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SUBMODULAR MEAN FIELD GAMES: EXISTENCE AND APPROXIMATION OF SOLUTIONS

BY JODI DIANETTI[1,*], GIORGIO FERRARI[1,†], MARKUS FISCHER[2] AND MAX NENDEL[1,‡]

[1]*Center for Mathematical Economics (IMW), Bielefeld University, [*]jodi.dianetti@uni-bielefeld.de; [†]giorgio.ferrari@uni-bielefeld.de; [‡]max.nendel@uni-bielefeld.de*

[2]*Department of Mathematics "Tullio Levi-Civita", University of Padua, fischer@math.unipd.it*

We study mean field games with scalar Itô-type dynamics and costs that are *submodular* with respect to a suitable order relation on the state and measure space. The submodularity assumption has a number of interesting consequences. First, it allows us to prove existence of solutions via an application of Tarski's fixed point theorem, covering cases with discontinuous dependence on the measure variable. Second, it ensures that the set of solutions enjoys a lattice structure: in particular, there exist minimal and maximal solutions. Third, it guarantees that those two solutions can be obtained through a simple learning procedure based on the iterations of the best-response-map. The mean field game is first defined over ordinary stochastic controls, then extended to relaxed controls. Our approach also allows us to prove existence of a strong solution for a class of submodular mean field games with common noise, where the representative player at equilibrium interacts with the (conditional) mean of its state's distribution.

1. Introduction. In this paper, we study a representative class of mean field games with *submodular* costs. Mean field games (MFGs for short), as introduced by Lasry and Lions [23] and, independently, by Huang, Malhamé and Caines [20], are limit models for noncooperative symmetric N-player games with mean field interaction as the number of players N tends to infinity; see, for instance, [6, 10], and the recent two-volume work [9].

Submodular games were first introduced by Topkis in [31] in the context of static noncooperative N-player games. They are characterized by costs of the players that have decreasing differences with respect to a partial order induced by a lattice on the set of strategy vectors. Because the notion of submodularity is related to that of substitute goods in Economics, submodular games have received large attention in the economic literature (see [2, 25], among many others). A systematic treatment of submodular games can be found in [32, 34], and in the survey [3].

The submodularity assumption has been applied to mean field games by Adlaka and Johari in [1] for a class of discrete time games with infinite horizon discounted costs, by Więcek in [35] for a class of finite state mean field games with total reward up to a time of first exit, and by Carmona, Delarue, and Lacker in [12] for mean field games of timing (optimal stopping), in order to study dynamic models of bank runs in a continuous time setting. It is also worth noticing that mean field games considered in recent works addressing the problem of nonuniqueness of solutions enjoy a submodular structure (see, e.g., [4, 13, 15]), even if the latter is not exploited therein.

Here, we consider a class of finite horizon mean field games with Itô-type dynamics. More specifically, the evolution of the state of the representative player is described by a one-

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dimensional Itô stochastic differential equation (SDE) with random (not necessarily Markovian) coefficients and controlled drift. The diffusion coefficient, while independent of state and control, is possibly degenerate. Deterministic dynamics are thus included as a special case. The measure variable, which represents the distribution of the continuum of "other" players, only appears in the (random, not Markovian) cost coefficients with running costs split into two parts, one depending on the control, the other on the measure. The measure-dependent costs are assumed to be submodular with respect to first order stochastic dominance on measures and the standard order relation on states (cf. Assumption 2.9 below).

The submodularity assumption has a number of remarkable consequences. It yields, in particular, an alternative way of establishing the existence of solutions and gives rise to a simple learning procedure. Existence of solutions to the mean field game can be obtained through Banach's fixed point theorem if the time horizon is small (cf. [20]). For arbitrary time horizons, a version of the Brouwer–Schauder fixed point theorem, including generalizations to multi-valued maps, can be used; cf. [6] and [22]. Under the submodularity assumption, existence of solutions can instead be deduced from Tarski's fixed point theorem [29]. This allows us to cover systems with coefficients that are possibly discontinuous in the measure variable. Another notable consequence of the submodularity is that the set of all solutions for a given initial distribution enjoys a lattice structure so that there are a minimal solution and a maximal solution with respect to the order relation. The existence of multiple solutions is in fact quite common in mean field games (see [4, 15] and the references therein), and the submodularity assumption is compatible with this nonuniqueness of solutions. In particular (yet relevant) cases, we can also prove the existence of MFG solutions when the dynamics of the state process depends on the measure (see Section 4.4). Furthermore, our lattice-theoretical approach allows us to deal with a class of MFGs with common noise in which the representative agent faces a mean field interaction through the conditional mean of its state given the common noise (see Section 4.5). Such MFGs have been studied in [15] and [30]—where the issue of uniqueness and selection of equilibria is addressed in a linear-quadratic setting—and for them we are able to show existence of a strong solution, a kind of result which is still relatively infrequent in the literature (cf. Remark 4.6 and Section 6 in [11]). Finally, although our results strongly hinge on the one-dimensional nature of the setting, suitable multidimensional cases can also be considered. In particular, if the dependence on the measure is only through one of its one-dimensional marginals, existence and approximation of MFG solutions can still be obtained in some settings (cf. Section 4.1).

The problem of how to find solutions to a mean field game in a constructive way has been addressed by Cardaliaguet and Hadikhanloo [7]. They analyze a learning procedure, similar to what is known as fictitious play (cf. [19] and the references therein), where the representative agent, starting from an arbitrary flow of measures, computes a new flow of measures by updating the average over past measure flows according to the best response to that average. For potential mean field games, the authors establish convergence of this kind of fictitious play. A simpler learning procedure consists in directly iterating the best response map, thus computing a new flow of measures as best response to the previous measure flow. Under the submodularity assumption, we show that this procedure converges to a mean field game solution for appropriately chosen initial measure flows (cf. Remark 2.20), while it need not converge for potential or other classes of mean field games.

The rest of this paper is organized as follows. In Section 2.1, we introduce the controlled system dynamics and costs, together with our standing assumptions, and give the definition of a mean field game, where we take ordinary stochastic open-loop controls as admissible strategies. In Section 2.2, we define the order relation on probability measures which is crucial for the submodularity assumption on the cost coefficients of the game. That assumption is stated and discussed in Section 2.3, while Section 2.4 deals with properties of the best response map. Section 2.5 contains our main results, namely Theorem 2.14 on the existence and lattice structure of MFG solutions and Theorem 2.17 on the convergence of the simple learning procedure. In Section 3, we extend the analysis of Section 2 to submodular mean field games defined over stochastic relaxed controls. This allows us to re-obtain the existence and, especially, the convergence result under more general conditions. Section 4 concludes with comments on the multidimensional setting, the linear-quadratic case, systems with multiplicative and mean field dependent dynamics, and mean field games with common noise. Some auxiliary results on first order stochastic dominance are collected in the Appendix A.

Notation. Throughout the rest of this paper, given $x, y \in \mathbb{R}$, we set $x \wedge y := \min\{x, y\}$ and $x \vee y := \max\{x, y\}$. Moreover, given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a random variable $X\colon \Omega \to \mathbb{R}$, we use the notation $\mathbb{P} \circ X^{-1}$ for the law of X under $\mathbb{P}$, that is, we set $(\mathbb{P} \circ X^{-1})[E] := \mathbb{P}[X \in E]$ for each Borel set E of $\mathbb{R}$. Finally, for a given $T \in (0, \infty)$ and a stochastic process $X = (X_t)_{t\in[0,T]}$, with a slight abuse of notation, we denote by $\mathbb{P} \circ X^{-1}$ the flow of measures associated to X; that is, we set $\mathbb{P} \circ X^{-1} := (\mathbb{P} \circ X_t^{-1})_{t\in[0,T]}$.

2. The submodular mean field game. In this section we develop our set up for submodular mean field games. This set up allows us to prove the existence of MFG solutions without using a weak formulation or the notion of relaxed controls. Instead, we combine probabilistic arguments together with a lattice-theoretical approach in order to prove the existence and approximation of MFG solutions.

2.1. *The mean field game problem.* Let $T > 0$ be a fixed time horizon and $W = (W_t)_{t\in[0,T]}$ be a Brownian Motion on a complete filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$. Let $\xi \in L^2(\Omega, \mathcal{F}_0, \mathbb{P})$ and $(\sigma_t)_{t\in[0,T]} \subset [0, \infty)$ be a progressively measurable square integrable stochastic process. Notice that we allow the volatility process to be zero on a progressively measurable set $E \subset [0, T] \times \Omega$ with positive measure, thus leading to a degenerate dynamics. For a closed and convex set $U \subset \mathbb{R}$, define the the set of admissible controls $\mathcal{A}$ as the set of all square integrable progressively measurable processes $\alpha\colon \Omega \times [0, T] \to U$. For a measurable function $b\colon \Omega \times [0, T] \times \mathbb{R} \times U \to \mathbb{R}$ and an admissible process α, we consider the controlled SDE (SDE(α), in short)

$$dX_t = b(t, X_t, \alpha_t)\, dt + \sigma_t\, dW_t, \quad t \in [0, T], X_0 = \xi. \tag{2.1}$$

With no further reference, throughout this paper we will assume that for each $(x, a) \in \mathbb{R} \times U$ the process $b(\cdot, \cdot, x, a)$ is progressively measurable and that the usual Lipschitz continuity and growth conditions are satisfied; that is, there exists a constant $C_1 > 0$ such that for each $(\omega, t, a) \in \Omega \times [0, T] \times U$ we have

$$\begin{aligned} \bigl|b(\omega, t, x, a) - b(\omega, t, y, a)\bigr| &\le C_1|x - y|, \quad \forall x, y \in \mathbb{R}, \\ \bigl|b(\omega, t, x, a)\bigr| &\le C_1\bigl(1 + |x| + |a|\bigr), \quad \forall x \in \mathbb{R}. \end{aligned} \tag{2.2}$$

Under the standing assumption, by standard SDE theory, for each $\alpha \in \mathcal{A}$ there exists a unique strong solution $X^\alpha := (X_t^\alpha)_{t\in[0,T]}$ to the controlled SDE(α) (2.1).

Let $\mathcal{P}(\mathbb{R})$ denote the space of all probability measures on the Borel σ-algebra $\mathcal{B}(\mathbb{R})$, endowed with the classical (C_b-)weak topology, that is, the topology induced by the weak convergence of probability measures. The costs of the problem are given by three measurable functions

$$\begin{aligned} f &: \quad \Omega \times [0, T] \times \mathbb{R} \times \mathcal{P}(\mathbb{R}) \to \mathbb{R}, \\ l &: \quad \Omega \times [0, T] \times \mathbb{R} \times U \to \mathbb{R}, \\ g &: \quad \Omega \times \mathbb{R} \times \mathcal{P}(\mathbb{R}) \to \mathbb{R}, \end{aligned} \tag{2.3}$$

such that, for each $(x, \mu, a) \in \mathbb{R} \times \mathcal{P}(\mathbb{R}) \times U$, the processes $f(\cdot, \cdot, x, \mu)$, $l(\cdot, \cdot, x, a)$ are progressively measurable and the random variable $g(\cdot, x, \mu)$ is $\mathcal{F}_T$-measurable. We underline that the cost processes f and g are not necessarily Markovian.

For any given and fixed measurable flow $\mu = (\mu_t)_{t\in[0,T]}$ of probability measures on $\mathcal{B}(\mathbb{R})$, we introduce the cost functional

$$J(\alpha, \mu) := \mathbb{E}\biggl[\int_0^T \bigl[f\bigl(t, X_t^\alpha, \mu_t\bigr) + l\bigl(t, X_t^\alpha, \alpha_t\bigr)\bigr]\,dt + g\bigl(X_T^\alpha, \mu_T\bigr)\biggr], \quad \alpha \in \mathcal{A}, \tag{2.4}$$

and consider the optimal control problem $\inf_{\alpha\in\mathcal{A}} J(\alpha, \mu)$.

We say that (X^μ, α^μ) is an *optimal pair* for the flow μ if $-\infty < J(\alpha^\mu, \mu) \le J(\alpha, \mu)$ for each admissible $\alpha \in \mathcal{A}$ and $X^\mu = X^{\alpha^\mu}$.

REMARK 2.1. The subsequent results of this paper remain valid if we consider a geometric dynamics for X (cf. Section 4.3 below). Moreover, for suitable choices of the costs, we can also allow for geometric or mean-reverting state processes with dependence on the measure in the dynamics (see Section 4.4 for more details).

We make the following standing assumption.

ASSUMPTION 2.2.

1. For each measurable flow μ of probability measures on $\mathcal{B}(\mathbb{R})$, there exists a unique (up to indistinguishability) optimal pair (X^μ, α^μ).
2. There exists a continuous and strictly increasing function $\psi : [0, \infty) \to [0, \infty)$ with $\lim_{s\to\infty} \psi(s) = \infty$ and a constant $M > \psi(0)$ such that

$$\mathbb{E}\bigl[\psi\bigl(\bigl|X_t^\mu\bigr|\bigr)\bigr] \le M \quad \text{for all measurable flows of probabilities } \mu \text{ and } t \in [0, T]. \tag{2.5}$$

REMARK 2.3. To underline the flexibility of our set up, condition (1) in Assumption 2.2 is stated at an informal level. Condition (1) holds, for example, in the case of a linear-convex setting in which $b(t, x, a) = c_t + p_t x + q_t a$, for suitable processes c_t, p_t, q_t, $l(t, \cdot, \cdot)$ is strictly convex and lower semicontinuous, $f(t, \cdot, \mu)$ and $g(\cdot, \mu)$ are lower semicontinuous, and U is convex and compact. More general conditions ensuring existence and uniqueness of an optimal pair in the strong formulation of the control problem can be found in [17] and in Chapter II of [8], among others.

REMARK 2.4. Notice that condition (2) in Assumption 2.2 is equivalent to the tightness of the family of laws $\{\mathbb{P} \circ (X_t^\mu)^{-1} : \mu \text{ is a measurable flow}, t \in [0, T]\}$ (cf. [14, 24] or [27]). The latter is satisfied, for example, if U is compact or if b is bounded in a. Alternatively, one can assume that U is closed and convex and that there exist exponents $p' > p \ge 1$ and constants $\kappa, K > 0$ such that $\mathbb{E}[|\xi|^{p'}] < \infty$ and

$$\begin{gathered}\bigl|f(t, x, \mu)\bigr| + \bigl|g(x, \mu)\bigr| \le K\bigl(1 + |x|^p\bigr),\\ \kappa|a|^{p'} - K\bigl(1 + |x|^p\bigr) \le l(t, x, a) \le K\bigl(1 + |x|^p + |a|^{p'}\bigr),\end{gathered} \tag{2.6}$$

for all $(t, x, \mu, a) \in [0, T] \times \mathbb{R} \times \mathcal{P}(\mathbb{R}) \times U$. Indeed, following the proof of Lemma 5.1 in [22], these conditions allow us to have an a priori bound on the p-moments of the minimizers independent of the measure μ; that is, there exists a constant $M > 0$ such that

$$\mathbb{E}\bigl[\bigl|X_t^\mu\bigr|^{p'}\bigr] \le M \quad \text{for all measurable flows of probabilities } \mu \text{ and } t \in [0, T]. \tag{2.7}$$

REMARK 2.5 (On the topology on $\mathbb{R} \times \mathcal{P}(\mathbb{R})$ and the noncontinuity of the costs). We point out that the space $\mathbb{R}$ is endowed with the usual Euclidean distance, while the set $\mathcal{P}(\mathbb{R})$ is endowed with the classical (C_b-)weak topology, that is, the topology induced by the weak convergence of probability measures. Also, we say that sequence of probability measures converges weakly if it converges in the (C_b-)weak topology. Unless otherwise stated, the set $\mathbb{R} \times \mathcal{P}(\mathbb{R})$ will always be endowed with the product topology, and the continuity of f, g will mean continuity with respect to this topology.

Alternatively, for $p \geq 1$, one could work on the space $\mathcal{P}_p(\mathbb{R}) := \{\mu \in \mathcal{P}(\mathbb{R}) \mid \int_{\mathbb{R}} |y|^p \, d\mu(y) < \infty\}$ endowed with the p-Wasserstein distance

$$W_p(\mu, \nu) := \left(\inf_{\gamma \in \Gamma(\mu,\nu)} \int_{\mathbb{R}^2} |x - y|^p \, d\gamma(x, y) \right)^{1/p}, \quad \mu, \nu \in \mathcal{P}_p(\mathbb{R}),$$

where $\Gamma(\mu, \nu)$ denotes the set of probability measures γ on the Borel sets of $\mathbb{R}^2$, such that $\gamma(E \times \mathbb{R}) = \mu(E)$ and $\gamma(\mathbb{R} \times E) = \nu(E)$ for each $E \in \mathcal{B}(\mathbb{R})$. The latter distance is usually used in the literature to address the continuity of the costs (see, e.g., [22]).

Differently from the standard conditions in the literature on mean field games, our existence result (Theorem 2.14) does not require any continuity of the costs f and g with respect to the measure μ. In fact, f and g can be discontinuous with respect to the weak topology or with respect to any Wasserstein distance.

For each measurable flow μ of probability measures on $\mathcal{B}(\mathbb{R})$, we now define the *best-response* by $R(\mu) := \mathbb{P} \circ (X^\mu)^{-1}$, where we set $\mathbb{P} \circ (X^\mu)^{-1} := (\mathbb{P} \circ (X_t^\mu)^{-1})_{t \in [0,T]}$. The map $\mu \mapsto R(\mu)$ is called the *best-response-map*.

DEFINITION 1 (MFG solution). A measurable flow μ^* of probability measures on $\mathcal{B}(\mathbb{R})$ is a mean field game solution if it is a fixed point of the best-response-map R; that is, if $R(\mu^*) = \mu^*$.

2.2. *The lattice structure.* In this section, we endow the space of measurable flows with a suitable lattice structure, which is fundamental for the subsequent analysis. We start by identifying the set of probability measures $\mathcal{P}(\mathbb{R})$ with the set of distribution functions on $\mathbb{R}$, setting $\mu(s) := \mu(-\infty, s]$ for each $s \in \mathbb{R}$ and $\mu \in \mathcal{P}(\mathbb{R})$. On $\mathcal{P}(\mathbb{R})$ we then consider the order relation $\leq^{\text{st}}$ given by the *first order stochastic dominance*, that is, we write

$$\mu \leq^{\text{st}} \nu \quad \text{for } \mu, \nu \in \mathcal{P}(\mathbb{R}) \quad \text{if and only if} \quad \mu(s) \geq \nu(s) \quad \text{for each } s \in \mathbb{R}. \tag{2.8}$$

The partially ordered set $(\mathcal{P}(\mathbb{R}), \leq^{\text{st}})$ is then endowed with a lattice structure by defining

$$(\mu \wedge^{\text{st}} \nu)(s) := \mu(s) \vee \nu(s) \quad \text{and} \quad (\mu \vee^{\text{st}} \nu)(s) := \mu(s) \wedge \nu(s) \quad \text{for each } s \in \mathbb{R}. \tag{2.9}$$

Observe that (see, e.g., [28]), for $\mu, \nu \in \mathcal{P}(\mathbb{R})$, we have

$$\mu \leq^{\text{st}} \nu \quad \text{if and only if} \quad \langle \varphi, \mu \rangle \leq \langle \varphi, \nu \rangle \tag{2.10}$$

for any increasing function $\varphi \colon \mathbb{R} \to \mathbb{R}$ such that $\langle \varphi, \mu \rangle$ and $\langle \varphi, \nu \rangle$ are finite, where $\langle \varphi, \mu \rangle := \int_{\mathbb{R}} \varphi(y) \, d\mu(y)$.

Recall that by (2.5),

$$\mathbb{E}[\psi(|X_t^\mu|)] \leq M \quad \text{for all measurable flows } \mu \text{ and } t \in [0, T].$$

Then, following the arguments in the proof of Lemma A.2, we can define $\mu^{\text{Min}}, \mu^{\text{Max}} \in \mathcal{P}(\mathbb{R})$ with

$$\mu^{\text{Min}} \leq^{\text{st}} \mathbb{P} \circ (X_t^\mu)^{-1} \leq^{\text{st}} \mu^{\text{Max}} \quad \text{for all measurable flows } \mu \text{ and } t \in [0, T],$$

where, extending ψ to $(-\infty,0)$ by $\psi(s):=\psi(0)$ for $s<0$, μ^{Min} and μ^{Max} are given by

$$\mu^{\mathrm{Min}}(s):=\frac{M}{\psi(-s)}\wedge 1 \quad \text{and} \quad \mu^{\mathrm{Max}}(s):=\left(1-\frac{M}{\psi(s)}\right)\vee 0 \quad \text{for all } s\in\mathbb{R}. \tag{2.11}$$

This observation suggests to consider the interval

$$[\mu^{\mathrm{Min}},\mu^{\mathrm{Max}}]=\{\mu\in\mathcal{P}(\mathbb{R})\mid \mu^{\mathrm{Min}}\leq^{\mathrm{st}}\mu\leq^{\mathrm{st}}\mu^{\mathrm{Max}}\}$$

endowed with the Borel σ-algebra induced by the weak topology, that is, the topology related to the weak convergence of probability measures. We consider the finite measure $\pi:=\delta_0+dt+\delta_T$ on the Borel σ-algebra $\mathcal{B}([0,T])$ of the interval $[0,T]$, where δ_t denotes the Dirac measure at time $t\in[0,T]$. Notice that we include δ_0 into the definition of the measure π in order to prescribe the initial law $\mathbb{P}\circ\xi^{-1}$. We then define the set L of feasible flows of measures as the set of all equivalence classes (w.r.t. π) of measurable flows $(\mu_t)_{t\in[0,T]}$ with $\mu_t\in[\mu^{\mathrm{Min}},\mu^{\mathrm{Max}}]$ for π-almost all $t\in(0,T]$ and $\mu_0=\mathbb{P}\circ\xi^{-1}$. On L we consider the order relation $\leq^L$ given by $\mu\leq^L\nu$ if and only if $\mu_t\leq^{\mathrm{st}}\nu_t$ for π-a.a. $t\in[0,T]$. This order relation implies that L can be endowed with the lattice structure given by

$$(\mu\wedge^L\nu)_t:=\mu_t\wedge^{\mathrm{st}}\nu_t \quad \text{and} \quad (\mu\vee^L\nu)_t:=\mu_t\vee^{\mathrm{st}}\nu_t \quad \text{for } \pi\text{-a.a. } t\in[0,T].$$

Notice that $(\mathbb{P}\circ(X_t^{\mu})^{-1})_{t\in[0,T]}\in L$ for every $\mu\in L$. In particular, the best-response-map $R\colon L\to L$ is well defined.

REMARK 2.6. We point out that if $\psi(x)=x^2$, then each element of $[\mu^{\mathrm{Min}},\mu^{\mathrm{Max}}]$ has finite first-order moment, that is, $\int_{\mathbb{R}}|y|\,d\mu(y)<\infty$ for each $[\mu^{\mathrm{Min}},\mu^{\mathrm{Max}}]$. This follows directly from Lemma A.3. Notice also that a higher integrability requirement in (2.5) implies the existence and uniform boundedness of higher moments for the elements of $[\mu^{\mathrm{Min}},\mu^{\mathrm{Max}}]$. More precisely, if $\psi(x)=x^{p'}$ for some $p'\in(1,\infty)$, then

$$\sup_{\mu\in[\mu^{\mathrm{Min}},\mu^{\mathrm{Max}}]}\int_{\mathbb{R}}|y|^p\,d\mu(y)<\infty \quad \text{for all } p\in(1,p').$$

We now turn our focus on the main result of this subsection, which is the following lemma. Its proof follows from the more general Proposition A.4, which is relegated to the Appendix A.

LEMMA 2.7. *The lattice $(L,\leq^L)$ is complete. That is, each subset of L has a least upper bound and a greatest lower bound.*

REMARK 2.8. We underline that, in general, $\inf L$ and $\sup L$ are given by

$$(\inf L)_t:=\mathbb{1}_{\{0\}}(t)\mathbb{P}\circ\xi^{-1}+\mathbb{1}_{(0,T]}(t)\mu^{\mathrm{Min}}, \qquad (\sup L)_t:=\mathbb{1}_{\{0\}}(t)\mathbb{P}\circ\xi^{-1}+\mathbb{1}_{(0,T]}(t)\mu^{\mathrm{Max}},$$

with μ^{Min}, μ^{Max} defined in (2.11) in terms of ψ and M. In particular, according to Remark 2.4, if U is compact, if b is bounded or if Condition (2.6) is satisfied, then condition (2) in Assumption 2.2 is satisfied with $\psi(s)=s^p$ for $s\geq 0$ and some $p\geq 1$. In this case, $\inf L$ and $\sup L$ are explicitly given by

$$(\inf L)_t(s):=\mathbb{1}_{\{0\}}(t)\mathbb{P}\circ\xi^{-1}(s)+\mathbb{1}_{(0,T]}(t)\left[\mathbb{1}_{\{s<0\}}\left(\frac{M}{(-s)^p}\wedge 1\right)+\mathbb{1}_{\{s\geq 0\}}\right], \tag{2.12}$$

for a.a. $t\in[0,T]$ and for each $s\in\mathbb{R}$, and

$$(\sup L)_t(s):=\mathbb{1}_{\{0\}}(t)\mathbb{P}\circ\xi^{-1}(s)+\mathbb{1}_{(0,T]}(t)\left[\mathbb{1}_{\{s\leq 0\}}+\mathbb{1}_{\{s>0\}}\left(1-\frac{M}{(s)^p}\right)\vee 0\right], \tag{2.13}$$

for a.a. $t\in[0,T]$ and for each $s\in\mathbb{R}$.

2.3. *The submodularity condition.* Our subsequent results rely on the following key assumption.

ASSUMPTION 2.9 (Submodularity condition). For $\mathbb{P} \otimes dt$ a.a. $(\omega, t) \in \Omega \times [0, T]$, the functions $f(t, \cdot, \cdot)$ and g have decreasing differences in (x, μ); that is, for $\phi \in \{f(t, \cdot, \cdot), g\}$,

$$\phi(\bar{x}, \bar{\mu}) - \phi(x, \bar{\mu}) \leq \phi(\bar{x}, \mu) - \phi(x, \mu),$$

for all $\bar{x}, x \in \mathbb{R}$ and $\bar{\mu}, \mu \in \mathcal{P}(\mathbb{R})$ s.t. $\bar{x} \geq x$ and $\bar{\mu} \geq^{\mathrm{st}} \mu$.

We list here two examples in which Assumption 2.9 is satisfied.

EXAMPLE 1 (Mean-field interaction of scalar type). Consider a mean-field interaction of scalar type; that is, $\phi(x, \mu) = \gamma(x, \langle \varphi, \mu \rangle)$ for given measurable maps $\gamma : \mathbb{R}^2 \to \mathbb{R}$ and $\varphi : \mathbb{R} \to \mathbb{R}$. If the map φ is increasing and the map $\gamma : \mathbb{R}^2 \to \mathbb{R}$ has decreasing differences in $(x, y) \in \mathbb{R}^2$, then Assumption 2.9 is satisfied. Observe that a function $\gamma \in \mathcal{C}^2(\mathbb{R}^2)$ has decreasing differences in (x, y) if and only if

$$\frac{\partial^2 \gamma}{\partial x \partial y}(x, y) \leq 0 \quad \text{for each } (x, y) \in \mathbb{R}^2.$$

EXAMPLE 2 (Mean-field interactions of order-1). Another example is provided by the interactions of order-1, that is, when ϕ is of the form

$$\phi(x, \mu) = \int_{\mathbb{R}} \gamma(x, y) \, d\mu(y).$$

It is easy to check that, thanks to (2.10), Assumption 2.9 holds when γ has decreasing differences in (x, y).

A natural and relevant question related to Assumption 2.9 concerns its link to the so-called *Lasry–Lions monotonicity condition*, that is, the condition

$$\int_{\mathbb{R}} \big(\phi(x, \bar{\mu}) - \phi(x, \mu)\big) \, d(\bar{\mu} - \mu)(x) \geq 0, \quad \forall \bar{\mu}, \mu \in \mathcal{P}(\mathbb{R}). \tag{2.14}$$

In general, there is no relation between the submodularity condition and (2.14). However, since Assumption 2.9 is equivalent to the fact that the map $\phi(\cdot, \bar{\mu}) - \phi(\cdot, \mu)$ is decreasing for $\mu, \bar{\mu} \in \mathcal{P}(\mathbb{R})$ with $\bar{\mu} \geq^{\mathrm{st}} \mu$, Assumption 2.9 and (2.10) imply that

$$\int_{\mathbb{R}} \big(\phi(x, \bar{\mu}) - \phi(x, \mu)\big) \, d(\bar{\mu} - \mu)(x) \leq 0, \quad \forall \bar{\mu}, \mu \in \mathcal{P}(\mathbb{R}) \text{ with } \bar{\mu} \geq^{\mathrm{st}} \mu;$$

the latter, roughly speaking, being sort of an opposite version of the Lasry–Lions monotonicity condition (2.14).

REMARK 2.10. Specific cost functions satisfying Assumption 2.9 are, for example,

$$f(t, x, \mu) \equiv 0, \qquad l(t, x, a) = \frac{a^2}{2}, \qquad g(x, \mu) = \big(x - \mathbb{1}_{[0,\infty)}\big(\langle \mathrm{id}, \mu \rangle\big)\big)^2,$$

where $\mathrm{id}(y) = y$. Notice that the function $\mu \mapsto g(x, \mu)$ is discontinuous, in contrast to the typical continuity requirement assumed in the literature (see, e.g., [22]).

2.4. *The best-response-map.* In the following lemma, we show that the set of admissible trajectories is a lattice.

LEMMA 2.11. *If α and $\bar{\alpha}$ are admissible controls, then there exists an admissible control $\alpha^{\vee}$ such that $X^{\alpha} \vee X^{\bar{\alpha}} = X^{\alpha^{\vee}}$. Moreover, there exists an admissible control $\alpha^{\wedge}$ such that $X^{\alpha} \wedge X^{\bar{\alpha}} = X^{\alpha^{\wedge}}$.*

PROOF. Let α and $\bar{\alpha}$ be admissible controls and define the process $\alpha^{\vee}$ by

$$\alpha_s^{\vee} := \begin{cases} \alpha_s & \text{on } \{X_s^{\alpha} > X_s^{\bar{\alpha}}\} \cup \{X_s^{\alpha} = X_s^{\bar{\alpha}}, b(s, X_s^{\alpha}, \alpha_s) \geq b(s, X_s^{\bar{\alpha}}, \bar{\alpha}_s)\}, \\ \bar{\alpha}_s & \text{on } \{X_s^{\alpha} < X_s^{\bar{\alpha}}\} \cup \{X_s^{\alpha} = X_s^{\bar{\alpha}}, b(s, X_s^{\alpha}, \alpha_s) < b(s, X_s^{\bar{\alpha}}, \bar{\alpha}_s)\}. \end{cases}$$

The process $\alpha^{\vee}$ is clearly progressively measurable and square integrable, hence admissible.

We want to show that $X^{\alpha} \vee X^{\bar{\alpha}} = X^{\alpha^{\vee}}$; that is,

$$X_t^{\alpha} \vee X_t^{\bar{\alpha}} = \xi + \int_0^t b(s, X_s^{\alpha} \vee X_s^{\bar{\alpha}}, \alpha_s^{\vee})\, ds + \int_0^t \sigma_s\, dW_s, \quad \forall t \in [0, T], \mathbb{P}\text{-a.s.} \tag{2.15}$$

In order to do so, observe that the process $X^{\alpha} \vee X^{\bar{\alpha}}$ satisfies, $\mathbb{P}$-a.s. for each $t \in [0, T]$, the following integral equation

$$X_t^{\alpha} \vee X_t^{\bar{\alpha}} = \xi + \int_0^t \sigma_s\, dW_s + \left(\int_0^t b(s, X_s^{\alpha}, \alpha_s)\, ds\right) \vee \left(\int_0^t b(s, X_s^{\bar{\alpha}}, \bar{\alpha}_s)\, ds\right). \tag{2.16}$$

Furthermore, defining the two processes A and $\bar{A}$ by

$$A_t := \int_0^t b(s, X_s^{\alpha}, \alpha_s)\, ds \quad \text{and} \quad \bar{A}_t := \int_0^t b(s, X_s^{\bar{\alpha}}, \bar{\alpha}_s)\, ds,$$

we see that the process S, defined by $S_t := A_t \vee \bar{A}_t$, is $\mathbb{P}$-a.s. absolutely continuous. Hence the time derivative of S exists a.e. in $[0, T]$ and, in view of (2.16), in order to prove (2.15) it suffices to show that $dS_t/dt = b(t, X_t^{\alpha} \vee X_t^{\bar{\alpha}}, \alpha_t^{\vee})$ for $\mathbb{P} \otimes dt$ a.a. $(\omega, t) \in \Omega \times [0, T]$.

Since the processes A, $\bar{A}$ and S are $\mathbb{P}$-a.s. absolutely continuous, for each ω in a set of full probability, the paths $A(\omega)$, $\bar{A}(\omega)$ and $S(\omega)$ admit time derivatives in a subset $E(\omega) \subset [0, T]$ with full Lebesgue measure. We now use a pathwise argument, without stressing the dependence on $\omega \in \Omega$. Take $t \in E$ such that $X_t^{\alpha} > X_t^{\bar{\alpha}}$. By continuity, there exists a (random) neighborhood I_t of t in $\mathbb{R}$ such that $X_s^{\alpha} > X_s^{\bar{\alpha}}$ for each $s \in I_t \cap [0, T]$, which, by (2.16), is true if and only if $A_s > \bar{A}_s$ for each $s \in I_t \cap [0, T]$. Hence, by definition of S, we have

$$\frac{dS_s}{ds} = \frac{dA_s}{ds} = b(s, X_s^{\alpha}, \alpha_s), \quad \forall s \in I_t \cap [0, T],$$

and, in particular, $dS_s/ds = b(s, X_s^{\alpha} \vee X_s^{\bar{\alpha}}, \alpha_s^{\vee})$ for each $s \in I_t \cap [0, T]$.

Take now $t \in E$ such that $X_t^{\alpha} = X_t^{\bar{\alpha}}$ and $b(t, X_t^{\alpha}, \alpha_t) \geq b(t, X_t^{\bar{\alpha}}, \bar{\alpha}_t)$. From (2.16) it follows that $A_t = \bar{A}_t$, which in turn implies that

$$\frac{dS_t}{dt} = \lim_{h \to 0} \frac{A_{t+h} \vee \bar{A}_{t+h} - A_t \vee \bar{A}_t}{h} \geq \frac{dA_t}{dt} \vee \frac{d\bar{A}_t}{dt}.$$

By construction,

$$\frac{dA_t}{dt} = b(t, X_t^{\alpha}, \alpha_t) \geq b(t, X_t^{\bar{\alpha}}, \bar{\alpha}_t) = \frac{d\bar{A}_t}{dt}. \tag{2.17}$$

If there exists a sequence $\{h^j\}_{j \in \mathbb{N}}$ converging to 0 such that $A_{t+h^j} \geq \bar{A}_{t+h^j}$ for each $j \in \mathbb{N}$, then clearly $dS_t/dt = dA_t/dt = b(t, X_t^{\alpha}, \alpha_t) = b(t, X_t^{\alpha} \vee X_t^{\bar{\alpha}}, \alpha_t^{\vee})$, as desired. On the other hand, if such a sequence does not exist, then there exists some $\delta > 0$ such that $A_{t+h} \leq$

$\bar{A}_{t+h}$ for each $h \in (-\delta, \delta)$. Recalling (2.17), this implies that $dA_t/dt \le dS_t/dt = d\bar{A}_t/dt \le dA_t/dt$, hence we obtain again that $dS_t/dt = dA_t/dt$.

Altogether, we have proved that for a.a. $t \in [0, T]$ with $X_t^\alpha > X_t^{\bar{\alpha}}$ or $X_t^\alpha = X_t^{\bar{\alpha}}$ and $b(t, X_t^\alpha, \alpha_t) \ge b(t, X_t^{\bar{\alpha}}, \bar{\alpha}_t)$, we have $dS_t/dt = b(t, X_t^\alpha, \alpha_t) = b(t, X_t^\alpha \vee X_t^{\bar{\alpha}}, \alpha_t^\vee)$. Analogously, one can prove that $dS_t/dt = b(t, X_t^{\bar{\alpha}}, \bar{\alpha}_t) = b(t, X_t^\alpha \vee X_t^{\bar{\alpha}}, \alpha_t^\vee)$ for a.a. $t \in [0, T]$ with $X_t^\alpha < X_t^{\bar{\alpha}}$ or $X_t^\alpha = X_t^{\bar{\alpha}}$ and $b(t, X_t^\alpha, \alpha_t) < b(t, X_t^{\bar{\alpha}}, \bar{\alpha}_t)$. Therefore $dS_t/dt = b(t, X_t^\alpha \vee X_t^{\bar{\alpha}}, \alpha_t^\vee)$ for $\mathbb{P} \otimes dt$ a.a. $(\omega, t) \in \Omega \times [0, T]$, which proves (2.15).

The arguments employed above allow us to prove that the process $X^\alpha \wedge X^{\bar{\alpha}}$ satisfies the SDE controlled by $\alpha^\wedge$; that is,

$$X_t^\alpha \wedge X_t^{\bar{\alpha}} = \xi + \int_0^t b(s, X_s^\alpha \wedge X_s^{\bar{\alpha}}, \alpha_s^\wedge)\, ds + \int_0^t \sigma_s\, dW_s, \quad \forall t \in [0, T], \mathbb{P}\text{-a.s.},$$

where $\alpha^\wedge$ is defined by

$$\alpha_s^\wedge := \begin{cases} \bar{\alpha}_s & \text{on } \{X_s^\alpha > X_s^{\bar{\alpha}}\} \cup \{X_s^\alpha = X_s^{\bar{\alpha}}, b(s, X_s^\alpha, \alpha_s) \ge b(s, X_s^{\bar{\alpha}}, \bar{\alpha}_s)\}, \\ \alpha_s & \text{on } \{X_s^\alpha < X_s^{\bar{\alpha}}\} \cup \{X_s^\alpha = X_s^{\bar{\alpha}}, b(s, X_s^\alpha, \alpha_s) < b(s, X_s^{\bar{\alpha}}, \bar{\alpha}_s)\}. \end{cases}$$

The proof of the lemma is therefore completed. □

We now prove the fundamental property of the best-response-map.

LEMMA 2.12. *The best-response-map R is increasing in $(L, \le^L)$.*

PROOF. Take $\bar{\mu}, \mu \in L$ such that $\mu \le^L \bar{\mu}$ and let $(X^{\bar{\mu}}, \alpha^{\bar{\mu}})$ and (X^μ, α^μ) be the optimal pairs related to $\bar{\mu}$ and μ, respectively. For $t \in [0, T]$, we define the event

$$B_t := \{X_t^\mu > X_t^{\bar{\mu}}\} \cup \{X_t^\mu = X_t^{\bar{\mu}}, b(t, X_t^\mu, \alpha_t^\mu) \ge b(t, X_t^{\bar{\mu}}, \alpha_t^{\bar{\mu}})\}. \tag{2.18}$$

As it is shown in Lemma 2.11, the process $X^\mu \vee X^{\bar{\mu}}$ is the solution to the dynamics (2.1) controlled by $\alpha_t^\vee := \alpha_t^\mu \mathbb{1}_{B_t} + \alpha_t^{\bar{\mu}} \mathbb{1}_{B_t^c}$, and the process $X^\mu \wedge X^{\bar{\mu}}$ is the solution to the dynamics controlled by $\alpha_t^\wedge := \alpha_t^\mu \mathbb{1}_{B_t^c} + \alpha_t^{\bar{\mu}} \mathbb{1}_{B_t}$.

By the admissibility of $\alpha^\vee$ and the optimality of $\alpha^{\bar{\mu}}$ we can write

$$\begin{aligned} 0 &\le J(\alpha^\vee, \bar{\mu}) - J(\alpha^{\bar{\mu}}, \bar{\mu}) \\ &= \mathbb{E}\left[\int_0^T [f(t, X_t^\mu \vee X_t^{\bar{\mu}}, \bar{\mu}_t) - f(t, X_t^{\bar{\mu}}, \bar{\mu}_t)]\, dt\right] \\ &\quad + \mathbb{E}\left[\int_0^T [l(t, X_t^\mu \vee X_t^{\bar{\mu}}, \alpha_t^\vee) - l(t, X_t^{\bar{\mu}}, \alpha_t^{\bar{\mu}})]\, dt\right] \\ &\quad + \mathbb{E}[g(X_T^\mu \vee X_T^{\bar{\mu}}, \bar{\mu}_T) - g(X_T^{\bar{\mu}}, \bar{\mu}_T)]. \end{aligned} \tag{2.19}$$

Next, from the definition of B_t in (2.18) and the trivial identity $1 = \mathbb{1}_{B_t} + \mathbb{1}_{B_t^c}$, we find

$$\begin{aligned} &\mathbb{E}\left[\int_0^T [f(t, X_t^\mu \vee X_t^{\bar{\mu}}, \bar{\mu}_t) - f(t, X_t^{\bar{\mu}}, \bar{\mu}_t)]\, dt\right] \\ &\quad = \mathbb{E}\left[\int_0^T \mathbb{1}_{B_t} [f(t, X_t^\mu, \bar{\mu}_t) - f(t, X_t^{\bar{\mu}}, \bar{\mu}_t)]\, dt\right] \\ &\quad = \mathbb{E}\left[\int_0^T [f(t, X_t^\mu, \bar{\mu}_t) - f(t, X_t^\mu \wedge X_t^{\bar{\mu}}, \bar{\mu}_t)]\, dt\right], \end{aligned}$$

as well as

$$\mathbb{E}[g(X_T^\mu \vee X_T^{\bar{\mu}}, \bar{\mu}_T) - g(X_T^{\bar{\mu}}, \bar{\mu}_T)] = \mathbb{E}[g(X_T^\mu, \bar{\mu}_T) - g(X_T^\mu \wedge X_T^{\bar{\mu}}, \bar{\mu}_T)].$$

In the same way, by the definition of $\alpha^\vee$ and $\alpha^\wedge$, we see that

$$\begin{aligned}&\mathbb{E}\Big[\int_0^T [l(t, X_t^\mu \vee X_t^{\bar\mu}, \alpha_t^\vee) - l(t, X_t^{\bar\mu}, \alpha_t^{\bar\mu})]\,dt\Big]\\ &\quad = \mathbb{E}\Big[\int_0^T \mathbb{1}_{B_t}[l(t, X_t^\mu, \alpha_t^\mu) - l(t, X_t^{\bar\mu}, \alpha_t^\wedge)]\,dt\Big]\\ &\quad = \mathbb{E}\Big[\int_0^T [l(t, X_t^\mu, \alpha_t^\mu) - l(t, X_t^\mu \wedge X_t^{\bar\mu}, \alpha_t^\wedge)]\,dt\Big].\end{aligned}$$

Now, the latter three equalities allow us to rewrite (2.19) as

$$\begin{aligned}0 &\le J(\alpha^\vee, \bar\mu) - J(\alpha^{\bar\mu}, \bar\mu)\\ &= \mathbb{E}\Big[\int_0^T [f(t, X_t^\mu, \bar\mu_t) - f(t, X_t^\mu \wedge X_t^{\bar\mu}, \bar\mu_t)]\,dt\Big]\\ &\quad + \mathbb{E}\Big[\int_0^T [l(t, X_t^\mu, \alpha_t^\mu) - l(t, X_t^\mu \wedge X_t^{\bar\mu}, \alpha_t^\wedge)]\,dt\Big]\\ &\quad + \mathbb{E}[g(X_T^\mu, \bar\mu_T) - g(X_T^\mu \wedge X_T^{\bar\mu}, \bar\mu_T)],\end{aligned} \tag{2.20}$$

which reads as

$$J(\alpha^\vee, \bar\mu) - J(\alpha^{\bar\mu}, \bar\mu) = J(\alpha^\mu, \bar\mu) - J(\alpha^\wedge, \bar\mu). \tag{2.21}$$

Finally, exploiting Assumption 2.9 in the expectations in (2.20), we deduce that

$$0 \le J(\alpha^\vee, \bar\mu) - J(\alpha^{\bar\mu}, \bar\mu) \tag{2.22}$$

$$\begin{aligned}&\le \mathbb{E}\Big[\int_0^T [f(t, X_t^\mu, \mu_t) - f(t, X_t^\mu \wedge X_t^{\bar\mu}, \mu_t)]\,dt\Big]\\ &\quad + \mathbb{E}\Big[\int_0^T [l(t, X_t^\mu, \alpha_t^\mu) - l(t, X_t^\mu \wedge X_t^{\bar\mu}, \alpha_t^\wedge)]\,dt\Big]\\ &\quad + \mathbb{E}[g(X_T^\mu, \mu_T) - g(X_T^\mu \wedge X_T^{\bar\mu}, \mu_T)]\end{aligned}$$

$$= J(\alpha^\mu, \mu) - J(\alpha^\wedge, \mu). \tag{2.23}$$

Hence the control $\alpha^\wedge$ is a minimizer for $J(\cdot, \mu)$, and, by uniqueness of the minimizer, we conclude that $X^\mu \wedge X^{\bar\mu} = X^\mu$; that is, $X_t^\mu \le X_t^{\bar\mu}$ for each $t \in [0, T]$ $\mathbb{P}$-a.s., which implies that $R(\mu) \le^L R(\bar\mu)$. □

REMARK 2.13. For later use, we point out that we have actually proved that for $\bar\mu, \mu \in L$ such that $\mu \le^L \bar\mu$ we have that $X_t^\mu \le X_t^{\bar\mu}$ for each $t \in [0, T]$, $\mathbb{P}$-a.s.

2.5. *Existence and approximation of MFG solutions*. We finally obtain an existence result for the mean field game solutions.

THEOREM 2.14. *Under Assumptions 2.2 and 2.9, the set of MFG solutions $(\mathcal{M}, \le^L)$ is a nonempty complete lattice: in particular there exist a minimal and a maximal MFG solution.*

PROOF. Combining Lemma 2.7 together with Lemma 2.12, we have that the best response map R is an increasing map from the complete lattice $(L, \le^L)$ into itself. The statement then follows from Tarski's fixed point theorem (see Theorem 1 in [29]). □

Following [31], we introduce *learning procedures* $\{\underline{\mu}^n\}_{n\in\mathbb{N}}, \{\overline{\mu}^n\}_{n\in\mathbb{N}} \subset L$ for the mean field game problem as follows:

- $\underline{\mu}^0 := \inf L$, $\overline{\mu}^0 := \sup L$;
- $\underline{\mu}^{n+1} = R(\underline{\mu}^n)$, $\overline{\mu}^{n+1} = R(\overline{\mu}^n)$ for each $n \geq 1$.

For simplicity, we make the following assumption. A discussion on the role of these conditions is postponed to Remark 2.19 below.

ASSUMPTION 2.15.

1. The control set $U \subset \mathbb{R}$ is compact and there exists some $p > 1$ such that $\mathbb{E}[|\xi|^p] < \infty$.
2. The dynamics of the system given by $b(t,x,a) = c_t + p_t x + q_t a$, where c_t, p_t and q_t are deterministic and continuous in t. The volatility σ is constant.
3. For $\mathbb{P} \otimes dt$-a.a. (ω, t) in $\Omega \times [0,T]$, the cost functions $f(t,\cdot,\cdot)$, g are continuous in (x,μ), and the cost function $l(t,\cdot,\cdot)$ is convex and lower semicontinuous in (x,a).
4. f, l and g have subpolynomial growth; that is, there exists a constant $C > 0$ such that for all $(\omega,t,x,a,\mu) \in \Omega \times [0,T] \times \mathbb{R} \times U \times [\mu^{\mathrm{Min}}, \mu^{\mathrm{Max}}]$,

$$\big|f(t,x,\mu)\big| + \big|l(t,x,a)\big| + \big|g(x,\mu)\big| \leq C\big(1 + |x|^p\big).$$

REMARK 2.16. Under Assumption 2.15 it can be easily verified that for each admissible control α the map $t \mapsto \mathbb{P} \circ (X_t^\alpha)^{-1}$ is continuous in the weak topology.

We then have the following convergence result.

THEOREM 2.17. *Under Assumptions 2.2, 2.9 and 2.15 we have*:

(i) *The sequence $\{\underline{\mu}^n\}_{n\in\mathbb{N}}$ is increasing in $(L, \leq^L)$ and it converges weakly to the minimal MFG solution, π-a.e.*
(ii) *The sequence $\{\overline{\mu}^n\}_{n\in\mathbb{N}}$ is decreasing in $(L, \leq^L)$ and it converges weakly to the maximal MFG solution, π-a.e.*

PROOF. We only prove the first claim, since the second follows by analogous arguments. By Lemma 2.12 the sequence $\{\underline{\mu}^n\}_{n\in\mathbb{N}}$ is clearly increasing. Moreover, the completeness of the lattice L allows us to define $\underline{\mu}^*$ as the least upper bound in the lattice $(L, \leq^L)$ of $\{\underline{\mu}^n\}_{n\in\mathbb{N}}$, and, by Remark A.5 in Appendix A, the sequence $\underline{\mu}^n$ converges weakly to $\underline{\mu}^*$ π-a.e.

Define now, for each $n \geq 1$, the optimal pairs $(X^n, \alpha^n) := (X^{\underline{\mu}^{n-1}}, \alpha^{\underline{\mu}^{n-1}})$. Since the controls α^n take values in the compact set U, the processes X^n are pathwise equicontinuous and equibounded. Moreover, by Remark 2.13, the sequence $(X^n)_{n\in\mathbb{N}}$ is increasing. Therefore, by Arzelà–Ascoli's theorem, we can find an adapted process X such that X^n converges uniformly on $[0,T]$ to X, $\mathbb{P}$-a.s.

We now prove that $\underline{\mu}^*$ is a MFG solution. Since $\underline{\mu}_t^n = R(\underline{\mu}^{n-1})_t = \mathbb{P} \circ (X_t^{\underline{\mu}^{n-1}})^{-1} = \mathbb{P} \circ (X_t^n)^{-1}$ and since X^n converges uniformly to X $\mathbb{P}$-a.s. and $\underline{\mu}_t^n$ converges weakly to $\underline{\mu}_t^*$ for π-a.a. $t \in [0,T]$, we deduce that $\underline{\mu}_t^* = \mathbb{P} \circ X_t^{-1}$ for π-a.a. $t \in [0,T]$. Hence, by the continuity of the map $t \mapsto \mathbb{P} \circ X_t^{-1}$ in the weak topology (see Remark 2.16), we can take $\mathbb{P} \circ X^{-1}$ as a continuous version of μ^*; that is, $\underline{\mu}_t^* = \mathbb{P} \circ X_t^{-1}$ for each $t \in [0,T]$. It remains to find an admissible control α such that $X = X^\alpha$ and (X,α) is the optimal pair for $\underline{\mu}^*$.

In order to do so, thanks to the compactness of U, we invoke the Banach–Saks theorem to find a subsequence of indexes $(n_j)_{j\in\mathbb{N}}$ such that the Cesàro means of (α^{n_j}) converge in L^2 to a process α. Up to a subsequence, we can assume that the convergence of the Cesàro means to the process α is pointwise; that is,

$$\beta_t^m := \frac{1}{m}\sum_{j=1}^m \alpha_t^{n_j} \to \alpha_t \quad \text{as } m \to \infty, \mathbb{P} \otimes dt\text{-a.e.} \tag{2.24}$$

Moreover, observe that, by Assumption 2.15(2), we have $X^{\beta^m} = \frac{1}{m}\sum_{j=1}^m X^{n_j}$. Hence, because we already know that X^{n_j} converges to X uniformly in $[0,T]$, $\mathbb{P}$-a.s. as $n_j \to \infty$, we deduce that X^{β^m} converges uniformly to X $\mathbb{P}$-a.s. as $m \to \infty$, and that

$$X_t = \xi + \int_0^t (c_s + p_s X_s + q_s \alpha_s)\, ds + \sigma W_t, \quad \forall t \in [0,T], \mathbb{P}\text{-a.s.};$$

that is, the process X is the solution to the dynamics controlled by α. Furthermore, by the subpolynomial growth of the costs, we have $-\infty < J(\alpha, \underline{\mu}^*)$.

We now prove that the pair (X, α) is optimal for the flow $\underline{\mu}^*$. Observe that, for each admissible ζ and each $n_j \geq 1$, by the optimality of the pair (X^{n_j}, α^{n_j}) for the flow $\underline{\mu}^{n_j-1}$, we have

$$J(\alpha^{n_j}, \underline{\mu}^{n_j-1}) \leq J(\zeta, \underline{\mu}^{n_j-1}).$$

Summing over $j \leq m$, we write

$$\frac{1}{m}\sum_{j=1}^m \mathbb{E}\Big[\int_0^T [f(t, X_t^{n_j}, \underline{\mu}_t^{n_j-1}) + l(t, X_t^{n_j}, \alpha_t^{n_j})]\, dt + g(X_T^{n_j}, \underline{\mu}_T^{n_j-1})\Big] \leq \frac{1}{m}\sum_{j=1}^m J(\zeta, \underline{\mu}^{n_j-1}),$$

which, by convexity of l, in turn implies that

$$\begin{aligned} &\mathbb{E}\Big[\int_0^T l(t, X_t^{\beta^m}, \beta_t^m)\, dt\Big] + \frac{1}{m}\sum_{j=1}^m \mathbb{E}\Big[\int_0^T f(t, X_t^{n_j}, \underline{\mu}_t^{n_j-1})\, dt + g(X_T^{n_j}, \underline{\mu}_T^{n_j-1})\Big] \\ &\quad \leq \frac{1}{m}\sum_{j=1}^m J(\zeta, \underline{\mu}^{n_j-1}). \end{aligned} \tag{2.25}$$

By the compactness of U and the subpolynomial growth of l, the sequence $l(t, X_t^{\beta^m}, \beta_t^m)$ is clearly uniformly integrable with respect to the measure $\mathbb{P} \otimes dt$. Moreover, by the convergence of X^{β^m} and β^m, thanks to the lower semi-continuity of l, we obtain the pointwise limit

$$l(t, X_t, \alpha_t) \leq \liminf_m l(t, X_t^{\beta^m}, \beta_t^m), \quad \mathbb{P} \otimes dt\text{-a.e.}$$

Therefore, we can take limits as $m \to \infty$ in the first expectation in (2.25) to find that

$$\mathbb{E}\Big[\int_0^T l(t, X_t, \alpha_t)\, dt\Big] \leq \liminf_m \mathbb{E}\Big[\int_0^T l(t, X_t^{\beta^m}, \beta_t^m)\, dt\Big]. \tag{2.26}$$

Furthermore, by the convergence of X^n and of $\underline{\mu}^n$ and the continuity of the costs f and g, we can use the subpolynomial growth of f and g and the boundedness of the sequence μ^n (cf. Remark 2.6) to deduce that

$$\begin{aligned} &\mathbb{E}\Big[\int_0^T f(t, X_t, \underline{\mu}_t^*)\, dt + g(X_T, \underline{\mu}_T^*)\Big] \\ &\quad = \lim_m \frac{1}{m}\sum_{j=1}^m \mathbb{E}\Big[\int_0^T f(t, X_t^{n_j}, \underline{\mu}_t^{n_j-1})\, dt + g(X_T^{n_j}, \underline{\mu}_T^{n_j-1})\Big] \end{aligned} \tag{2.27}$$

and that

$$J(\zeta, \underline{\mu}^*) = \lim_m \frac{1}{m}\sum_{j=1}^m J(\zeta, \underline{\mu}^{n_j-1}). \tag{2.28}$$

Finally, using (2.26), (2.27) and (2.28) in (2.25), we conclude that $J(\alpha, \underline{\mu}^*) \leq J(\zeta, \underline{\mu}^*)$, which, in turn, proves the optimality of (X, α) for $\underline{\mu}^*$, by arbitrariness of ζ. Hence, $\underline{\mu}^*$ is a MFG solution.

It only remains to prove the minimality of $\underline{\mu}^*$. Suppose that $\nu^* \in L$ is another MFG solution. By definition, $\inf L = \underline{\mu}^0 \leq^L \nu^*$. Since R is increasing, we have $\underline{\mu}^1 = R(\underline{\mu}^0) \leq^L R(\nu^*) = \nu^*$ and, by induction, we conclude that $\underline{\mu}^n \leq^L \nu^*$ for each $n \in \mathbb{N}$. This implies that the same inequality holds for the least upper bound of $\{\underline{\mu}^n\}_{n\in\mathbb{N}}$; that is, $\underline{\mu}^* \leq^L \nu^*$, which completes the proof of the claim. $\square$

2.6. *Remarks and examples.* In this subsection we collect some remarks and some examples concerning the previous theorems.

REMARK 2.18. In light of Theorem 2.17, a natural question is whether the minimal (resp. maximal) MFG solution is associated to the minimal expected cost. In fact, this relation does not hold in general (see Example 3 below). Nevertheless, it is easy to see that whenever $f(t, x, \cdot)$ and $g(x, \cdot)$ are increasing (resp. decreasing) in μ for each $(t, x) \in [0, T] \times \mathbb{R}$, the minimal (resp. maximal) solution leads to the minimal expected cost and can be approximated via the learning procedure above.

REMARK 2.19 (On Assumption 2.15). We point out that the linear-convex structure required in conditions (2) and (3) of Assumption 2.15 is crucial for our proof of Theorem 2.17. Indeed, the linear-convex structure is employed, together with a Banach–Sacks compactification argument, in order to characterize the limit points of the learning procedure as MFG solutions. In the next section, we extend Theorem 2.17 to a nonconvex setting, by employing a weak formulation of the problem (see also Remark 3.6). Clearly, also the continuity of the costs f and g in the measure μ plays an essential role in the proof of Theorem 2.17. Alternatively, one could require the continuity of f and g with respect to a Wasserstein distance (see Remark 2.5).

On the other hand, conditions (1) and (4) can be replaced by the growth condition (2.6) (when $p' \geq 2$), unless to slightly extend some of the arguments. Also, if the a priori estimate (2.7) is satisfied, one can see that the continuity of f and g in the weak topology can be replaced by the continuity in the p-Wasserstein distance, where $p' > p \geq 1$ are as in Remark 2.4.

REMARK 2.20 (On the initialization of the learning procedure). Theorem 2.17 assumes a more concrete meaning observing that, according to Remark 2.8, the initial conditions of the learning procedure can be written in terms of the data of the problem. In particular, if U is compact, if b is bounded or if the growth condition (2.6) is satisfied (see also Remark 2.19), (2.12) and (2.13) provides an explicit expression for $\inf L$ and $\sup L$, respectively.

Moreover, let μ be a generic flow of probabilities, which is not necessarily an element of L. Define the sequence $\mu^0 := \mu$ and $\mu^{n+1} := R(\mu^n)$ for $n \in \mathbb{N}$. Following the proof of Theorem 2.17 we see that, if $\mu^0 \leq^L R(\mu^0) = \mu^1$ (resp. $\mu^0 \geq^L R(\mu^0) = \mu^1$), then the sequence $\{\mu^n\}_{n\in\mathbb{N}}$ is increasing (resp. decreasing) in $(L, \leq^L)$ and it converges to a MFG equilibrium. In other words, if the learning procedure of Theorem 2.17 starts from an arbitrary element, then it converges to a MFG equilibrium whenever the first and the second element of the sequence are comparable. In particular, in order to approximate the minimal (resp. the maximal) MFG equilibrium, it is sufficient to start the learning procedure from a generic flow of measures μ^0 such that $\mu^0 \leq^L \inf L$ (resp. $\geq^L \sup L$).

EXAMPLE 3. We discuss here the setting studied in [15] in order to draw a connection between the solutions selected therein and our maximal and minimal solutions. Consider the case $U=\mathbb{R}$, $\xi=0$, $b(t,x,a)=cx+a$, $c\in\mathbb{R}$, σ constant, $f(t,x,\mu)=0$, $l(t,x,a)=(x^2+a^2)/2$ and $g(x,\mu)=(x+\varphi(\langle \mathrm{id},\mu\rangle))^2/2$. Here φ is defined as

$$\varphi(y):=-\frac{y}{r_\delta}\mathbb{1}_{\{|y|\le r_\delta\}}-\operatorname{sign}(y)\mathbb{1}_{\{|y|>r_\delta\}},\quad y\in\mathbb{R},\delta\in(0,T),r_\delta:=\int_\delta^T w_s^{-2}\,ds,$$

with $w_t:=\exp[\int_t^T(-c+\eta_s)\,ds]$, η solution to the Riccati equation $\frac{d\eta_t}{dt}=\eta_t^2-2c\eta_t-1$, $\eta_T=1$.

By the monotonicity of φ (see also Example 1), we can easily verify that g satisfies the Submodularity Assumption 2.9, while existence and uniqueness of optimal pairs is a consequence of the strict convexity of the costs, and of the linearity of b (we refer to [15] for more datails). Moreover, by the boundedness of φ, we have that $g(x,\mu)\le x^2+1$. Hence, for any flow of measures μ we see that the optimal control α^μ must satisfy

$$\mathbb{E}\biggl[\int_0^T\frac{(\alpha_t^\mu)^2}{2}\,dt\biggr]\le J(\alpha^\mu,\mu)\le J(0,\mu)\le 1+\mathbb{E}\biggl[\int_0^T\frac{(X_t^0)^2}{2}\,dt+\bigl(X_T^0\bigr)^2\biggr]<\infty,$$

where 0 denotes the control constantly equal to zero. From the latter estimate, and a standard use of Grönwall's inequality, we deduce that (2.7) is satisfied with $p'=2$. All the requirements of Theorem 2.14 are then fulfilled. Moreover, the proof of Theorem 2.17 can be easily modified to fit the example under consideration (see also Remark 2.19). Therefore, the set of MFG solutions is a nonempty complete lattice, and the minimal and maximal MFG solutions can be selected by the learning procedure introduced in the previous subsection.

It is shown in [15] that the set of MFG solutions $\mathcal{M}$ has exactly three elements, namely $\mathcal{M}=\{\mu^{-1},\mu^0,\mu^1\}$, satisfying

$$\langle \mathrm{id},\mu_t^A\rangle:=Aw_t\int_0^t w_s^{-2}\,ds\quad\text{for each } t\in[0,T], A\in\{-1,0,1\}.\tag{2.29}$$

Since $w>0$, we immediately see that $\langle \mathrm{id},\mu_t^{-1}\rangle<\langle \mathrm{id},\mu_t^0\rangle<\langle \mathrm{id},\mu_t^1\rangle$ for each $t\in[0,T]$, which can happen only if $\mu^{-1}\le^L\mu^0\le^L\mu^1$. We finally draw a connection between the solutions selected in [15] and our maximal and minimal solutions, recalling from [15] the following facts:

- The equilibrium with minimal cost is μ^0.
- The "zero-noise limit" and the "N-player game limit" select a randomized equilibrium, given by a combination of the maximal and the minimal MFG solution, both with probability 1/2; that is, with law $\frac{1}{2}\delta_{\mu^{-1}}+\frac{1}{2}\delta_{\mu^1}$.

3. Relaxed submodular mean field games.

In this section we aim at allowing for multiple solutions of the individual optimization problem, and at overcoming the linear-convex setting in the convergence result. This comes with the price of pushing the analysis to a more technical level, by working with a *weak formulation* of the problem and with the so-called *relaxed controls*.

3.1. *The relaxed mean field game.* Let b,σ,f,l,g,U be given as in Section 2 (see (2.2) and (2.3)), with the additional assumption that b,f,l,g are deterministic and, for simplicity, that σ is constant. Let $\mathcal{C}$ denote the set of continuous functions on $[0,T]$. In view of a weak formulation of the problem, the initial value of the dynamics will be described through an initial fixed probability distribution $\nu_0\in\mathcal{P}(\mathbb{R})$.

Let Λ denote the set of *deterministic relaxed controls* on $[0,T]\times U$; that is, the set of positive measures λ on $[0,T]\times U$ such that $\lambda([s,t]\times U)=t-s$ for all $s,t\in[0,T]$ with $s<t$.

DEFINITION 2. A 7-tuple $\rho = (\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}, \xi, W, \lambda)$ is said to be an admissible relaxed control if:

1. $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ is a filtered probability space satisfying the usual conditions;
2. ξ is an $\mathcal{F}_0$-measurable $\mathbb{R}$-valued random variable (r.v.) such that $\mathbb{P} \circ \xi^{-1} = \nu_0$;
3. $W = (W_t)_{t\in[0,T]}$ is a standard $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$-Brownian motion;
4. λ is a Λ-valued r.v. defined on Ω such that $\sigma\{\lambda([0,t] \times E) \mid E \in \mathcal{B}(U)\} \subset \mathcal{F}_t$, $\forall t \in [0, T]$.

We denote by $\widetilde{\mathcal{A}}$ the set of admissible relaxed controls.

The set of admissible ordinary controls is naturally included in the set of relaxed controls via the map $\alpha \mapsto \lambda^\alpha(dt, da) := \delta_{\alpha_t}(da)\,dt$. Any admissible relaxed control $\rho = (\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}, \xi, W, \lambda) \in \widetilde{\mathcal{A}}$, on the other hand, can be factorized in that one can find an adapted process $\lambda\colon \Omega \times [0, T] \to \mathcal{P}(U)$ such that $\lambda(dt, da) = \lambda_t(da)\,dt$ $\mathbb{P}$-almost surely.

Furthermore, since b is assumed to satisfy the usual Lipschitz continuity and growth conditions, there exists a unique process $X^\rho : \Omega \times [0, T] \to \mathbb{R}$, solving the system's dynamics equation that now reads as

$$X_t^\rho = \xi + \int_0^t \int_U b(t, X_t^\rho, a)\lambda_t(da)\,dt + \sigma W_t, \quad t \in [0, T]. \tag{3.1}$$

Then, for a measurable flow of probability measures μ, we define the cost functional

$$\widetilde{J}(\rho, \mu) := \mathbb{E}^{\mathbb{P}}\left[\int_0^T \int_U \left[f(t, X_t^\rho, \mu_t) + l(t, X_t^\rho, a)\right]\lambda_t(da)\,dt + g(X_T^\rho, \mu_T)\right], \quad \rho \in \widetilde{\mathcal{A}},$$

and we say that $\rho \in \widetilde{\mathcal{A}}$ is an *optimal relaxed control* for the flow of measures μ if it solves the optimal control problem related to μ; that is, if $-\infty < \widetilde{J}(\rho, \mu) = \inf \widetilde{J}(\cdot, \mu)$.

We now make the following assumptions, which will be employed in the existence result of Theorem 3.5.

ASSUMPTION 3.1.

1. The control space U is compact.
2. The costs $f(t, \cdot, \mu)$, $l(t, \cdot, \cdot)$ and $g(\cdot, \mu)$ are lower semicontinuous in (x, a) for each $(t, \mu) \in [0, T] \times \mathcal{P}(\mathbb{R})$.
3. There exist exponents $p' > p \geq 1$ and a constant $K > 0$ such that $|\nu_0|^{p'} := \int_{\mathbb{R}} |y|^{p'}\,d\nu_0(y) < \infty$ and such that, for all $(t, x, \mu, a) \in [0, T] \times \mathbb{R} \times \mathcal{P}(\mathbb{R}) \times U$,

$$\left|g(x, \mu)\right| \leq K\left(1 + |x|^p + |\mu|^p\right),$$
$$\left|f(t, x, \mu)\right| + \left|l(t, x, a)\right| \leq K\left(1 + |x|^p + |\mu|^p\right),$$

where $|\mu|^p = \int_{\mathbb{R}} |y|^p\,d\mu(y)$.

4. f and g satisfy the Submodularity Assumption 2.9.

REMARK 3.2. Alternatively, as discussed also in Remark 2.4, we can replace (1) in Assumption 3.1 by requiring U to be closed and the growth condition (2.6) to be satisfied.

REMARK 3.3. Under Assumption 3.1, it is well known that for each measurable flow μ, $\arg\min \widetilde{J}(\cdot, \mu)$ is nonempty. This can be proved using the so-called "compactification-method" (see, e.g., [16] and [18], among others). For later use, we now sketch the main argument. Let $(\rho_n)_{n\in\mathbb{N}}$ be a minimizing sequence for $\widetilde{J}(\cdot, \mu)$, with $\rho_n = (\Omega^n, \mathcal{F}^n, \mathbb{F}^n, \mathbb{P}^n, \xi^n, W^n, \lambda^n)$. Then, since U is compact, thanks to the growth conditions on b, the sequence

$\mathbb{P}^n \circ (\xi^n, W^n, \lambda^n, X^{\rho_n})^{-1}$ is tight in $\mathcal{P}(\mathbb{R} \times \mathcal{C} \times \Lambda \times \mathcal{C})$, so that, up to a subsequence, $\mathbb{P}^n \circ (\xi^n, W^n, \lambda^n, X^{\rho_n})^{-1}$ converges weakly to a probability measure $\bar{\mathbb{P}} \in \mathcal{P}(\mathbb{R} \times \mathcal{C} \times \Lambda \times \mathcal{C})$. Moreover, through a Skorokhod representation argument, we can find an admissible relaxed control $\rho_* = (\Omega_*, \mathcal{F}_*, \mathbb{F}_*, \mathbb{P}_*, \xi_*, W_*, \lambda_*)$ such that $\bar{\mathbb{P}} = \mathbb{P}_* \circ (\xi_*, W_*, \lambda_*, X^{\rho_*})^{-1}$. Finally, the continuity assumptions on the costs together with their polynomial growth, allows us to conclude that

$$\widetilde{J}(\rho_*, \mu) \leq \liminf_n \widetilde{J}(\rho_n, \mu) = \inf \widetilde{J}(\cdot, \mu);$$

that is, $\rho_* \in \arg\min \widetilde{J}(\cdot, \mu)$. In particular, this argument shows that for any sequence $(\rho_n)_{n\in\mathbb{N}} \subset \arg\min \widetilde{J}(\cdot, \mu)$ we can find an admissible relaxed control $\rho_* = (\Omega_*, \mathcal{F}_*, \mathbb{F}_*, \mathbb{P}_*, \xi_*, W_*, \lambda_*) \in \arg\min \widetilde{J}(\cdot, \mu)$ such that, up to a subsequence, $\mathbb{P}^n \circ (X^{\rho_n})^{-1}$ converges weakly to $\mathbb{P}_* \circ (X^{\rho_*})^{-1}$ in $\mathcal{P}(\mathcal{C})$.

The compactness of U and (2.2) immediately imply that there exists a constant $M > 0$ such that,

$$\mathbb{E}^{\mathbb{P}}\big[|X_t^\rho|^{p'}\big] \leq M, \quad \forall t \in [0, T], \rho \in \widetilde{\mathcal{A}}.$$

Hence, Lemma A.2 in the Appendix A allows us to find $\mu^{\mathrm{Min}}, \mu^{\mathrm{Max}} \in \mathcal{P}(\mathbb{R})$ with

$$\mu^{\mathrm{Min}} \leq^{\mathrm{st}} \mathbb{P} \circ \big(X_t^\rho\big)^{-1} \leq^{\mathrm{st}} \mu^{\mathrm{Max}}, \quad \forall t \in [0, T], \rho \in \widetilde{\mathcal{A}}.$$

Moreover, as it is shown in Remark 2.6, we have uniform boundedness of the moments

$$\sup_{\mu \in [\mu^{\mathrm{Min}}, \mu^{\mathrm{Max}}]} |\mu|^q < \infty, \quad \forall q < p'. \tag{3.2}$$

Next, define the set of feasible flows of measures L as the set of all equivalence classes (w.r.t. $\pi := \delta_0 + dt + \delta_T$) of measurable flows $(\mu_t)_{t\in[0,T]}$ with $\mu_t \in [\mu^{\mathrm{Min}}, \mu^{\mathrm{Max}}]$ for π-almost all $t \in (0, T]$ and $\mu_0 = \nu_0$. Let 2^L be the set of all subset of L, and define the best-response-correspondence $\mathcal{R} : L \to 2^L$ by

$$\mathcal{R}(\mu) := \Big\{\mathbb{P} \circ \big(X^\rho\big)^{-1} \mid \rho \in \arg\min \widetilde{J}(\cdot, \mu)\Big\} \subset L, \quad \mu \in L. \tag{3.3}$$

We can then give the following definition.

DEFINITION 3. The flow of measures μ^* is a relaxed mean field game solution if $\mu^* \in \mathcal{R}(\mu^*)$.

3.2. *Existence and approximation of relaxed MFG solutions.* We now move on to proving the existence and approximation of relaxed mean field game solutions. In order to keep a self-contained but concise analysis, the proofs of the subsequent results will be only sketched whenever their arguments follow along the same lines of those employed in the proofs of Section 2.

LEMMA 3.4. *Under Assumption 3.1, the best-response-correspondence satisfies the following*:

(i) *For all* $\mu \in L$, *we have that* $\inf \mathcal{R}(\mu), \sup \mathcal{R}(\mu) \in \mathcal{R}(\mu)$.
(ii) $\inf \mathcal{R}(\mu) \leq^L \inf \mathcal{R}(\overline{\mu})$ *and* $\sup \mathcal{R}(\mu) \leq^L \sup \mathcal{R}(\overline{\mu})$ *for all* $\mu, \overline{\mu} \in L$ *with* $\mu \leq^L \overline{\mu}$.

PROOF. We prove the two claims separately.

Proof of (i). Take $\mu \in L$. In order to show that $\inf \mathcal{R}(\mu) \in \mathcal{R}(\mu)$, we recall that, as it is shown in the proof of Lemma A.4 in the Appendix A, we can select a sequence of relaxed controls $(\rho_n)_{n\in\mathbb{N}} \subset \arg\min \widetilde{J}(\cdot,\mu)$ such that $\inf\{\mathbb{P}^n \circ X^{\rho_n} \mid n \in \mathbb{N}\} = \inf \mathcal{R}(\mu)$.

Without loss of generality, we can assume that the relaxed controls ρ_n are defined on the same stochastic basis; that is, $\rho^n = (\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}, \xi, W, \lambda^n)$ for each $n \in \mathbb{N}$. Indeed, we can choose

$$\Omega := \mathbb{R} \times \mathcal{C} \times \Lambda^{\mathbb{N}}$$

as sample space and take ξ, W, λ^n, $n \in \mathbb{N}$, as the canonical projections. Let $\hat{\mathcal{F}}$ be the Borel σ-algebra on Ω (w.r.t. the product topology), and let $\hat{\mathbb{F}}$ be the natural filtration induced by ξ, W, λ^n, $n \in \mathbb{N}$; that is, $\hat{\mathbb{F}}_t := \sigma(\xi, W(s), \lambda^n(C) : s \in [0,t], C \in \mathcal{B}([0,t] \times U), n \in \mathbb{N})$, $t \in [0,T]$. Thus, W corresponds to a continuous real-valued $\hat{\mathbb{F}}$-adapted process, while λ^n can be identified with a $\mathcal{P}(U)$-valued $\hat{\mathbb{F}}$-predictable process (see, for instance, Lemma 3.2 in [22]). Recall that ν_0 denotes the common initial distribution. Let γ denote standard Wiener measure on $\mathcal{B}(\mathcal{C})$. If $\bar{\rho}^n = (\Omega_n, \mathcal{F}_n, \mathbb{F}^n, \mathbb{P}_n, \xi^n, W^n, \bar{\lambda}^n)$, $n \in \mathbb{N}$, are stochastic relaxed controls with $\mathbb{P}_n \circ (\xi^n)^{-1} = \nu_0$, hence $\mathbb{P}_n \circ (\xi^n, W^n)^{-1} = \nu_0 \otimes \gamma$, then let Q_n denote the Markov kernel from $\mathbb{R} \times \mathcal{C}$ to Λ that corresponds to (a version of) the regular conditional distribution of $\bar{\lambda}^n$ given (ξ^n, W^n). Let $\mathbb{P}$ be the probability measure on $\hat{\mathcal{F}}$ determined by

$$\mathbb{P}\biggl(\{\xi \in B_0\} \cap \{W \in B\} \cap \bigcap_{i\in I}\{\lambda^i \in C_i\}\biggr) := \int_{B_0 \times B}\biggl(\prod_{i\in I} Q_i(x, w; C_i)\biggr)\nu_0 \otimes \gamma(dx, dw)$$

for any choice of $B_0 \in \mathcal{B}(\mathbb{R})$, $B \in \mathcal{B}(\mathcal{C})$, $I \subset \mathbb{N}$ a finite subset, and $C_i \in \mathcal{B}(\Lambda)$, $i \in I$. Then $\mathbb{P} \circ (\xi, W, \lambda^n)^{-1} = \mathbb{P}_n \circ (\xi^n, W^n, \bar{\lambda}^n)^{-1}$ for all $n \in \mathbb{N}$. As a last step, define $\mathcal{F}$ to be the $\mathbb{P}$-completion of $\hat{\mathcal{F}}$, and let $\mathbb{F}$ be the right-continuous $\mathbb{P}$-augmentation of $\hat{\mathbb{F}}$.

We will now employ an inductive scheme. Let ρ^1, ρ^2 be the first two elements of the sequence $(\rho_n)_{n\in\mathbb{N}}$. As in Lemma 2.11, we can define two Λ-valued r.v.'s $\lambda^\vee$ and $\lambda^\wedge$ and two admissible relaxed controls $\rho^\vee = (\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}, \xi, W, \lambda^\vee)$ and $\rho^\wedge = (\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}, \xi, W, \lambda^\wedge)$ such that $X^{\rho_1} \vee X^{\rho_2} = X^{\rho^\vee}$ and $X^{\rho_1} \wedge X^{\rho_2} = X^{\rho^\wedge}$. In fact, set

$$\lambda_s^\wedge := \begin{cases} \lambda_s^2 & \text{on } \{X_s^{\rho_1} > X_s^{\rho_2}\} \\ & \qquad \cup \biggl\{X_s^{\rho_1} = X_s^{\rho_2}, \int_U b(s, X_s^{\rho_1}, a)\lambda_s^1(da) \geq \int_U b(s, X_s^{\rho_2}, a)\lambda_s^2(da)\biggr\}, \\ \lambda_s^1 & \text{on } \{X_s^{\rho_1} < X_s^{\rho_2}\} \\ & \qquad \cup \biggl\{X_s^{\rho_1} = X_s^{\rho_2}, \int_U b(s, X_s^{\rho_1}, a)\lambda_s^1(da) < \int_U b(s, X_s^{\rho_2}, a)\lambda_s^2(da)\biggr\}, \end{cases}$$

where $\lambda^1(ds, da) = \lambda_s^1(da)\,ds$, $\lambda^2(ds, da) = \lambda_s^2(da)\,ds$, and $\lambda^\wedge(ds, da) := \lambda_s^\wedge(da)\,ds$. The definition of $\lambda^\vee$ is analogous. Repeating the same arguments which lead to (2.21) in the proof of Lemma 2.12, we see that

$$0 \leq \widetilde{J}(\rho^\vee, \mu) - \widetilde{J}(\rho_1, \mu) = \widetilde{J}(\rho_2, \mu) - \widetilde{J}(\rho^\wedge, \mu) = 0,$$

which implies that $\mathbb{P} \circ (X^{\rho^\wedge})^{-1} = \mathbb{P} \circ (X^{\rho_1} \wedge X^{\rho_2})^{-1} \in \mathcal{R}(\mu)$. Moreover, since $X^{\rho_1} \wedge X^{\rho_2} = X^{\rho^\wedge}$, we obviously have $\mathbb{P} \circ (X^{\rho^\wedge})^{-1} \leq^L \mathbb{P} \circ (X^{\rho^1})^{-1} \wedge^L \mathbb{P} \circ (X^{\rho^2})^{-1}$. Repeating this construction inductively, for each $n \in \mathbb{N}$ we find an admissible relaxed control $\rho^{\wedge n} = (\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}, \xi, W, \lambda^{\wedge n})$ such that $\mathbb{P} \circ (X^{\rho^{\wedge n}})^{-1} \in \mathcal{R}(\mu)$ and $\mathbb{P} \circ (X^{\rho^{\wedge n}})^{-1} \leq^L \mathbb{P} \circ (X^{\rho^1} \wedge^L \cdots \wedge^L \mathbb{P} \circ X^{\rho^n})^{-1}$. Furthermore, the sequence $\mathbb{P} \circ (X^{\rho^{\wedge n}})^{-1}$ is decreasing in L, since for each n we have $X_t^{\rho^{\wedge n}} = X_t^1 \wedge \cdots \wedge X_t^n \leq X_t^1 \wedge \cdots \wedge X_t^{n-1}$ for each $t \in [0,T]$ $\mathbb{P}$-a.s. Hence,

$$\inf \mathcal{R}(\mu) = \inf\{\mathbb{P} \circ (X^{\rho_n})^{-1} \mid n \in \mathbb{N}\} = \inf\{\mathbb{P} \circ (X^{\rho^{\wedge n}})^{-1} \mid n \in \mathbb{N}\},$$

which implies that the sequence $\mathbb{P}\circ(X^{\rho^{\wedge n}})^{-1}$ converges weakly to $\inf\mathcal{R}(\mu)$, π-a.e. Since $(\mathbb{P}\circ(X^{\rho^{\wedge n}})^{-1})_{n\in\mathbb{N}}\subset\mathcal{R}(\mu)$, by the closure property of $\mathcal{R}(\mu)$ (see Remark 3.3), we conclude that $\inf\mathcal{R}(\mu)\in\mathcal{R}(\mu)$.

Analogously, it can be shown that $\sup\mathcal{R}(\mu)\in\mathcal{R}(\mu)$.

Proof of (ii). Let $\mu,\bar{\mu}\in L$ with $\mu\leq^L\bar{\mu}$ and $\rho,\bar{\rho}\in\widetilde{\mathcal{A}}$ with $\rho\in\arg\min\widetilde{J}(\cdot,\mu)$ and $\bar{\rho}\in\arg\min\widetilde{J}(\cdot,\bar{\mu})$. As in the proof of claim (i), we may assume that ρ and $\bar{\rho}$ are defined on the same stochastic basis; that is, $\rho=(\Omega,\mathcal{F},\mathbb{F},\mathbb{P},\xi,W,\lambda)$ and $\rho=(\Omega,\mathcal{F},\mathbb{F},\mathbb{P},\xi,W,\bar{\lambda})$. As above, we can then define two Λ-valued r.v.'s $\lambda^\vee$ and $\lambda^\wedge$ and two admissible relaxed controls $\rho^\vee=(\Omega,\mathcal{F},\mathbb{F},\mathbb{P},\xi,W,\lambda^\vee)$ and $\rho^\wedge=(\Omega,\mathcal{F},\mathbb{F},\mathbb{P},\xi,W,\lambda^\wedge)$ such that $X^\rho\vee X^{\bar{\rho}}=X^{\rho^\vee}$ and $X^\rho\wedge X^{\bar{\rho}}=X^{\rho^\wedge}$.

Repeating the arguments which lead to (2.22) in the proof of Lemma 2.12, we exploit the submodularity of the costs and the definitions of $\lambda^\vee$ and $\lambda^\wedge$ to find

$$0\leq\widetilde{J}(\rho^\vee,\bar{\mu})-\widetilde{J}(\bar{\rho},\bar{\mu})\leq\widetilde{J}(\rho^\vee,\mu)-\widetilde{J}(\bar{\rho},\mu)=\widetilde{J}(\rho,\mu)-\widetilde{J}(\rho^\wedge,\mu)\leq 0, \tag{3.4}$$

where the first and the last inequality hold because of the optimality of ρ and $\bar{\rho}$.

By claim (i), we have that $\sup\mathcal{R}(\mu)\in\mathcal{R}(\mu)$ and $\sup\mathcal{R}(\bar{\mu})\in\mathcal{R}(\bar{\mu})$, therefore, we can choose ρ and $\bar{\rho}$ such that $\mathbb{P}\circ(X^{\bar{\rho}})^{-1}=\sup\mathcal{R}(\bar{\mu})$ and $\mathbb{P}\circ(X^{\rho})^{-1}=\sup\mathcal{R}(\mu)$. From (3.4) we see that $\rho^\vee\in\arg\min\widetilde{J}(\cdot,\bar{\mu})$, which implies that $\mathbb{P}\circ(X^{\rho^\vee})^{-1}\leq^L\sup\mathcal{R}(\bar{\mu})$. This, by construction of $\rho^\vee$, in turn implies that

$$\sup\mathcal{R}(\mu)=\mathbb{P}\circ(X^\rho)^{-1}\leq^L\mathbb{P}\circ(X^\rho)^{-1}\vee^L\mathbb{P}\circ(X^\rho)^{-1}\leq^L\mathbb{P}\circ(X^{\rho^\vee})^{-1}\leq^L\sup\mathcal{R}(\bar{\mu});$$

that is, $\sup\mathcal{R}(\mu)\leq^L\sup\mathcal{R}(\bar{\mu})$. In the same way, choosing ρ and $\bar{\rho}$ such that $\mathbb{P}\circ(X^{\bar{\rho}})^{-1}=\inf\mathcal{R}(\bar{\mu})$ and $\mathbb{P}\circ(X^{\rho})^{-1}=\inf\mathcal{R}(\mu)$ we conclude that $\inf\mathcal{R}(\mu)\leq^L\inf\mathcal{R}(\bar{\mu})$. $\square$

THEOREM 3.5. *Under Assumption 3.1, we have that*:

(i) *The set of mean field game solutions $\mathcal{M}$ is nonempty and admits a minimal and a maximal element.*

Assume moreover that the costs $f(t,\cdot,\cdot)$ and $g(\cdot,\cdot)$ are continuous in (x,μ). Then:

(ii) *For $\underline{\mu}^0:=\inf L$ and $\underline{\mu}^n:=\inf\mathcal{R}(\underline{\mu}^{n-1})$ for $n\in\mathbb{N}$, we have that the learning procedure $(\underline{\mu}^n)_{n\in\mathbb{N}}$ is increasing and it converges weakly to* $\inf\mathcal{M}$, π*-a.e.*

(iii) *For $\overline{\mu}^0:=\sup L$ and $\overline{\mu}^n:=\sup\mathcal{R}(\overline{\mu}^{n-1})$ for $n\in\mathbb{N}$, we have that the learning procedure $(\overline{\mu}^n)_{n\in\mathbb{N}}$ is decreasing and it converges weakly to* $\sup\mathcal{M}$, π*-a.e.*

PROOF. Claim (i) follows from Lemma 3.4 and Theorem 4.1 in [33].

We only prove (ii), since the proof of (iii) is similar. By Lemma 3.4 the sequence $(\underline{\mu}^n)_{n\in\mathbb{N}}$ is increasing, hence it converges weakly to its least upper bound $\underline{\mu}_*$, π-a.e. For each $n\in\mathbb{N}$, let $\rho^n=(\Omega^n,\mathcal{F}^n,\mathbb{F}^n,\mathbb{P}^n,\xi^n,W^n,\lambda^n)$ be an admissible relaxed control such that $\mathbb{P}^n\circ(X^{\rho^n})^{-1}=\inf\mathcal{R}(\underline{\mu}^{n-1})$. As in Remark 3.3, the sequence $(\mathbb{P}\circ(\xi^n,W^n,\lambda^n,X^{\rho^n})^{-1})_{n\in\mathbb{N}}$ is tight, so that, up to a subsequence, we can assume that the sequence $\mathbb{P}^n\circ(\xi,W,\lambda^n,X^{\rho^n})^{-1}$ converges weakly to a probability measure $\bar{\mathbb{P}}\in\mathcal{P}(\mathbb{R}\times\mathcal{C}\times\Lambda\times\mathcal{C})$. Moreover, we can find an admissible relaxed control $\rho_*=(\Omega_*,\mathcal{F}_*,\mathbb{F}_*,\mathbb{P}_*,\xi_*,W_*,\lambda_*)$ such that $\bar{\mathbb{P}}=\mathbb{P}_*\circ(\xi_*,W_*,\lambda_*,X^{\rho_*})^{-1}$, and this implies that $\mu_*=\mathbb{P}_*\circ(X^{\rho_*})^{-1}$.

By the optimality of ρ^n for the flow of measures $\underline{\mu}^{n-1}$, we have

$$\widetilde{J}(\rho^n,\underline{\mu}^{n-1})\leq\widetilde{J}(\rho,\underline{\mu}^{n-1}),\quad\forall\rho\in\widetilde{\mathcal{A}}. \tag{3.5}$$

Now, the continuity of the costs f, l and g, together with their polynomial growth and the uniform integrability condition (3.2), allow us to show the continuity of the functional $\widetilde{J}$ along the sequences $(\rho^n,\underline{\mu}^{n-1})_{n\in\mathbb{N}}$ and $(\rho,\underline{\mu}^{n-1})_{n\in\mathbb{N}}$. This in turn enables us to take limits as $n \to \infty$ in (3.5) and to deduce that $\widetilde{J}(\rho_*, \underline{\mu}_*) \leq \widetilde{J}(\rho, \underline{\mu}_*)$ for each $\rho \in \widetilde{\mathcal{A}}$. Hence, X^{ρ_*} is an optimal trajectory for the flow $\underline{\mu}_*$ and, since $\underline{\mu}_* = \mathbb{P}_* \circ (X^{\rho_*})^{-1}$, we have $\underline{\mu}_* \in \mathcal{R}(\underline{\mu}_*)$; that is, $\underline{\mu}_*$ is a mean field game solution.

It remains to show that $\underline{\mu}_* = \inf \mathcal{M}$. Let $\nu \in \mathcal{M}$. By definition, we have $\underline{\mu}^0 = \inf L \leq^L \nu$. Since $\inf \mathcal{R}$ is increasing by (ii) in Lemma 3.4, $\underline{\mu}^1 = \inf \mathcal{R}(\underline{\mu}^0) \leq^L \inf \mathcal{R}(\nu) \leq^L \nu$, where the last inequality follows from $\nu \in \mathcal{R}(\nu)$. By induction, we deduce that $\underline{\mu}^n \leq^L \nu$ for each $n \in \mathbb{N}$. Recalling that $\underline{\mu}_* = \sup\{\underline{\mu}^n \mid n \in \mathbb{N}\}$, we conclude that $\underline{\mu}_* \leq^L \nu$, which completes the proof. □

REMARK 3.6. Notice that the role of the compactification through the problem's weak formulation and the use of relaxed controls is twofold. On the one hand, it ensures that the sets of best responses $\mathcal{R}(\cdot)$ admit minimal and maximal elements, which is essential for our arguments in the case in which $\mathcal{R}(\cdot)$ are not singletons. On the other hand, regarding the convergence of the learning procedure, it replaces the compactification via Banach–Saks' theorem used in the proof of Theorem 2.17, for which the additional linear-convex structure (enforced in Assumption 2.15) is necessary (see also Remark 2.19).

4. Concluding remarks and further extensions. In the following, we provide some comments on our assumptions and further extensions of the techniques elaborated in the previous sections.

4.1. *On the multidimensional case.* Our approach can be extended only to some particular multidimensional cases. Indeed, although the first order stochastic dominance induces a lattice structure on $\mathcal{P}(\mathbb{R})$, it does not induce a lattice order on $\mathcal{P}(\mathbb{R}^d)$ for $d > 1$ (cf. [21] and [26]). Also, Lemma 2.11 does not hold, in general, for multidimensional settings, as the following counterexample shows.

EXAMPLE 4. Consider a two-dimensional Brownian motion $W = (W^1, W^2)$. For any $\mathbb{R}$-valued integrable progressively measurable process α, let $X^\alpha = (X^{1,\alpha}, X^{2,\alpha})$ be the solution to

$$X_t^{1,\alpha} = \int_0^t \alpha_s \, ds + W_t^1, \qquad X_t^{2,\alpha} = -\int_0^t \alpha_s \, ds + W_t^2.$$

Taking a positive α, we see that $X^{1,\alpha} \vee X^{1,-\alpha} = X^{1,\alpha}$, while $X^{2,\alpha} \vee X^{2,-\alpha} = X^{2,-\alpha}$. This means that the first component of $X^\alpha \vee X^{-\alpha}$ should be controlled by α, while the second component should be controlled by $-\alpha$. Therefore, $X^\beta \neq X^\alpha \vee X^{-\alpha}$ for any control β.

Nevertheless, the results in this paper can be extended to suitable multidimensional settings where the actual dependence on the measure is only through one of its one-dimensional marginals, and Lemma 2.11 and Proposition 2.12 hold.

For example, take $d > 1$ and a d-dimensional Brownian motion $W = (W^1, \dots, W^d)$, on a complete filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$. Consider closed sets $U^i \subset \mathbb{R}$, $i = 1, \dots, d$. Admissible controls are d-dimensional square integrable progressively measurable processes $\alpha = (\alpha^1, \dots, \alpha^d)$ taking values in $U^1 \times \cdots \times U^d$. Take measurable functions

$$b^i, l^i : \quad \Omega \times [0, T] \times \mathbb{R} \times U^i \to \mathbb{R}, \quad i = 1, \dots, d,$$

$$f : \quad \Omega \times [0, T] \times \mathbb{R}^d \times \mathcal{P}(\mathbb{R}) \to \mathbb{R}, \qquad g : \quad \Omega \times \mathbb{R}^d \times \mathcal{P}(\mathbb{R}) \to \mathbb{R},$$

and a d-dimensional $\mathcal{F}_0$-measurable square integrable random variable $\xi = (\xi^1, \dots, \xi^d)$. For each admissible control α, let the process $X^\alpha = (X^{1,\alpha}, \dots, X^{d,\alpha})$ denote the solution to the system

$$dX_t^{i,\alpha} = b^i(t, X_t^{i,\alpha}, \alpha_t^i)\,dt + dW_t^i, \quad t \in [0,T], X_0^{i,\alpha} = \xi^i, i = 1, \dots, d.$$

Next, for any given measurable flow $\mu = (\mu_t)_{t\in[0,T]}$ of probability measures on $\mathcal{B}(\mathbb{R})$, we consider the cost functional

$$J(\alpha, \mu) := \mathbb{E}\left[\int_0^T \left[f(t, X_t^\alpha, \mu_t) + \sum_{i=1}^d l^i(t, X_t^{i,\alpha}, \alpha_t^i)\right] dt + g(X_T^\alpha, \mu_T)\right].$$

We enforce an analogous of Assumption 2.2; that is, we assume that for each flow μ the exists a unique optimal pair (X^μ, α^μ) with X^μ satisfying some tightness condition uniformly in μ.

Notice that we assume that the minimization problem depends on a measure on $\mathbb{R}$, not on $\mathbb{R}^d$. For example, the problem can depend only on one fixed marginal, say the first. In this spirit, a MFG solution is a measurable flow $\mu^* = (\mu_t^*)_{t\in[0,T]}$ of probabilities such that

$$\mu_t^* = \mathbb{P} \circ (X_t^{1,\mu^*})^{-1} \quad \text{for each } t \in [0,T].$$

Now, since the components of X^α are decoupled, we easily see that Lemma 2.11 can be recovered. However, in order to deal with the multidimensional setting, we need to enforce a stronger version of Assumption 2.9.

ASSUMPTION 4.1. For $\mathbb{P} \otimes dt$ a.a. $(\omega, t) \in \Omega \times [0,T]$, for $\phi \in \{f(t,\cdot,\cdot), g\}$, we have

$$\phi(\bar{x} \vee x, \mu) - \phi(\bar{x}, \mu) \le \phi(x, \mu) - \phi(\bar{x} \wedge x, \mu),$$

for all $\bar{x}, x \in \mathbb{R}^d$ and $\mu \in \mathcal{P}(\mathbb{R})$, and

$$\phi(\bar{x}, \bar{\mu}) - \phi(x, \bar{\mu}) \le \phi(\bar{x}, \mu) - \phi(x, \mu),$$

for all $\bar{x}, x \in \mathbb{R}^d$ and $\bar{\mu}, \mu \in \mathcal{P}(\mathbb{R})$ s.t. $\bar{x} \ge x$ and $\bar{\mu} \ge^{\mathrm{st}} \mu$.

By the additive structure of the running cost involving the controls, using Assumption 4.1 we can adapt the proof of Proposition 2.12 to prove that the best reply map is increasing. Therefore, for this particular set up, the arguments of Section 2 can be recovered, and Theorems 2.14 and (by making an analogous of Assumption 2.15) 2.17 can be extended.

4.2. *On linear-quadratic MFG.* Assumption 2.9 is fulfilled in the linear-quadratic case

$$b(t,x,a) = c_t + p_t x + q_t a,$$
$$f(t,x,\mu) + l(t,x,a) = \frac{1}{2} n_t a^2 + \frac{1}{2}(m_t x + \widehat{m}_t \langle \mathrm{id}, \mu\rangle)^2,$$
$$g(x,\mu) = \frac{1}{2}(h_t x + \widehat{h}_t \langle \mathrm{id}, \mu\rangle)^2,$$

where $\mathrm{id}(y) = y$, and for deterministic continuous functions $c_t, p_t, q_t, n_t, m_t, \widehat{m}_t, h_t$ and $\widehat{h}_t$ such that $\inf_{t\in[0,T]} q_t > 0$, $\inf_{t\in[0,T]} n_t > 0$, $n_t \widehat{m}_t \le 0$ and $h_t \widehat{h}_t \le 0$ for each $t \in [0,T]$.

However, the tightness condition (2) in Assumption 2.2 is not satisfied unless we consider a compact control set U. In fact, when U is not compact, there is a counterexample in Section 7 of [22], which shows that a mean field game solution may not exist.

Nevertheless, our approach allows us to treat nonstandard linear-quadratic mean field games as, for example, the one considered in Section 2.2 in [15] (see also [5] and [10]).

4.3. *On a geometric dynamics.* Our results still hold true if we replace (2.1) with a dynamics of the geometric form

$$dX_t = b(t, X_t, \alpha_t)X_t\,dt + \sigma_t X_t\,dW_t, \quad t \in [0, T], X_0 = \xi, \tag{4.1}$$

for some square-integrable positive r.v. ξ, a bounded drift b and a bounded stochastic process σ. Indeed, for each square-integrable process α there exists a unique strong solution X^α to the latter SDE, and classical estimates show that there exists a constant $M > 0$ such that

$$\sup_{t\in[0,T]} \mathbb{E}\big[|X_t^\alpha|^2\big] \le M;$$

hence, the tightness condition in Assumption 2.2 is satisfied. Moreover, the solution to (4.1) can be represented as

$$X_t^\alpha = \xi \exp\biggl(\int_0^t \Bigl(b\bigl(s, X_s^\alpha, \alpha_s\bigr) - \frac{1}{2}\sigma_s^2\Bigr)\,ds + \int_0^t \sigma_s\,dW_s\biggr), \quad t \in [0, T],$$

and the mapping $x \mapsto \exp(x)$ is monotone. Hence, since ξ is positive, for any couple of admissible controls $\alpha, \bar{\alpha}$, we have that for each $t \in [0, T]$ $\mathbb{P}$-a.s.

$$X_t^{\bar{\alpha}} \ge X_t^\alpha \quad \text{if and only if} \quad \int_0^t b\bigl(s, X_s^{\bar{\alpha}}, \bar{\alpha}_s\bigr)\,ds \ge \int_0^t b\bigl(s, X_s^\alpha, \alpha_s\bigr)\,ds.$$

The latter property allows us to repeat all the arguments employed in the proof of Lemma 2.11 and (mutatis mutandis) to carry on the analysis that lead to the existence results of Theorems 2.14 and 3.5.

4.4. *On mean field dependent dynamics.* For a suitable choice of the costs f, g and l, Theorem 2.14 still holds if we have a "sufficiently simple" mean field dependence in the dynamics of the system. For the sake of illustration, we discuss here two examples.

Let U be a compact subset of $\mathbb{R}$. For any admissible process α and any measurable flow of probability measures μ, consider a state process given by

$$dX_t = X_t\bigl(\alpha_t + m(\mu_t)\bigr)\,dt + \sigma X_t\,dW_t, \quad t \in [0, T], X_0 = \xi, \tag{4.2}$$

where ξ is a positive square-integrable r.v. and $m\colon \mathcal{P}(\mathbb{R}) \to \mathbb{R}$ is a bounded function which is measurable with respect to the Borel σ-algebra associated to the topology of weak convergence of probability measures. Assume moreover that m is increasing with respect to the first order stochastic dominance.

Notice that, for each measurable flow μ and for each admissible α, the SDE (4.2) admits the explicit solution

$$X_t^{\alpha,\mu} = E_t(\alpha)M_t(\mu), \tag{4.3}$$

where

$$E_t(\alpha) := \xi \exp\biggl(\int_0^t \Bigl(\alpha_s - \frac{\sigma^2}{2}\Bigr)\,ds + \sigma W_t\biggr) \quad \text{and} \quad M_t(\mu) := \exp\biggl(\int_0^t m(\mu_s)\,ds\biggr).$$

Since U is compact and m is bounded, we can find a constant $K > 0$ which is independent of μ, such that

$$\sup_{t\in[0,T]} \mathbb{E}\big[|X_t^{\alpha,\mu}|^2\big] \le K.$$

The latter implies the tightness condition in Assumption 2.2. As in Section 2.2, this allows us to define a set L of feasible flows of measures, and to show that $(L, \le^L)$ is a complete lattice.

Given $\mu \in L$ and two admissible controls α and $\bar{\alpha}$, as in Lemma 2.11 we can construct $\alpha^{\vee}$ and $\alpha^{\wedge}$ such that $X_t^{\alpha,\mu} \vee X_t^{\bar{\alpha},\mu} = X_t^{\alpha^{\vee},\mu}$ and $X_t^{\alpha,\mu} \wedge X_t^{\bar{\alpha},\mu} = X_t^{\alpha^{\wedge},\mu}$. Moreover, due to the particular structure of (4.2), the construction of $\alpha^{\vee}$ and $\alpha^{\wedge}$ does not depend on μ.

Consider now cost functions $l(t,x,a) = a^2/2$ and $f(t,x,\mu) = x\psi(\mu)$, for a measurable function $\psi : \mathcal{P}(\mathbb{R}) \to \mathbb{R}_-$ which is decreasing w.r.t. the first order stochastic dominance. With such a choice of the costs, the functional J is strictly convex w.r.t. α. Hence, for each $\mu \in L$, there exists a unique minimizer α of $J(\cdot,\mu)$ (see, e.g., Theorem 5.2 in [36]). We then have the following result.

LEMMA 4.2. *The best-response-map $R : L \to L$ is increasing.*

PROOF. Take $\mu, \bar{\mu} \in L$ with $\mu \leq^L \bar{\mu}$. Let $\alpha \in \arg\min J(\cdot,\mu)$ and $\bar{\alpha} \in \arg\min J(\cdot,\bar{\mu})$. Similar to Lemma 2.12, we first see that

$$0 \geq J(\alpha,\mu) - J(\alpha^{\wedge},\mu) = J(\alpha^{\vee},\mu) - J(\bar{\alpha},\mu). \tag{4.4}$$

We also observe that, exploiting (4.3), the monotonicity of m and the fact that ψ is negative and decreasing, one has

$$\begin{aligned}(X_t^{\alpha^{\vee},\mu} - X_t^{\bar{\alpha},\mu})\psi(\mu_t) &= (E_t(\alpha^{\vee}) - E_t(\bar{\alpha}))M_t(\mu)\psi(\mu_t) \\ &\geq (E_t(\alpha^{\vee}) - E_t(\bar{\alpha}))M_t(\bar{\mu})\psi(\bar{\mu}_t) = (X_t^{\alpha^{\vee},\bar{\mu}} - X_t^{\bar{\alpha},\bar{\mu}})\psi(\bar{\mu}_t).\end{aligned} \tag{4.5}$$

Thus, combining (4.4) and (4.5), we obtain

$$\begin{aligned}0 \geq J(\alpha^{\vee},\mu) - J(\bar{\alpha},\mu) &= \mathbb{E}\left[\int_0^T \left(\frac{(\alpha_t^{\vee})^2}{2} - \frac{\bar{\alpha}_t^2}{2} + (X_t^{\alpha^{\vee},\mu} - X_t^{\bar{\alpha},\mu})\psi(\mu_t)\right) dt\right] \\ &\geq \mathbb{E}\left[\int_0^T \left(\frac{(\alpha_t^{\vee})^2}{2} - \frac{\bar{\alpha}_t^2}{2} + (X_t^{\alpha^{\vee},\bar{\mu}} - X_t^{\bar{\alpha},\bar{\mu}})\psi(\bar{\mu}_t)\right) dt\right] = J(\alpha^{\vee},\bar{\mu}) - J(\bar{\alpha},\bar{\mu}).\end{aligned}$$

Hence $\alpha^{\vee} \in \arg\min J(\cdot,\bar{\mu})$, which, by uniqueness, implies that $\alpha^{\vee} = \bar{\alpha}$. This in turn implies that $E_t(\alpha^{\vee}) = E_t(\alpha) \vee E_t(\bar{\alpha}) = E_t(\bar{\alpha})$. Hence, $E_t(\alpha) \leq E_t(\bar{\alpha})$ and, by monotonicity of m, we find $X_t^{\alpha,\mu} = E_t(\alpha)M_t(\mu) \leq E_t(\bar{\alpha})M_t(\bar{\mu}) = X_t^{\bar{\alpha},\bar{\mu}}$ and $R(\mu) = \mathbb{P} \circ (X_t^{\alpha,\mu})^{-1} \leq^L \mathbb{P} \circ (X_t^{\bar{\alpha},\bar{\mu}})^{-1} = R(\bar{\mu})$, which completes the proof. □

Thanks to Lemma 4.2, we can invoke Tarski's fixed point theorem in order to deduce that the set of mean field game equilibria is a nonempty and complete lattice.

REMARK 4.3. Statements analogous to the previous ones still hold if we consider a controlled Ornstein–Uhlenbeck process with mean field term in the dynamics; that is, if the state process is given by

$$dX_t = (\kappa X_t + \alpha_t + m(\mu_t))\,dt + \sigma\,dW_t, \quad t \in [0,T], X_0 = \xi \text{ with } \kappa \in \mathbb{R} \text{ and } \sigma \geq 0, \tag{4.6}$$

for a measurable bounded increasing function $m : \mathcal{P}(\mathbb{R}) \to \mathbb{R}$.

4.5. *On a class of MFGs with common noise.* Our approach allows us also to treat a class of submodular mean field games with common noise, in which the representative player interacts with the population through the conditional mean of its state given the common noise. We refer to the recent works [11, 15] and [30] for a related set up. In the following, we provide the main ingredients of the setting and we show that the set of solutions to the considered class of MFGs with common noise is a nonempty complete lattice.

Let $(W_t)_{t\in[0,T]}$ and $(B_t)_{t\in[0,T]}$ be two independent Brownian motions on a complete filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$. Let $\xi \in L^2(\Omega, \mathcal{F}_0, \mathbb{P})$, $\sigma \geq 0$ and $\sigma_0 > 0$. For each $\alpha \in \mathcal{A}$ (see the beginning of Section 2.1), consider a dynamics of the system given by

$$dX_t = b(t, X_t, \alpha_t)\, dt + \sigma\, dW_t + \sigma_0\, dB_t, \quad t \in [0, T], X_0 = \xi, \tag{4.7}$$

for some measurable function b satisfying the requirements in (2.2). Here, the Brownian motion B stands for the common noise, while W represents the idiosyncratic noises affecting the state processes in the pre-limit N-player game.

Notice that, if we assume the control set U to be compact, then, by a standard use of Grönwall inequality, we can find a constant $C > 0$ such that the solution X^α to the SDE (4.7) satisfies ($\mathbb{P}$-a.s.) the estimate

$$\left|X_t^\alpha\right| \leq C\Big(1 + |\xi| + \sigma \sup_{s\in[0,t]} |W_s| + \sigma_0 \sup_{s\in[0,t]} |B_s|\Big) =: Y_t \quad \text{for all } t \in [0, T] \text{ and } \alpha \in \mathcal{A}.$$

Moreover, notice that the process $(Y_t)_{t\in[0,T]}$ belongs to $L^2(\Omega \times [0, T])$.

Let $\mathbb{F}^B = (\mathcal{F}_t^B)$ be the natural filtration generated by B augmented by all $\mathbb{P}$-null sets, and define L^B to be the set of all real-valued $\mathbb{F}^B$-progressively measurable processes $m = (m_t)_{t\in[0,T]}$ such that $|m_t| \leq Y_t$ $\mathbb{P}$-a.s., for each $t \in [0, T]$. Then, for any given process $m \in L^B$, consider the optimization problem $\inf J(\cdot, m)$, with J defined by

$$J(\alpha, m) := \mathbb{E}\bigg[\int_0^T \big[f\big(t, X_t^\alpha, m_t\big) + l\big(t, X_t^\alpha, \alpha_t\big)\big]\, dt + g\big(X_T^\alpha, m_T\big)\bigg], \quad \alpha \in \mathcal{A},$$

for appropriately measurable functions $f : \Omega \times [0, T] \times \mathbb{R}^2 \to \mathbb{R}$, $l : \Omega \times [0, T] \times \mathbb{R} \times U \to \mathbb{R}$ and $g : \Omega \times \mathbb{R}^2 \to \mathbb{R}$. Notice that f and g are now functions of the process m, which represents the conditional mean of the population given the common noise B.

ASSUMPTION 4.4.

1. The control space U is compact.
2. For each process $m \in L^B$, there exists a unique optimal pair (X^m, α^m).
3. For $\mathbb{P} \otimes dt$ a.a. $(\omega, t) \in \Omega \times [0, T]$, the functions $f(t, \cdot, \cdot)$ and g have decreasing differences in (x, y); that is, for $\phi \in \{f(t, \cdot, \cdot), g\}$,

$$\phi(\bar{x}, \bar{y}) - \phi(x, \bar{y}) \leq \phi(\bar{x}, y) - \phi(x, y),$$

for all $\bar{x}, x, \bar{y}, y \in \mathbb{R}$ s.t. $\bar{x} \geq x$ and $\bar{y} \geq y$.

Next, introduce the map $R : L^B \to L^B$ defined by

$$R(m)_t := \mathbb{E}\big[X_t^m \mid \mathcal{F}_T^B\big] \quad \text{for } t \in [0, T].$$

Notice that $R(m)$ is $\mathbb{F}^B$-adapted (see Remark 1 in [30]) and continuous in t, and therefore $\mathbb{F}^B$-progressively measurable.

DEFINITION 4. A process $m^* \in L^B$ is a strong MFG solution to the MFG with common noise if

$$m_t^* = \mathbb{E}\big[X_t^{m^*} \mid \mathcal{F}_T^B\big] \quad \text{for each } t \in [0, T].$$

Consider on L^B the order relation given by $m \leq \overline{m}$ if and only if $m_t \leq \overline{m}_t$ $\mathbb{P} \otimes dt$-a.e. Since L^B is a bounded subset of the Dedekind complete lattice $L^2(\Omega \times [0, T])$, it is a complete lattice. Moreover, as in Remark 2.13, for $\overline{m}, m \in L^B$ with $m \leq \overline{m}$ we have that $X_t^m \leq X_t^{\overline{m}}$ for each $t \in [0, T]$, $\mathbb{P}$-a.s., and hence

$$R(m)_t = \mathbb{E}\big[X_t^m \mid \mathcal{F}_T^B\big] \leq \mathbb{E}\big[X_t^{\overline{m}} \mid \mathcal{F}_T^B\big] = R(\overline{m})_t, \quad \mathbb{P} \otimes dt\text{-a.e.},$$

which implies that $R : L^B \to L^B$ is increasing. Once more, using Tarski's fixed point theorem, we have proved the following result.

THEOREM 4.5. *Under Assumption 4.4, the set of strong solutions of the MFG with common noise is a nonempty complete lattice.*

REMARK 4.6. We point out that Theorem 4.5 guarantees existence of a strong solution to the MFG; that is, a solution which is adapted to the common noise. As a matter of fact, results of existence of strong solutions are still relatively limited in the literature on MFGs with common noise, and they are usually proved through uniqueness results (see, e.g., Section 6 in [11]), in the spirit of the Yamada–Watanabe theory for weak and strong solutions to standard SDEs.

REMARK 4.7. Notice that the crucial step in order to obtain Theorem 4.5 is the inequality $X_t^m \leq X_t^{\overline{m}}$, for each $t \in [0, T]$, whenever $m \leq \overline{m}$. Following the arguments developed in Section 4.4 for MFG without common noise, a similar relation can be established also in the case of mean field dependent dynamics as in (4.2) or (4.6) with an additional common noise term $\sigma_0 \, dB_t$. Note that the latter mean-reverting dynamics is exactly the one considered in [15] and [30].

APPENDIX A: SOME RESULTS ON FIRST ORDER STOCHASTIC DOMINANCE

In this section, we derive some technical results concerning the first order stochastic dominance introduced in Section 2.2. As in Section 2.2, we identify the set of probability measures $\mathcal{P}(\mathbb{R})$ with the set of distribution functions on $\mathbb{R}$, setting $\mu(s) := \mu(-\infty, s]$ for each $s \in \mathbb{R}$ and $\mu \in \mathcal{P}(\mathbb{R})$. On $\mathcal{P}(\mathbb{R})$ we then consider the lattice ordering of first order stochastic dominance given by (2.8) and (2.9). In the following remark, we collect some fundamental observations that are crucial for the analysis in this section.

REMARK A.1.

(a) Notice that by identifying μ by its distribution function, $\mathcal{P}(\mathbb{R})$ coincides with the set of all nondecreasing right continuous functions $F : \mathbb{R} \to [0, 1]$ with $\lim_{s \to -\infty} F(s) = 0$ and $\lim_{s \to \infty} F(s) = 1$. Moreover, we would like to recall that the weak topology is metrizable and that the weak convergence coincides with the pointwise convergence of distribution functions at every continuity point, that is, $\mu_n \to \mu$ if and only if

$$\mu_n(s) \to \mu(s) \quad \text{as } n \to \infty \text{ for every continuity point } s \in \mathbb{R} \text{ of } \mu.$$

Therefore, the weak convergence behaves well with the pointwise lattice operations $\vee^{\text{st}}$ and $\wedge^{\text{st}}$. In particular, the maps $(\mu, \nu) \mapsto \mu \vee^{\text{st}} \nu$ and $(\mu, \nu) \mapsto \mu \wedge^{\text{st}} \nu$ are continuous $\mathcal{P}(\mathbb{R}) \times \mathcal{P}(\mathbb{R}) \to \mathcal{P}(\mathbb{R})$.

(b) Recall that a nondecreasing function $\mathbb{R} \to \mathbb{R}$ is right continuous if and only if it is upper semicontinuous (usc). Hence, for a sequence $(\mu^n)_{n \in \mathbb{N}} \in \mathcal{P}(\mathbb{R})$ which is bounded above, the supremum $\sup_{n \in \mathbb{N}} \mu^n$ is exactly the pointwise infimum of the distribution functions $(\mu^n)_{n \in \mathbb{N}}$.

(c) For a nondecreasing function $F : \mathbb{R} \to \mathbb{R}$, we define its usc-envelope $F^* : \mathbb{R} \to \mathbb{R}$ by

$$F^*(s) := \inf_{\delta > 0} F(s + \delta) \quad \text{for all } s \in \mathbb{R}.$$

Notice that

$$F(s) \leq F^*(s) \leq F(s + \varepsilon) \quad \text{for all } s \in \mathbb{R} \text{ and } \varepsilon > 0. \tag{A.1}$$

Intuitively speaking, F^* is the right continuous version of F. That is, F^* differs from F only at discontinuity points of F. For a sequence $(\mu^n)_{n \in \mathbb{N}} \in \mathcal{P}(\mathbb{R})$ which is bounded below, the infimum $\inf_{n \in \mathbb{N}} \mu^n$ is then given by the usc-envelope of the pointwise supremum of the distribution functions $(\mu^n)_{n\in\mathbb{N}}$. That is, one has to modify the pointwise supremum at all its discontinuity points in order to be right continuous. In fact, let $\underline{\mu} = F^*$ denote the usc-envelope of the pointwise supremum F of $(\mu^n)_{n\in\mathbb{N}}$. By equation (A.1), $\underline{\mu}(s) \le F(s+\varepsilon) \le \underline{\mu}(s+\varepsilon)$ for all $s \in \mathbb{R}$ and $\varepsilon > 0$, that is, $\underline{\mu}$ is nondecreasing and $\underline{\mu} \le^{\mathrm{st}} \mu^n$ for all $n \in \mathbb{N}$. Moreover, by definition, μ is usc, and thus right-continuous. Since $\underline{\mu} \le^{\mathrm{st}} \mu^1$, $\underline{\mu}(s) \ge \mu^1(s) \to 1$ as $s \to \infty$. Let ν be a lower bound of $(\mu^n)_{n\in\mathbb{N}}$. Then, $\underline{\mu}(s) \le F(s+\varepsilon) \le \nu(s+\varepsilon)$ for all $s \in \mathbb{R}$ and $\varepsilon > 0$. Taking the limit $\varepsilon \to 0$, we may conclude that $\underline{\mu}(s) \le \nu(s)$ for all $s \in \mathbb{R}$. In particular, $\underline{\mu}(s) \le \nu(s) \to 0$ as $s \to -\infty$. Altogether, we have shown that $\underline{\mu}$ is a distribution function with $\nu \le^{\mathrm{st}} \underline{\mu} \le^{\mathrm{st}} \mu^n$ for all $n \in \mathbb{N}$ and every lower bound ν of $(\mu^n)_{n\in\mathbb{N}}$.

(d) Combining the previous remarks, leads to the following insight: If $(\mu^n)_{n\in\mathbb{N}} \subset \mathcal{P}(\mathbb{R})$ is a bounded and nondecreasing or nonincreasing sequence, then $(\mu^n)_{n\in\mathbb{N}}$ converges weakly to its supremum or infimum, respectively. In fact, we have seen that the supremum $\bar{\mu}$ of $(\mu^n)_{n\in\mathbb{N}}$ exists, and that its distribution function is given by the pointwise supremum of the sequence of distribution functions of $(\mu^n)_{n\in\mathbb{N}}$. In particular, $\mu^n(s) \to \bar{\mu}(s)$ as $n \to \infty$ for all $s \in \mathbb{R}$. Moreover, it is shown that infimum $\underline{\mu}$ of $(\mu^n)_{n\in\mathbb{N}}$ exists, and its distribution function is given by the usc-envelope of the pointwise supremum of the sequence of distribution functions of $(\mu^n)_{n\in\mathbb{N}}$. Therefore, the distribution function of $\underline{\mu}$ coincides with the pointwise supremum of the sequence of distribution functions of $(\mu^n)_{n\in\mathbb{N}}$ at every continuity point of the distribution function of $\underline{\mu}$. In particular, $\mu^n(s) \to \underline{\mu}(s)$ as $n \to \infty$ for every continuity point $s \in \mathbb{R}$ of the distribution function of $\underline{\mu}$. Since the weak convergence of probability measures is equivalent to the pointwise convergence of the distribution functions at every continuity point of the distribution function of the limit, we obtain that $\mu^n \to \bar{\mu}$ and $\mu^n \to \underline{\mu}$ weakly as $n \to \infty$.

LEMMA A.2. *Let $K \subset \mathcal{P}(\mathbb{R})$ and $\psi\colon [0,\infty) \to [0,\infty)$ be continuous and strictly increasing with $\psi(s) \to \infty$ as $s \to \infty$ and*

$$\sup_{\mu\in M} \int_{\mathbb{R}} \psi\big(|x|\big)\, d\mu(x) < \infty.$$

Then, there exist $\mu^{\mathrm{Min}}, \mu^{\mathrm{Max}} \in \mathcal{P}(\mathbb{R})$ with $\mu^{\mathrm{Min}} \le^{\mathrm{st}} \mu \le^{\mathrm{st}} \mu^{\mathrm{Max}}$ for all $\mu \in K$.

PROOF. We extend ψ to $(-\infty, 0)$ by $\psi(s) := \psi(0)$ for $s < 0$. Moreover, let $C \ge \psi(0)$ with

$$\sup_{\mu\in K} \int_{\mathbb{R}} \psi\big(|x|\big)\, d\mu(x) \le C.$$

Then, we define $\mu^{\mathrm{Min}}, \mu^{\mathrm{Max}}\colon \mathbb{R} \to [0,1]$ by

$$\mu^{\mathrm{Min}}(s) := \frac{C}{\psi(-s)} \wedge 1 \quad \text{and} \quad \mu^{\mathrm{Max}}(s) := \left(1 - \frac{C}{\psi(s)}\right) \vee 0 \tag{A.2}$$

for all $s \in \mathbb{R}$. Since ψ is strictly increasing with $\psi(s) \to \infty$ as $s \to \infty$, $\mu^{\mathrm{Min}}(s) = 1$ for $s \ge -\psi^{-1}(C)$ and $\mu^{\mathrm{Max}} = 0$ for $s \le \psi^{-1}(C)$. In particular, $\lim_{s\to-\infty} \mu^{\mathrm{Min}}(s) = 0$ and $\lim_{s\to\infty} \mu^{\mathrm{Max}}(s) = 1$. Moreover, μ^{Min} and μ^{Min} are nondecreasing and (right) continuous, which shows that $\mu^{\mathrm{Min}}, \mu^{\mathrm{Max}} \in \mathcal{P}(\mathbb{R})$. Now, let $\mu \in K$. Then, recalling that ψ is nondecreasing, one has

$$1 - \mu(s) \le \frac{1}{\psi(s)} \int_s^\infty \psi\big(|x|\big)\, d\mu(x) \le \frac{1}{\psi(s)} \int_{\mathbb{R}} \psi\big(|x|\big)\, d\mu(x) \le \frac{C}{\psi(s)} = 1 - \mu^{\mathrm{Max}}(s)$$

for all $s \in \mathbb{R}$ with $\psi(s) > C$. Since $\mu^{\mathrm{Max}}(s) = 0$ for all $s \in \mathbb{R}$ with $\psi(s) \le C$, it follows that $\mu \le \mu^{\mathrm{Max}}$. On the other hand,

$$\mu(s) \le \frac{1}{\psi(-s)} \int_{-\infty}^s \psi\big(|x|\big)\, d\mu(x) \le \frac{1}{\psi(-s)} \int_{\mathbb{R}} \psi\big(|x|\big)\, d\mu(x) \le \frac{C}{\psi(-s)} = \mu^{\mathrm{Min}}(s)$$

for all $s \in \mathbb{R}$ with $\psi(-s) > C$. Since $\mu^{\mathrm{Min}}(s) = 1$ for all $s \in \mathbb{R}$ with $\psi(-s) \le C$, it follows that $\mu \ge \mu^{\mathrm{Min}}$. $\square$

LEMMA A.3. *Let $K \subset \mathcal{P}(\mathbb{R})$ and $\psi\colon [0,\infty) \to [0,\infty)$ be continuous and strictly increasing with $\psi(s) \to \infty$ as $s \to \infty$ and*

$$\sup_{\mu \in M} \int_{\mathbb{R}} \psi(|x|)\, d\mu(x) < \infty.$$

Further, let μ^{Min} and μ^{Max} be given by (A.2) *and $0 \le \alpha < 1$. Then, the map $x \mapsto \psi(|x|)^\alpha$ is u.i for $[\mu^{\mathrm{Min}}, \mu^{\mathrm{Max}}]$, that is,*

$$\sup_{\mu \in [\mu^{\mathrm{Min}}, \mu^{\mathrm{Max}}]} \int_{\mathbb{R}} 1_{(M,\infty)}\big(|x|\big) \cdot \psi\big(|x|\big)^\alpha\, d\mu(x) \to 0 \quad \textit{as } M \to \infty.$$

PROOF. Let $\beta \in (\alpha, 1)$. Then, by (A.2),

$$\begin{aligned} \psi(s) &= \frac{C}{1 - \mu^{\mathrm{Max}}(s)} \quad \text{for } s \ge \psi^{-1}(C) \quad \text{and} \\ \psi(-s) &= \frac{C}{\mu^{\mathrm{Min}}(s)} \quad \text{for } s \le -\psi^{-1}(C). \end{aligned} \tag{A.3}$$

Recall $\psi^{-1}(C) = \max\{s \in \mathbb{R} \mid (\mu^{\mathrm{Max}})(s) = 0\}$ and $-\psi^{-1}(C) = \min\{s \in \mathbb{R} \mid (\mu^{\mathrm{Min}})(s) = 1\}$. This together with (A.3) implies that

$$\int_0^\infty \psi(s)^\beta\, d\mu^{\mathrm{Max}}(s) = \int_{\psi^{-1}(C)}^\infty \left(\frac{C}{1 - \mu^{\mathrm{Max}}(s)}\right)^\beta d\mu^{\mathrm{Max}}(s) = \int_0^1 \left(\frac{C}{1-u}\right)^\beta du < \infty$$

and

$$\int_{-\infty}^0 \psi(-s)^\beta\, d\mu^{\mathrm{Min}}(s) = \int_{-\infty}^{-\psi^{-1}(C)} \left(\frac{C}{\mu^{\mathrm{Min}}(s)}\right)^\beta d\mu^{\mathrm{Min}}(s) = \int_0^1 \left(\frac{C}{u}\right)^\beta du < \infty,$$

where, in both equalities, we used the transformation lemma. It follows that

$$\sup_{\mu \in [\mu^{\mathrm{Min}}, \mu^{\mathrm{Max}}]} \int_{\mathbb{R}} \psi\big(|x|\big)^\beta\, d\mu(x) \le \int_0^\infty \psi(s)^\beta\, d\mu^{\mathrm{Max}}(s) + \int_{-\infty}^0 \psi(-s)^\beta\, d\mu^{\mathrm{Min}}(s).$$

By the De La Vallée–Poussin Lemma, it follows that $|x| \mapsto \psi(|x|)^\alpha$ is u.i. for $[\mu^{\mathrm{Min}}, \mu^{\mathrm{Max}}]$. In particular, if $\psi(s) \ge s^p$ for some $p \in (0,\infty)$, then, $x \mapsto |x|^q$ is u.i. for $[\mu^{\mathrm{Min}}, \mu^{\mathrm{Max}}]$ for all $q \in (0, p)$. $\square$

We now turn our focus on measurable flows of probability measures. The following proposition is the starting point in order to apply Tarski's fixed point theorem in the proof of the existence of mean field game solutions. We start by building up the setup. Let $\underline{\mu}, \overline{\mu} \in \mathcal{P}(\mathbb{R})$ with $\underline{\mu} \le^{\mathrm{st}} \overline{\mu}$ and $(S, \mathcal{S}, \pi)$ be a finite measure space. We denote by $\mathcal{B}$ the Borel σ-algebra on $\mathcal{P}(\mathbb{R})$ generated by the weak topology. We denote the lattice of all equivalence classes of $\mathcal{S}$-$\mathcal{B}$-measurable functions $S \to [\underline{\mu}, \overline{\mu}]$ by $L = L^0(S, \pi; [\underline{\mu}, \overline{\mu}])$. An arbitrary element μ of L will be denoted in the form $\mu = (\mu_t)_{t \in S}$. On L we consider the order relation $\le^L$ given by $\mu \le^L \nu$ if and only if $\mu_t \le^{\mathrm{st}} \nu_t$ for π-a.a. $t \in S$. The following proposition can be found in a more general form in [27]. However, for the sake of a self-contained exposition, we provide a short proof below.

PROPOSITION A.4. *The lattice L is complete.*

PROOF. Let $M \subset L$ be a nonempty subset of L. Then, for every countable set $\Psi \subset M$, we denote by $\mu^{\Psi} := \sup_{\mu \in \Psi} \mu$. Let Ψ be a countable subset of M, and $(\Psi^n)_{n\in\mathbb{N}}$ be a sequence of finite subsets of Ψ with $\Psi^n \subset \Psi^{n+1}$ for all $n \in \mathbb{N}$ and $\bigcup_{n\in\mathbb{N}} \Psi^n = \Psi$. As Ψ^n is finite, by Remark A.1(b), $\mu^{\Psi^n} \in L$ with $\mu^{\Psi^n} \leq^L \mu^{\Psi^{n+1}}$ for all $n \in \mathbb{N}$. By Remark A.1(d), if follows that $(\mu^{\Psi_n})_{n\in\mathbb{N}}$ converges weakly π-a.e. to μ^{Ψ}. As a consequence, $\mu^{\Psi} \in L$ for every countable set $\Psi \subset M$. Let

$$c := \sup\left\{\int_S \int_{\mathbb{R}} \arctan(x)\, d\mu_t^{\Psi}(x)\, d\pi(t) \,\middle|\, \Psi \subset M \text{ countable}\right\}.$$

Notice that the map $t \mapsto \int_{\mathbb{R}} \arctan(x)\, d\mu_t$ is measurable for every $\mu \in L$ since $\arctan \in C_b(\mathbb{R})$ induces a continuous (w.r.t. to the weak topology) linear functional $\mathcal{P}(\mathbb{R}) \to \mathbb{R}$. By definition of the constant c, there exists a sequence $(\Psi^n)_{n\in\mathbb{N}}$ of countable subsets of M with

$$\int_S \int_{\mathbb{R}} \arctan(x)\, d\mu_t^{\Psi^n}(x)\, d\pi(t) \to c \quad \text{as } n \to \infty.$$

Let $\Psi^* := \bigcup_{n\in\mathbb{N}} \Psi^n$ and $\mu^* := \mu^{\Psi^*}$. We now show that $\mu \leq^L \mu^*$ π-a.s. for all $\mu \in M$. In order to see this, fix some $\mu \in M$ and let $\Psi' := \Psi^* \cup \{\mu\}$. Then, it follows that

$$c = \int_S \int_{\mathbb{R}} \arctan(x)\, d\mu_t^*(x)\, d\pi(t) \leq \int_S \int_{\mathbb{R}} \arctan(x)\, d\mu_t^{\Psi'}(x)\, d\pi(t) \leq c.$$

Since arctan is strictly increasing it follows that $\mu^{\Psi'} = \mu^*$, that is, $\mu \leq^L \mu^*$. Moreover, for any upper bound $\mu \in L$ of M it is easily seen that $\mu^* \leq^L \mu$. Altogether, we have shown that $\mu^* = \sup M$. In a similar way, one shows that M has an infimum. □

REMARK A.5. Let $M \subset L$ be nonempty. Then, we say that M is *directed upwards* or *directed downwards* if for all $\mu, \nu \in M$ there exists some $\eta \in M$ with $\mu \vee \nu \leq^L \eta$ or $\eta \leq^L \mu \wedge \nu$, respectively.

(a) The proof of the previous theorem shows that if M is directed upwards, then there exists a nondecreasing sequence $(\mu^n)_{n\in\mathbb{N}} \subset M$ with $\mu^n \to \sup M$ weakly π-a.e. as $n \to \infty$. The analogous statement holds for the infimum if M is directed downwards. In particular, if $(\mu^n)_{n\in\mathbb{N}}$ is a nondecreasing or nonincreasing sequence in L, then it converges weakly π-a.e. to its least upper bound or greatest lower bound, respectively.

(b) Assume that S is a singleton with $\pi(S) > 0$. Then, the previous remark implies the following: For any nonempty set $K \subset \mathcal{P}(\mathbb{R})$ that is bounded above and directed upwards, its supremum exists and can be weakly approximated by a monotone sequence. An analogous statement holds for the infimum if the set K is bounded below and directed downwards.

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