|\mathbf{z}-\mathbf{w}|^2=\rho^2&+\rho'^2\\&-2\rho\rho'(\cos(\varphi_1-\varphi_1')\sin\theta_{n-1}\sin\theta_{n-1}'\ldots\sin\theta_1\sin\theta_1'\\&\quad+\cos(\varphi_2-\varphi_2')\sin\theta_{n-1}\sin\theta_{n-1}'\ldots\cos\theta_1\cos\theta_1'+\cdots\\&\quad\cdots+\cos(\varphi_n-\varphi_n')\cos\theta_{n-1}\cos\theta_{n-1}').\end{aligned} \tag{3.10}$$

Now, we denote by $\Phi^m_{k,k}$ the joint eigenfunction for $\mathcal{L}$ and $i^{-1}\partial_t$ (for more details and an explicit expression see, for example, [FH, Chapitre V]). Orthogonality of joint spectral projectors yields

$$\begin{aligned}\|P_{m,k}f\|^2_{L^2(h^n)}&=\langle P_{m,k}f,f\rangle_{L^2(h^n)}=\int_{h^n} f*\Phi^m_{k,k}(\mathbf{z},t)\,\overline{f(\mathbf{z},t)}\,d\mathbf{z}\,dt\\&=\int_{h^n}\Big(\int_{h^n}\Phi^m_{k,k}(\mathbf{z}-\mathbf{w},t-t'+\operatorname{Im}\mathbf{z}\cdot\overline{\mathbf{w}})f(\mathbf{w},t')\,d\mathbf{w}\,dt'\Big)\overline{f(\mathbf{z},t)}\,d\mathbf{z}\,dt\\&=m^n\int_{h^n}\Big(\int_{h^n}e^{im(t-t'+\operatorname{Im}\mathbf{z}\cdot\overline{\mathbf{w}})}L_k^{n-1}(m|\mathbf{z}-\mathbf{w}|^2)e^{-\frac12 m|\mathbf{z}-\mathbf{w}|^2}f(\mathbf{w},t')\,d\mathbf{w}\,dt'\Big)\\&\qquad\cdot\overline{f(\mathbf{z},t)}\,d\mathbf{z}\,dt.\end{aligned}$$

Now we shall deal with the right-hand side in (3.7). For brevity we set

$$d\Phi_{(n)}:=d\varphi_1\ldots d\varphi_n\quad\text{and}\quad d\Theta_{(n-1)}:=\prod_{j=1}^{n-1}\sin^{2j-1}\theta_j\cos\theta_j\,d\theta_j.$$

From the orthogonality of the joint spectral projectors $\pi_{l,l'}$ in $L^2(S^{2n+1})$ and from (3.5) we deduce

$$\|\pi_{k,N(\nu)-k}f_\nu\|^2_{L^2(S^{2n+1})} = \langle \pi_{k,N(\nu)-k}f_\nu, f_\nu\rangle_{L^2(S^{2n+1})}$$
$$= \int_{S^{2n+1}} (\pi_{k,N(\nu)-k}f_\nu)(\Theta_{n-1},\rho,\Phi_n,t)\,\overline{f_\nu(\Theta_{n-1},\rho,\Phi_n,t)}\; d\sigma_{S^{2n+1}}$$
$$= \frac{n!}{2\pi^{n+1}\nu}\int_{A_\nu} (\pi_{k,N(\nu)-k}f_\nu)\Big(\Theta_{n-1},\frac{\rho}{\sqrt{\nu}},\Phi_n,\frac{t}{\nu}\Big)\overline{\tilde f(\Theta_{n-1},\rho,\Phi_n,t)}\Bigg(\frac{\sin\frac{\rho}{\sqrt{\nu}}}{\frac{\rho}{\sqrt{\nu}}}\Bigg)^{2n-1}$$
$$\cdot\cos\Big(\frac{\rho}{\sqrt{\nu}}\Big)\rho^{2n-1}\,d\rho\,d\Theta_{(n-1)}\,d\Phi_{(n)}\,dt$$
$$= \frac{n!^2}{4\pi^{2n+2}\nu^2}\int_{A_\nu}\Bigg(\int_{A_\nu} {}^b\mathbb{Z}_{k,N(\nu)-k}\Big(\Big\langle\Big(\Theta_{n-1},\frac{\rho}{\sqrt{\nu}},\Phi_n,\frac{t}{\nu}\Big),\Big(\Theta'_{n-1},\frac{\rho'}{\sqrt{\nu}},\Phi'_n,\frac{t'}{\nu}\Big)\Big\rangle\Big)$$
$$\cdot\tilde f(\Theta'_{n-1},\rho',\Phi'_n,t')\Bigg(\frac{\sin\frac{\rho'}{\sqrt{\nu}}}{\frac{\rho'}{\sqrt{\nu}}}\Bigg)^{2n-1}\cos\Big(\frac{\rho'}{\sqrt{\nu}}\Big)\,\rho'^{2n-1}\,d\rho'\,d\Theta'_{(n-1)}\,d\Phi'_{(n)}\,dt'\Bigg)$$
$$\cdot\overline{\tilde f(\Theta_{n-1},\rho,\Phi_n,t)}\Bigg(\frac{\sin\frac{\rho}{\sqrt{\nu}}}{\frac{\rho}{\sqrt{\nu}}}\Bigg)^{2n-1}\cos\Big(\frac{\rho}{\sqrt{\nu}}\Big)\,\rho^{2n-1}\,d\rho\,d\Theta'_{(n-1)}\,d\Phi_{(n)}\,dt$$

where the integration set A_ν is given by

$$A_\nu := \{(\Theta_{n-1},\rho,\Phi_n,t) : 0\le\rho\le\pi\sqrt{\nu}/2, 0\le\varphi_k\le 2\pi,\ k=1,\dots,n,\ 0\le\theta_j\le\pi/2,\ j=1,\dots,n-1,\ -\pi\nu\le t\le\pi\nu\}. \tag{3.11}$$

Now by using (3.3) we compute the inner product in $\mathbb{C}^{n+1}$:

$$\Big\langle\Big(\Theta_{n-1},\frac{\rho}{\sqrt{\nu}},\Phi_{n-1},\frac{t}{\nu}\Big),\Big(\Theta'_{n-1},\frac{\rho'}{\sqrt{\nu}},\Phi'_{n-1},\frac{t'}{\nu}\Big)\Big\rangle$$
$$= e^{i(\varphi_1-\varphi'_1)}\sin\Big(\frac{\rho}{\sqrt{\nu}}\Big)\sin\Big(\frac{\rho'}{\sqrt{\nu}}\Big)\sin\theta_{n-2}\sin\theta'_{n-2}\dots\sin\theta_1\sin\theta'_1$$
$$+ e^{i(\varphi_2-\varphi'_2)}\sin\Big(\frac{\rho}{\sqrt{\nu}}\Big)\sin\Big(\frac{\rho'}{\sqrt{\nu}}\Big)\sin\theta_{n-2}\sin\theta'_{n-2}\dots\cos\theta_1\cos\theta'_1$$
$$+\dots+ e^{i(\varphi_{n-1}-\varphi'_{n-1})}\sin\Big(\frac{\rho}{\sqrt{\nu}}\Big)\sin\Big(\frac{\rho'}{\sqrt{\nu}}\Big)\cos\theta_{n-2}\cos\theta'_{n-2}$$
$$+ e^{i(t-t')\frac{1}{\nu}}\cos\Big(\frac{\rho}{\sqrt{\nu}}\Big)\cos\Big(\frac{\rho'}{\sqrt{\nu}}\Big)$$
$$= R_\nu e^{i\psi_\nu},$$

where

$$R_\nu = 1 - \frac{1}{2\nu}\big(\rho^2 + \rho'^2 - 2\rho\rho'(\cos(\varphi_1 - \varphi_1')\sin\theta_{n-1}\sin\theta_{n-1}' \ldots \sin\theta_1 \sin\theta_1' \\ + \cos(\varphi_2 - \varphi_2')\sin\theta_{n-1}\sin\theta_{n-1}' \ldots \cos\theta_1\cos\theta_1' \\ + \cdots + \cos(\varphi_n - \varphi_n')\cos\theta_{n-1}\cos\theta_{n-1}')\big) + o\Big(\frac{1}{\nu}\Big),$$

$$\psi_\nu = \arctan\bigg(\frac{1}{\nu}\,\rho\rho'(\sin(\varphi_1 - \varphi_1')\sin\theta_{n-1}\sin\theta_{n-1}' \ldots \sin\theta_1\sin\theta_1' \\ + \sin(\varphi_2 - \varphi_2')\sin\theta_{n-1}\sin\theta_{n-1}' \ldots \cos\theta_1\cos\theta_1' \\ + \cdots + \sin(\varphi_n - \varphi_n')\cos\theta_{n-1}\cos\theta_{n-1}') + \frac{t-t'}{\nu} + o\Big(\frac{1}{\nu}\Big)\bigg)$$

as $\nu \to \infty$. Thus as a consequence of (3.9) and (3.10) we have

$$R_\nu = \cos\Big(\frac{1}{\sqrt{\nu}}\,|\mathbf{z}-\mathbf{w}|\Big) + o\Big(\frac{1}{\nu}\Big) \quad \text{and} \quad \psi_\nu = \frac{1}{\nu}\,(t-t') + \frac{1}{\nu}\,\operatorname{Im}\,\mathbf{z}\cdot\overline{\mathbf{w}} + o\Big(\frac{1}{\nu}\Big),$$

so that formula (2.3) for the zonal function yields

$$ {}^b\mathbb{Z}_{k,N(\nu)-k}\bigg(\bigg\langle\Big(\Theta_{n-1}, \frac{\rho}{\sqrt{\nu}}, \Phi_n, \frac{t}{\nu}\Big), \Big(\Theta_{n-1}', \frac{\rho'}{\sqrt{\nu}}, \Phi_n', \frac{t'}{\nu}\Big)\bigg\rangle\bigg) \\ = \frac{N(\nu)^n}{\omega_{2n+1}}\, e^{i(N(\nu)-2k)\frac{1}{\nu}(t-t'+\operatorname{Im}\mathbf{z}\cdot\overline{\mathbf{w}}+o(1))}\bigg(\cos\Big(\frac{1}{\sqrt{\nu}}\,|\mathbf{z}-\mathbf{w}|\Big)\bigg)^{|N(\nu)-2k|} \\ \cdot P_k^{(n-1,|N(\nu)-2k|)}\bigg(\cos\Big(\frac{2}{\sqrt{\nu}}\,|\mathbf{z}-\mathbf{w}|\Big)\bigg) + o\Big(\frac{1}{\nu}\Big), \qquad \nu \to \infty.$$

By using condition (3.6) and the Mean Value Theorem, we easily check that

$$\frac{1}{\nu^n}\,\|\pi_{k,N(\nu)-k} f_\nu\|^2_{L^2(S^{2n+1})} = \mathcal{I}_\nu^M + \mathcal{I}_\nu^R,$$

where the remainder term $\mathcal{I}_\nu^R$ satisfies $\lim_{\nu\to\infty}\mathcal{I}_\nu^R = 0$, while the main term $\mathcal{I}_\nu^M$ is given by

$$\mathcal{I}_\nu^M = \frac{n!^2}{4\omega_{2n+1}\pi^{2n+2}\nu^2} \\ \cdot \int_{A_\nu}\bigg(\int_{A_\nu}\Big(\frac{N(\nu)}{\nu}\Big)^n e^{im(t-t'+\operatorname{Im}\mathbf{z}\cdot\overline{\mathbf{w}})}\bigg(\cos\Big(\frac{1}{\sqrt{\nu}}\,|\mathbf{z}-\mathbf{w}|\Big)\bigg)^{|N(\nu)-2k|} \\ \cdot P_k^{(n-1,|N(\nu)-2k|)}\bigg(\cos\Big(\frac{2}{\sqrt{\nu}}\,|\mathbf{z}-\mathbf{w}|\Big)\bigg)\tilde{f}(\Theta_{n-1}', \rho', \Phi_n', t')\bigg(\frac{\sin\frac{\rho'}{\sqrt{\nu}}}{\frac{\rho'}{\sqrt{\nu}}}\bigg)^{2n-1}$$

$$\cdot \cos\left(\frac{\rho'}{\sqrt{\nu}}\right) \rho'^{2n-1}\, d\rho'\, d\Theta'_{(n-1)}\, d\Phi'_{(n)}\, dt'\Big) \overline{\tilde{f}(\Theta_{n-1},\rho,\Phi_n,t)} \left(\frac{\sin\frac{\rho}{\sqrt{\nu}}}{\frac{\rho}{\sqrt{\nu}}}\right)^{2n-1}$$
$$\cdot \cos\left(\frac{\rho}{\sqrt{\nu}}\right) \rho^{2n-1}\, d\rho\, d\Theta_{(n-1)}\, d\Phi_{(n)}\, dt, \qquad \nu\to\infty.$$

We shall now treat $\mathcal{I}_\nu^M$ by means of the Lebesgue dominated convergence theorem. First of all, we extend the integration set in $\mathcal{I}_\nu^M$ (this may be done, since f has compact support and the integrand is periodic with respect to t), and we obtain

$$\mathcal{I}_\nu^M = \frac{n!^2}{4\pi^{2n+2}\omega_{2n+1}} \int_0^\infty \int_0^{\pi/2} \cdots \int_0^{\pi/2} \int_0^{2\pi} \cdots \int_0^{2\pi} \int_{-\pi}^{\pi} \tag{3.12}$$
$$\Bigg(\int_0^\infty \int_0^{\pi/2} \cdots \int_0^{\pi/2} \int_0^{2\pi} \cdots \int_0^{2\pi} \int_{-\pi}^{\pi} \left(\frac{N(\nu)}{\nu}\right)^n e^{im(t-t'-\operatorname{Im}\mathbf{w}\cdot\bar{\mathbf{z}})}$$
$$\cdot \left(\cos\left(\frac{1}{\sqrt{\nu}}\,|\mathbf{z}-\mathbf{w}|\right)\right)^{|N(\nu)-2k|} P_k^{(n-1,|N(\nu)-2k|)}\left(\cos\left(\frac{2}{\sqrt{\nu}}|\mathbf{z}-\mathbf{w}|\right)\right)$$
$$\cdot f(\Theta'_{n-1},\rho',\Phi'_n,t')\left(\frac{\sin\frac{\rho'}{\sqrt{\nu}}}{\frac{\rho'}{\sqrt{\nu}}}\right)^{2n-1} \cos\left(\frac{\rho'}{\sqrt{\nu}}\right)\rho'^{2n-1}\, d\rho'\, d\Theta'_{(n-1)}\, d\Phi'_{(n)}\, dt'\Bigg)$$
$$\cdot \overline{f(\Theta_{n-1},\rho,\Phi_n,t)}\left(\frac{\sin\frac{\rho}{\sqrt{\nu}}}{\frac{\rho}{\sqrt{\nu}}}\right)^{2n-1} \cos\left(\frac{\rho}{\sqrt{\nu}}\right)\rho^{2n-1}\, d\rho\, d\Theta_{(n-1)}\, d\Phi_{(n)}\, dt.$$

By using Lemma 2.1 and the Mehler–Heine formula as stated in Lemma 2.2 (with $N = N(\nu)+j-k$, $j-k=n-1$ and $x=\sqrt{\frac{N(\nu)-2k}{\nu}}|\mathbf{z}-\mathbf{w}|$), we may conclude as in Proposition 4.4 of [Ca].

The proof for (3.8) is completely analogous. ∎

THEOREM 3.3. *Let $n>2$. Take $m\in\mathbb{Z}\setminus\{0\}$ and $k\in\mathbb{N}$. Then*

$$\|P_{m,k}\|_{(L^p(h^n),L^2(h^n))} \lesssim \begin{cases} C(2k+n)^{n(\frac{1}{p}-\frac{1}{2})-\frac{1}{2}}|m|^{n(\frac{1}{p}-\frac{1}{2})} & \text{if } 1\le p<\tilde{p},\\ C(2k+n)^{\frac{1}{4}-\frac{1}{2p}}|m|^{n(\frac{1}{p}-\frac{1}{2})} & \text{if } \tilde{p}\le p\le 2,\end{cases} \tag{3.13}$$

where $\tilde{p}=2(2n+1)/(2n+3)$. Moreover, the estimates are sharp.

Proof. Take $m>0$ (the other case being analogous). For every $\nu\in\mathbb{N}$ let $N(\nu)\in\mathbb{N}$ be such that

$$\lim_{\nu\to\infty}\frac{1}{\nu}N(\nu)=m.$$

Thus

$$\begin{aligned}\|P_{m,k}f\|_{L^2(h^n)} &= \lim_{\nu\to\infty}\frac{1}{\nu^{n/2}}\,\|\pi_{k,N(\nu)-k}f_\nu\|_{L^2(S^{2n+1})}\\ &\le \lim_{\nu\to\infty}\left(\frac{N(\nu)}{\nu}\right)^{n/2}\left(\frac{2k(N(\nu)-k)}{N(\nu)}+n\right)^{n/2}\|f_\nu\|_{L^1(S^{2n+1})}\\ &= m^{n/2}(2k+n)^{(n-1)/2}\lim_{\nu\to\infty}\|f_\nu\|_{L^1(S^{2n+1})}\\ &= m^{n/2}(2k+n)^{(n-1)/2}\|f\|_{L^1(h^n)},\end{aligned}$$

where we have used first (3.7) and then Theorem 2.3 and Lemma 3.1.

In the same way, we see that

$$\begin{aligned}\|P_{m,k}f\|_{L^2(h^n)} &= \lim_{\nu\to\infty}\frac{1}{\nu^{n/2}}\,\|\pi_{k,N(\nu)-k}f_\nu\|_{L^2(S^{2n+1})}\\ &\le \lim_{\nu\to\infty}\frac{1}{\nu^{n/2}}\left(\frac{2k\cdot(N(\nu)-k)}{N(\nu)}+n\right)^{-\frac{1}{2(2n+1)}}\\ &\qquad\qquad\cdot N(\nu)^{\frac{n}{2n+1}}\|f_\nu\|_{L^{2\frac{2n+1}{2n+3}}(S^{2n+1})}\\ &\le (2k+n)^{-\frac{1}{2(2n+1)}}\lim_{\nu\to\infty}\frac{1}{\nu^{n/2}}\,N(\nu)^{\frac{n}{2n+1}}\nu^{\frac{n(2n-1)}{2(2n+1)}}\|f\|_{L^{2\frac{2n+1}{2n+3}}(h^n)}\\ &= (2k+n)^{-\frac{1}{2(2n+1)}}m^{\frac{n}{2n+1}}\|f\|_{L^{2\frac{2n+1}{2n+3}}(h^n)}.\end{aligned}$$

An interpolation argument yields the conclusion. Finally, sharpness follows from arguments in [KoR]. ∎

4. A restriction theorem on h^n. By applying the bounds proved in Section 2 we obtain a restriction theorem for the spectral projectors associated to the sub-Laplacian L on h^n. Our theorem improves in some cases a previous result due to Thangavelu ([Th1]). More precisely, let Q_N be the spectral projection corresponding to the eigenvalue N associated to L on h^n, that is,

$$Q_N f := \sum_{(2k+n)|m|=N} P_{m,k}f,$$

where $P_{m,k}$ is the joint spectral projection operator introduced in the previous section. We look for estimates of the type

$$\|Q_N\|_{(L^p(h^n),L^2(h^n))} \le CN^{\sigma(p,n)} \tag{4.1}$$

for all $1 \le p \le 2$, where the exponent σ is in general a convex function of $1/p$. In [Th1] Thangavelu proved that

$$\|Q_N\|_{(L^p(h^n),L^2(h^n))} \leq CN^{n(1/p-1/2)} d(N)^{1/p-1/2}, \quad 1 \leq p \leq 2, \tag{4.2}$$

where $d(N)$ is the divisor-type function defined by

$$d(N) := \sum_{2k+n|N} \frac{1}{2k+n}, \tag{4.3}$$

and the estimate is sharp for $p = 1$. Here $a \,|\, b$ means that a divides b. Thangavelu also proved that when $N = nR$ with $R \in \mathbb{N}$, then

$$CN^{n(1/p-1/2)} \leq \|Q_N\|_{(L^p(h^n),L^2(h^n))}, \quad 1 \leq p \leq 2.$$

Here we show that there exist arithmetic progressions a_N in $\mathbb{N}$ such that the estimate for $\|Q_{a_N}\|_{(p,2)}$ is sharp and better than (4.2) for $1 < p < 2$.

Proposition 4.1. *Let $n \geq 1$. Let N be any positive integer. Then for every $1 \leq p \leq 2$,*

$$\|Q_N\|_{(L^p(h^n),L^2(h^n))} \leq CN^{n(1/p-1/2)} d(N)^{\rho(1/p,n)}, \tag{4.4}$$

where ρ is defined by

$$\rho\Big(\frac{1}{p}, n\Big) := \begin{cases} \dfrac{1}{2} & \text{if } 1 \leq p < \tilde{p}, \\ \Big(n + \dfrac{1}{2}\Big)\Big(\dfrac{1}{p} - \dfrac{1}{2}\Big) & \text{if } \tilde{p} \leq p \leq 2, \end{cases} \tag{4.5}$$

with $\tilde{p} = 2(2n+1)/(2n+3)$, and $d(N)$ is given by (4.3).

Proof. For $p = 1$ our estimate coincides with (4.2); nonetheless we give a different, simpler proof:

$$\begin{aligned} \|Q_N f\|^2_{L^2(h^n)} &= \Big\| \sum_{(2k+n)|m|=N} P_{m,k} f \Big\|^2_{L^2(h^n)} = \sum_{(2k+n)|m|=N} \|P_{m,k} f\|^2_{L^2(h^n)} \\ &\leq C \sum_{(2k+n)|m|=N} m^n (2k+n)^{n-1} \|f\|^2_{L^1(h^n)}, \\ &\leq CN^n \sum_{2k+n|N} \frac{1}{2k+n} \|f\|^2_{L^1(h^n)}, \end{aligned}$$

whence

$$\|Q_N\|_{(L^1,L^2)} \leq CN^{n/2} d(N)^{1/2}. \tag{4.6}$$

For $p=2$ the bound is obvious, since Q_N is an orthogonal projector. Finally, for $p=\tilde{p}$ one has

$$\begin{aligned}\|Q_N f\|^2_{L^2(h^n)} &= \sum_{(2k+n)|m|=N} \|P_{m,k}f\|^2_{L^2(h^n)} \\ &\le C \sum_{(2k+n)|m|=N} (2k+n)^{-1/(2n+1)}|m|^{2n/(2n+1)}\|f\|^2_{L^{\tilde{p}}(h^n)} \\ &= CN^{2n/(2n+1)} \sum_{2k+n|N} (2k+n)^{-1}\|f\|^2_{L^{\tilde{p}}(h^n)},\end{aligned}$$

whence

$$\|Q_N\|_{(L^{\tilde{p}},L^2)} \le CN^{n/(2n+1)} d(N)^{1/2}. \tag{4.7}$$

Then by applying the Riesz–Thorin interpolation theorem to (4.6) and to (4.7) we get (4.4). ∎

Remark 4.2. Observe that estimate (4.4) is better than (4.2) only when $d(N)<1$.

Thus, on the one hand, we are led to seek arithmetic progressions $\{N_m\}$ on which the divisor function $d(N_m)$, whose behaviour is in general highly irregular, is strictly smaller than one. On the other hand, we are led to inquire about the average size of the norm of Q_N.

We remark that if $n=1$ then $d(N)$ is necessarily greater than one.

Remark 4.3. Proposition 4.1 reveals the existence of a critical point $\tilde{p}\in(1,2)$ where the form of the exponent of the eigenvalue N in (4.1) changes.

In the following we list some cases in which estimate (4.4) really improves the result in [Th1]. First of all, when $n\ge 2$ and N is a prime number, Proposition 4.1 yields the following sharp result.

Proposition 4.4. *Let $n>2$ be odd. Let N be a prime number. Then for every $1\le p\le 2$,*

$$\|Q_N\|_{(L^p(h^n),L^2(h^n))} \le \begin{cases} C\,N^{n(\frac{1}{p}-\frac{1}{2})-\frac{1}{2}} & \text{if } 1\le p<\tilde{p},\\ C\,N^{-\frac{1}{2}(\frac{1}{p}-\frac{1}{2})} & \text{if } \tilde{p}\le p\le 2,\end{cases} \tag{4.8}$$

with $\tilde{p}=2(2n+1)/(2n+3)$. Moreover, the above estimate is sharp.

Proof. (4.8) follows directly from (4.4). Furthermore, since in this case

$$\|Q_N\|_{(L^p(h^n),L^2(h^n))} \sim \|P_{1,(N-n)/2}\|_{(L^p(h^n),L^2(h^n))}, \qquad 1\le p\le 2,$$

sharpness follows from Theorem 3.3. ∎

Proposition 4.4 may be generalized to the case $N=r^{k_0}$, where $k_0\in\mathbb{N}$ and r varies in the set of all prime numbers.

PROPOSITION 4.5. *Let $n \ge 2$ be odd. Fix a positive integer k_0. Set $N_r = r^{k_0}$, where r varies in the set of all prime numbers. Then for every $1 \le p \le 2$,*

$$\|Q_{N_r}\|_{(L^p(h^n),L^2(h^n))} \le \begin{cases} CN_r^{n(\frac{1}{p}-\frac{1}{2})-\frac{1}{2k_0}} & \text{if } 1 \le p < \tilde{p}, \\ CN_r^{(n-\frac{1}{2k_0}(2n+1))(\frac{1}{p}-\frac{1}{2})} & \text{if } \tilde{p} \le p \le 2, \end{cases} \tag{4.9}$$

with $\tilde{p} = 2(2n+1)/(2n+3)$. Moreover, (4.9) *is sharp.*

Proof. (4.9) follows directly from (4.4), since

$$d(N_r) = \frac{1}{r} + \frac{1}{r^2} + \cdots + \frac{1}{r^{k_0}} \le \frac{2}{r}.$$

To prove that (4.9) is sharp, take the joint eigenfunction f_0 for L and $i^{-1}\partial_t$ with eigenvalues, respectively, $(2k+n)m = N_r$ and $m = r^{k_0-1}$, yielding the sharpness for the joint spectral projection $P_{r^{k_0-1},(r-n)/2}$, that is, such that

$$\|P_{r^{k_0-1},(r-n)/2}\|_{(p,2)} \sim \frac{\|f_0\|_{p'}}{\|f_0\|_2}.$$

Now we have

$$\begin{aligned} \|Q_N\|_{(L^2(h^n),L^{p'}(h^n))} &\ge \frac{\|Q_N f_0\|_{L^{p'}}}{\|f_0\|_{L^2}} = \frac{\|f_0\|_{L^{p'}}}{\|f_0\|_{L^2}} \sim \|P_{r^{k_0-1},(r-n)/2}\|_{(p,2)} \\ &\sim Cr^{n(\frac{1}{p}-\frac{1}{2})-\frac{1}{2}}(r^{k_0-1})^{n(\frac{1}{p}-\frac{1}{2})} \sim Cr^{-\frac{1}{2}}r^{k_0 n(\frac{1}{p}-\frac{1}{2})} \\ &\sim CN_r^{n(\frac{1}{p}-\frac{1}{2})-\frac{1}{2k_0}} \end{aligned}$$

for all $1 \le p \le \tilde{p}$. For $\tilde{p} \le p \le 2$ an analogous estimate holds, so that (4.9) is sharp. ∎

We shall now consider integers of the form $N_l := q_0^l$, where q_0 is a fixed prime number and $l \in \mathbb{N}$. The argument of the previous proposition also proves the following.

PROPOSITION 4.6. *Let $n = 2$ or $n > 2$ odd. For $n = 2$ let $q_0 = 2$, for $n > 2$ let q_0 be a prime number strictly greater than 2. Set $N_l := q_0^l$, $l \in \mathbb{N}$. Then*

$$\|Q_{N_l}\|_{(L^p(h^n),L^2(h^n))} \le CN_l^{n(1/p-1/2)} \quad \text{if } 1 \le p \le 2. \tag{4.10}$$

Moreover, (4.10) *is sharp.*

The above examples show the highly irregular behaviour of $d(N)$, and therefore of $\|Q_N\|_{p,2}$. In order to smooth out fluctuations we introduce appropriate averages of joint spectral projectors. More precisely, for $N \in \mathbb{N}$ we define

$$\Pi_N f := \sum_{L=n}^{N} \sum_{(2k+n)|m|=L} P_{m,k} f \tag{4.11}$$

and ask what is the behaviour of $\|M_N\|_{(p,2)}$, where

$$M_N f := \frac{1}{N}\, \Pi_N f. \tag{4.12}$$

For $p = 1$ Theorem 3.3 and orthogonality yield

$$\begin{aligned}
\|\Pi_N f\|^2_{L^2(h^n)} &= \Big\| \sum_{L=n}^{N} \sum_{(2k+n)|m|=L} P_{m,k} f \Big\|^2_{L^2(h^n)} \\
&= \sum_{(k,m)\colon (2k+n)|m|\le N} \|P_{m,k} f\|^2_{L^2(h^n)} \\
&\le C \sum_{(k,m)\colon (2k+n)|m|\le N} (2k+n)^{n-1} |m|^n \|f\|^2_{L^1(h^n)} \\
&\le C \sum_{m=1}^{N} m^n \sum_{2k+n=n}^{[N/m]} (2k+n)^{n-1} \|f\|^2_{L^1(h^n)} \le C N^n \cdot N \|f\|^2_{L^1(h^n)},
\end{aligned}$$

whence

$$\|\Pi_N\|_{(1,2)} \le N^{(n+1)/2}. \tag{4.13}$$

The trivial L^2-L^2 estimate and Riesz–Thorin interpolation yield

$$\|\Pi_N\|_{(p,2)} \le C N^{(n+1)(1/p-1/2)}, \qquad 1 \le p \le 2. \tag{4.14}$$

Observe that by using Theorem 3.3 we may obtain the following estimate at the critical point $\tilde{p}$:

$$\begin{aligned}
\|\Pi_N f\|^2_{L^2(h^n)} &= \sum_{(k,m)\colon (2k+n)|m|\le N} \|P_{m,k} f\|^2_{L^2(h^n)} \\
&\le C \sum_{(k,m)\colon (2k+n)|m|\le N} (2k+n)^{2\alpha} m^{2\beta} \|f\|^2_{L^{\tilde{p}}(h^n)} \\
&= C \sum_{m=1}^{N} m^{2\beta} \sum_{2k+n=n}^{[N/m]} (2k+n)^{2\alpha} \|f\|^2_{L^{\tilde{p}}(h^n)} \\
&= N^{2\alpha+1} \sum_{m=1}^{N} m^{2\beta-2\alpha-1} \|f\|^2_{L^{\tilde{p}}(h^n)} \\
&\le C N^{2\alpha+2} \|f\|^2_{L^{\tilde{p}}(h^n)},
\end{aligned}$$

where we have used the fact that $2\beta - 2\alpha = 1$ for all $1 \le p \le \tilde{p}$, with $\alpha = \alpha(1/p, n)$ and $\beta = \beta(1/p, n)$ given by (2.6) and (2.7).

Thus

$$\|\Pi_N\|_{(\tilde{p},2)} \le C N^{\alpha+1} = C N^{(2n+1/2)/(2n+1)}. \tag{4.15}$$

A comparison between (4.14) and (4.15) shows that at the critical point the estimate given by Riesz–Thorin interpolation is better than the bound obtained by summing up the estimates for joint spectral projections.

Thus we obtain the following result.

PROPOSITION 4.7. *Let $n \geq 1$. The following L^p-L^2 bounds hold for Π_N and for the average projection operators M_N:*

$$\|\Pi_N\|_{(L^p(h^n),L^2(h^n))} \leq CN^{(n+1)(1/p-1/2)} \quad \text{if } 1 \leq p \leq 2.$$

and

$$\|M_N\|_{(L^p(h^n),L^2(h^n))} \leq CN^{(n+1)(1/p-1/2)-1} \quad \text{if } 1 \leq p \leq 2.$$

A similar proof also yields the following result about the operators E_{N_1,N_2}, where

$$E_{N_1,N_2} := \Pi_{N_2} - \Pi_{N_1}, \quad N_1, N_2 \in \mathbb{N},\ N_2 > N_1.$$

PROPOSITION 4.8. *Let $n \geq 1$. Then*

$$\|E_{N_1,N_2}\|_{(L^p(h^n),L^2(h^n))} \leq C(N_2^n(N_2 - N_1))^{1/p-1/2} \quad \text{for all } 1 \leq p \leq 2.$$

REMARK 4.9. This should be compared with Proposition 3.8 in [M], which shows that this estimate is sharp.

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