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Stable and Efficient Allocation of Tasks in Teamwork

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Introduction

Matching theory was born as a branch of graph theory between the end of the XIX and the beginning of XX century in the seminal works of Petersen (1891), König (1916), and their followers. In these early works, the main research questions were the existence of a matching in a given graph, i.e. a subset of edges such that every vertex in the graph is the endpoint of exactly one edge in the subset, or the finding of a matching of maximal weight in a weighted graph such as in Egerváry (1931).[1] A very relevant innovation for the economic applications of the theory was the introduction of the notion of stable matching by Gale and Shapley (1962), which is defined in a context where every vertex of the graph has a preference on the vertices it can be matched with. In this setting, a matching is said to be stable when there is no pair of vertices in which both prefer the other vertex of the pair rather than their respective partners in the matching. In this context, vertices of the graph are usually referred to as agents. Gale and Shapley (1962) proved that there always exists a stable matching when the underlying graph is bipartite (the existence of such a matching is called the Stable Marriage Problem), while there could be none if the underlying graph is complete (the existence of such a matching is called the Stable Roommate Problem).
Matching theory had several economic applications in recent years, the most famous are the organization of labor market for medical interns (Roth (1984),Roth and Peranson (1999)), house allocation (Abdulkadiroğlu and Sönmez (1999)), school choice (Abdulkadiroğlu and Sönmez (2003)) and kidney exchange (Roth et al. (2004)).

Following these illustrious examples, our work focuses on matching models, in a Stable Roommate Problem style, aimed to represent problems of task assignment to pair of agents. We focus on matchings that guarantee Pareto efficiency and stability. The original notion of stability introduced by Gale and Shapley (1962) can not be applied directly in our settings, as it was defined only on preferences over possible partners, without taking in consideration tasks. Therefore, an ad-hoc notion of stability is defined for each model that we discuss.
In every model we present, we show that a stable and Pareto efficient outcome exists for every admissible preference profile. Proofs of these existence results are constructive: we present algorithms whose outcome always respects stability and efficiency, plus other desirable properties that depend on the specific model.

[1] See Lovász and Plummer (2009) for a more detailed discussion about early works and history of matching theory.

In the first chapter, Stable Sharing, we propose a simple model in which agents are matched in pairs in order to complete a task. Here, tasks are assumed to be identical and are represented by a unit value interval of time or effort that is needed to complete them. This value, when a task is assigned to a pair, is split between the two agents. We assume that the agents care only about their share of the task and not about the partner. We assume that the effort or time to complete the task is rewarded proportionally. It is then natural to consider that the preferences of agents are single-peaked and continuous on the amount of time they devote to it. Our model combines features of two models: assignment games (Shapley and Shubik (1971)) and the division problem (Sprumont (1991)). We provide an algorithm (Select-Allocate-Match) that generates a stable and Pareto efficient allocation. We show that stable allocations may fail to exist if either the single-peakedness or the continuity assumption fail.

In the second chapter, Welfare Maximisation in Stable Sharing, we further elaborate on the model proposed in the first chapter to study the utilitarian welfare of the outcomes of the algorithm. Thus, agents are matched in pairs in order to complete a job of unit size. To have a meaningful discussion about the welfare of an allocation, we will restrict the preference domain and consider a case where agents' utility functions have a cardinal interpretation and agents' utility decreases proportionally with the distance from the best option, called the agent's peak. We will show that the Select-Allocate-Match algorithm in this framework can be re-expressed in a more compact form and that it always generate a welfare maximising allocation. We will then discuss about incentive compatibility and present further results for welfare functions different from the utilitarian welfare.

Finally, in the third chapter, Stable and Efficient Task Assignment to Pairs, we study another model in which agents are matched in pairs in order to undertake a task. This model, however, is very different from the models presented in the first two chapters. Here, tasks are indivisible and distinguishable objects taken from a finite set. Agents have preferences over both the partner and the task they are assigned to. Every agent has a set of tasks (possibly empty) that she likes to perform with a potential partner, and this may change depending on the partner. Moreover, preferences over tasks are assumed to be pairwise aligned: the set of tasks that agent i likes to perform with agent j coincides with the set of tasks that agent j would like to perform with agent i. Individual preferences partition the set of partner-task pairs in three indifference classes. The topmost indifference class consists of the pairs in which an agent is matched with a partner and a task they like to perform. The second class contains all the pairs in which the agent is matched with a partner that she likes, but the task assigned to the pair is not in the set of preferred tasks. Finally, the third class contains all pairs in which the agent is matched with someone with whom she has no commonly liked tasks. We also provide the possibility for agents to remain unmatched and receive no task.

We assume that this option is preferred with respect to being in the third indifference class. We characterise a class of algorithms that identify a Pareto efficient and stable assignment and satisfy a pair of additional axioms: Restricted Maskin Monotonicity and Invariance with Respect to Deleted Links.

1 Stable Sharing

Based on work by A. Nicolò, P. Salmaso, A. Sen, S. Yadav.

In this chapter we consider a model where agents are matched in pairs. Each pair has to provide a unitary amount of input or effort in order to generate an output of fixed value. Agents differ in their cost of effort. In particular, each agent's preference over the amount of effort supplied is single-peaked. Several examples come to mind. Consider for instance a joint venture to undertake a risky project with a fixed cost. The inputs are monetary amounts and the share of revenue is proportional to the money invested in the project. Agents are heterogeneous in their degree of risk aversion. Under standard assumptions, they have single-peaked preferences over the amount of money to invest in the venture. Another example is the problem of formation of two-person teams in order to perform tasks, each of which requires a fixed amount of time. Agents receive an hourly wage and have standard quasi-concave preferences on consumption and leisure.

In our model, an allocation consists of a matching that assigns all agents in pairs and a contribution for each agent. The sum of contributions for agents who are paired together by the matching, is one. We impose two standard requirements on allocations - stability and Pareto efficiency. A pair of agents can block an allocation by proposing contributions that sum to one that make them strictly better off than they were in the allocation. An allocation is stable if it cannot be blocked by any pair of agents. Stability is an important requirement in decentralized market design because it can be interpreted as a form of envy-freeness. The motivation for Pareto efficiency is, of course, evident.

Our main result is that a stable and Pareto efficient allocation exists for every profile of single-peaked and continuous preferences. Our existence proof is constructive - we provide an algorithm, the Select-Allocate-Match (SAM) algorithm that identifies a stable and Pareto efficient allocation at every preference profile. Stability fails if either of the assumptions on preferences, single-peakedness and continuity are violated.

The SAM algorithm proceeds by partitioning agents into high type (H) and low type (L) agents depending upon whether their peaks are greater than or less than 0.5. The algorithm relies on the key notion of an *improvement set* which is defined with respect to an agent's contribution. Roughly speaking, the improvement set for an L type agent is the set of her contributions in the interval $[0, 0.5]$ that would make her strictly better-off. The improvement set for an H type agent is the set of contributions of her partner in $[0, 0.5]$ that would make her strictly better-off. An important result that undergirds our algorithm is that an allocation is stable if the intersection of the improvement sets for each (L, H) pair is empty and 0.5 does not belong to the improvement set of any agent.

In an initializing step, an equal pool of H and L agents is created by removing the "excess" agents of one type. This is done by choosing agents whose "equivalent contribution" to 0.5 is closest to 0.5.[1] These agents are matched to each others with each contributing 0.5. In subsequent steps, an agent of one type, called the primary agent, is selected and her contribution chosen. This is done in a manner such that the primary agent does not want to block with any of the agents who have been assigned in a previous step, by considering their improvement sets. This may involve the primary agent being given a contribution equal to her peak. Then an agent of the opposite type, called the secondary agent, is selected based on the equivalent of the chosen contribution. We continue in this fashion until all agents are matched. It is worth emphasizing that the partitioning of agents into H and L types is artificial in the sense that blocking pairs can be formed by agents of the same type. In our procedure, we first *select* a primary agent, *allocate* a contribution to her and *match* her with an appropriate secondary agent. This motivates our use of the SAM term. The SAM algorithm works in polynomial time and its time complexity is $O(n^2)$, where n is the number of the agents.

The apparent simplicity of our model may suggest that more naive approaches to finding stable and Pareto efficient allocations exist. The examples in Section 1.2 show that several natural procedures fail. These include procedures based on rewarding agents on "the short side of the market", "top peak-bottom peak" matching and on the uniform rule (Sprumont (1991)).

The existence of a stable allocation in our problem is far from obvious for several reasons. Unlike the celebrated Shapley-Shubik assignment game (Shapley and Shubik (1971)), we do not have a bipartite structure on the set of agents. Instead, we have a roommate type problem (Gale and Shapley (1962)) where every pair of agents can potentially be matched. It is well-known that non-fractional stable matchings in the roommate model do not exist in general (see Teo and Sethuraman (1998) and Eriksson and Karlander (2001)). There are two additional features of our model that are absent in the Shapley-Shubik model. The first is that the free-disposal assumption is violated in ours. Agents cannot always be made better-off by giving them "more" - making an agent contribute more than her peak results in the agent being worse-off. The second feature is that the assumption of single-peaked

[1] For an L agent, this is the contribution in $[0, 0.5]$ that is indifferent to 0.5. The equivalent for a H type agent can be suitably defined. The notion of an equivalent contribution is used extensively in the algorithm. We are glossing over important details here which can be found in Section 1.3.

preferences implies that our model cannot be represented by a transferable utility game. The underlying non-transferable utility game is hard to analyze using standard techniques because of satiation in preferences, the roommate type structure and the non-convexity of stable allocations (see Section 1.5.5).

Our model is also related to the class of division problems first studied in Sprumont (1991). The uniform rule plays a central role in this setting and has been a characterized in a variety of ways (see Sprumont (1991), Sönmez (1994), Ching (1994), Schummer and Thomson (1997) etc.). However it does not appear to be important for the analysis of stability in our model. We provide an example where there is no agent matching that generates a stable allocation if the uniform rule is used to determine the contribution of the paired agents.[2]

In our model, agents do not have preferences over their partner as in the classical roommate problem. Nicolò et al. (2019) study a model where agents are matched in pairs and have preferences over both the partner and the project they are assigned to. They show that the existence of stable allocations cannot be guaranteed except when specific assumptions are made on an agent's ranking of partners and projects. The assumption that agents only care about the amount of time (or effort) they devote to the task, and not about their partner or the specific portion of the day or the week they work, is indeed a simplification.[3]

An interesting open question is whether the existence of stable allocations in our model can be derived from existing stability results such as Scarf (1967) and Shapley and Vohra (1991). Proving balancedness of the game and dealing with the absence of free-disposal appears to be non-trivial. In any case, we believe that our approach is more direct and illuminating since we provide an algorithm which generates a stable and Pareto efficient allocation.

The rest of the chapter is organized as follows. In Section 1.1 we introduce the model and basic definitions. Section 1.2 contains some illustrative examples. Section 1.3 introduces the concept of improvement sets while Section 1.4 presents the algorithm and the main result. Section 1.5 discusses various aspects of our model and results. The proof is contained in the Appendix.

[2] A more detailed discussion of the literature on stability in division problems can be found in Section 1.5.3.

[3] The model nevertheless captures relevant features of job sharing such as the demand for reduced working time and the need to find a compatible match. The assumption that workers are indifferent towards their matched partner is likely to be satisfied in routine jobs, or jobs in which the contribution of each worker is fully verifiable.

1.1 The Model

Let the set of agents be $N = \{1, \ldots, n\}$, where n is even. Agents have to be assigned in pairs and each pair has to complete a task of unit value. No agent can remain on her own[4] and each agent has only one partner.

An allocation σ is a collection of triples, (i, j, t_i) where $i, j \in N$ and $t_i \in [0, 1]$. We interpret t_i as the contribution of agent i. The contribution of agent i's partner j is $t_j = 1 - t_i$. We refer to (t_i, t_j) as the contribution vector associated with the matched pair (i, j).

We say $(i, j, t_i) \in \sigma$ if the pair (i, j) has the contribution vector (t_i, t_j) in σ. Let Σ denote the set of all feasible allocations.

Each agent i has a preference ordering $\succsim_i$ over her contribution.[5] We assume $\succsim_i$ is *single-peaked* and *continuous*. The ordering $\succsim_i$ is single-peaked if there exists a unique contribution $p_i \in [0, 1]$ such that for all $x, y \in [0, 1]$, if $x < y < p_i$ or $x > y > p_i$ then $y \succ_i x$.[6] The contribution p_i will be referred to as the peak of agent i in $\succsim_i$. A special instance of a single-peaked preference is a *symmetric* or *Euclidean* preference: $x \succsim_i y$ if and only if $|x - p_i| \leq |y - p_i|$. The ordering $\succsim_i$ is continuous if the sets $\{y : y \succsim_i x\}$ and $\{y : x \succsim_i y\}$ are closed for all $x \in [0, 1]$. A preference profile $\succsim$ is an n-tuple of preferences $(\succsim_1, \ldots, \succsim_n)$.

The fundamental property that an allocation should satisfy is *stability.*

Definition **1.** *Let σ be an allocation and $i, j \in N$ be agents with contributions t_i and t_j respectively in σ. Then the pair (i, j) blocks σ if there exists a contribution vector (t'_i, t'_j) with $t'_i + t'_j = 1$, $t'_i \succ_i t_i$ and $t'_j \succ_j t_j$. An allocation is stable if it cannot be blocked by any pair of agents.*

Blocking can occur in two ways. It is possible that i and j are matched together in σ, but can propose an alternative contribution vector which makes both of them better-off.[7] The other possibility is that i and j are not matched together in σ, but can abandon their partners and come together with a contribution vector which makes both better-off.

A more permissive notion of blocking is weak blocking, where only one of the blocking agents is better-off and the other one no worse-off.

Definition **2.** *Let σ be an allocation and $i, j \in N$ be agents with contributions t_i and t_j respectively in σ. Then the pair (i, j) weakly blocks σ if there exists a contribution vector (t'_i, t'_j) with $t'_i + t'_j = 1$, $t'_i \succsim_i t_i$ and $t'_j \succsim_j t_j$ with either $t'_i \succ_i t_i$ or $t'_j \succ_j t_j$. An allocation is strongly stable if it cannot be weakly blocked by any pair of agents.*

[4] For further discussion on this assumption, see Section 5.3.

[5] The asymmetric and symmetric components of $\succsim_i$ are denoted by $\succ_i$ and $\sim_i$ respectively.

[6] The notion of single-peaked preferences is standard - see Mas-Colell et al. (1995). It is used extensively in a variety of contexts such as political economy and axiomatic allocation theory.

[7] See Section 1.6.2 for a detailed discussion about blocking by a pair of agents who are matched together in an allocation.

We are also interested in *Pareto efficient* allocations. Note that we are using the stronger notion of Pareto efficiency.

Definition **3.** *Let σ be an allocation where the contribution of agent i is t_i^σ. The allocation τ Pareto dominates σ at preference profile $\succsim$ if $t_i^\tau \succsim_i t_i^\sigma$ for all $i \in N$ and $t_i^\tau \succ_i t_i^\sigma$ for some $i \in N$. The allocation σ is Pareto efficient at $\succsim$ if there does not exist $\tau \in \Sigma$ that Pareto dominates it.*

Stability and Pareto efficiency are independent properties in our model. Consider a problem with four agents, all of whom have symmetric preferences with their peak at 0.3. An allocation where one agent in each pair receives her peak is Pareto efficient. However, it is not stable because their partners contribute 0.7 and can strictly improve by forming a pair with each contributing 0.5.
To show that stability does not imply Pareto efficiency, consider the problem with four agents, 1, 2, 3 and 4, who have symmetric preferences with peaks 0.1, 0.2, 0.8 and 0.9, respectively. The allocation $(1, 3, 0.1)$ and $(2, 4, 0.2)$ is stable because 1 and 2 are receiving their peaks. It is not Pareto efficient because the allocation $(1, 4, 0.1)$ and $(2, 3, 0.2)$ dominates it.

A characterization of stable allocations is not straightforward. However the two propositions below identify some of their important features.

Fix a preference profile $\succsim$. Partition the set of agents into "high" type (H) and "low" type (L) agents depending upon whether their peaks are greater than or equal to or less than 0.5. Formally, $H = \{i \in N : p_i \geq 0.5\}$ and $L = \{i \in N : p_i < 0.5\}$. Furthermore, the set of strictly high type agents is $\hat{H} = \{i \in N : p_i > 0.5\}$. A mixed pair in an allocation is a pair consisting of an agent from L and an agent from $\hat{H}$. The next proposition shows that the number of mixed pairs in any stable allocation is maximal.

Proposition **1.** *In any stable allocation, the number of mixed pairs must be equal to* $\min\{|\hat{H}|, |L|\}$.

Proof: Assume for contradiction that there exists an agent $i_1 \in L$ who is matched to an agent $i_2 \notin \hat{H}$ and an agent $j_1 \in \hat{H}$ who is matched to an agent $j_2 \notin L$. There are two cases to consider.

The first case is when each of the agents i_1, i_2, j_1, j_2 contribute 0.5 in σ. There exists $\epsilon > 0$ small enough such that $p_{i_1} \leq 0.5 - \epsilon$ and $p_{j_1} \geq 0.5 + \epsilon$. Thus the pair (i_1, j_1) blocks σ with $(0.5 - \epsilon, 0.5 + \epsilon)$.

Suppose the first case does not hold. Then there exists at least one agent whose contribution is not 0.5. Suppose i_1 is one of these agents. We must also have $t_{i_2} \neq 0.5$. Clearly either t_{i_1} or t_{i_2} is greater than 0.5. Suppose $t_{i_2} > 0.5$. Consider the pair (j_1, j_2). There are two possibilities: $t_{j_1} \leq 0.5$ and $t_{j_1} > 0.5$. If $t_{j_1} \leq 0.5$, then there exists $\epsilon > 0$ small enough such that $0.5 + \epsilon \leq p_{j_1}$ and $0.5 - \epsilon \succ_{i_2} t_{i_2}$. Thus σ is blocked by (i_2, j_2) with $(0.5 - \epsilon, 0.5 - \epsilon)$. If $t_{j_1} > 0.5$, then $t_{j_2} < 0.5$ and the pair (i_2, j_2) blocks σ with $(0.5, 0.5)$.

The remaining case is where i_1 and i_2 contribute 0.5 in σ. Then j_1, j_2 have contributions not equal to 0.5. This case can be dealt with in a manner similar to the earlier case. ■

According to Proposition 2, surplus agents who do not belong to mixed pairs must contribute 0.5 in any stable allocation.

PROPOSITION **2.** *In any stable allocation, the following must hold:*

(a). If $|\hat{H}| - |L| > 2$, then every agent $i \in \hat{H}$ who is not matched to an agent in L must contribute 0.5*.*

(b). If $|L| - |\hat{H}| > 2$, then every agent $i \in L$ who is not matched to an agent in $\hat{H}$ must contribute 0.5*.*

Proof: We only prove Part (a) since the proof of Part (b) is the symmetric analogue. Let σ be a stable allocation. Assume for contradiction that there exists $(i_1, j_1, t_{i_1}) \in \sigma$ where $i_1 \in \hat{H}$, $j_1 \notin L$ and $t_{i_1} \neq 0.5$. By hypothesis, there exists at least another triple, say $(i_2, j_2, t_{i_2}) \in \sigma$ where $i_2 \in \hat{H}, j_2 \notin L$.

Since $t_{i_1} \neq 0.5$, either $t_{i_1} < 0.5$ or $t_{i_1} \geq 0.5$ (this implies $t_{j_1} < 0.5$) must hold. Suppose $t_{i_1} < 0.5$. There are two subcases to consider. If $t_{i_2} \leq 0.5$, there exists $\epsilon > 0$ and small enough such that $t_{i_1} < 0.5 - \epsilon$ and $0.5 + \epsilon \succ_{i_2} t_{i_2}$. Then the pair (i_1, i_2) blocks σ with $(0.5 - \epsilon, 0.5 + \epsilon)$. Otherwise, $t_{j_2} < 0.5$. In this case, there exists $\epsilon > 0$ and small enough such that $t_{i_1} < 0.5 - \epsilon$ and $0.5 + \epsilon \succ_{j_2} t_{j_2}$. Then (i_1, j_2) blocks σ with $(0.5 - \epsilon, 0.5 + \epsilon)$.

Suppose $t_{j_1} < 0.5$. Note that $p_{j_1} \geq 0.5$ whereas $p_{i_1} > 0.5$. However, it is easily verified that the argument in the previous paragraph works in this case as well. ■

1.2 Illustrative Examples

The purpose of this section is to highlight important features of our model with simple examples. The first example shows that strongly stable allocations may not exist.

EXAMPLE **1.** Let $N = \{1, 2, 3, 4\}$. Agents' preferences are symmetric and the peaks are summarized in Table 1.1.

p_1	p_2	p_3	p_4
0.8	0.3	0.3	0.3

Table 1.1: Peaks of agents in Example 1.

Consider an arbitrary allocation. Since agents 2, 3 and 4 have identical preferences, we can assume w.l.o.g. that $(1, 2)$ and $(3, 4)$ are the matched pairs. At least one of the agents in $\{3,4\}$ must have a contribution of at least 0.5. Suppose this agent is 3. The pair $(1,3)$ blocks with the contribution vector $(t_1,t_3)=(0.8,0.2)$. Agent 1 is at least as well-off as before while agent 3 is strictly better-off. Clearly there are no strongly stable allocations. □

A stable allocation does exist in Example 1, for instance, $(1,2,0.8)$ and $(3,4,0.5)$. In fact in any stable allocation, agent 1 must receive her peak. Otherwise agent 1 together with the agent who contributes at least 0.5 will block.

Agents whose peaks sum exactly to 1 are obviously perfect matches. This suggests the following procedure for generating stable allocations. First create as many perfect matches as possible. Then order all possible remaining pairs by the distance of the sum of their peaks from one and select the "best possible" pairs. Unfortunately this algorithm does not produce a stable matching as the next example shows.

Example **2.** Let $N=\{1,2,3,4,5,6\}$. Agents' preferences are symmetric and the peaks are summarized in Table 1.2.

p_1	p_2	p_3	p_4	p_5	p_6
0	0.4	0.41	0.41	0.41	0.75

Table 1.2: Peaks of agents in Example 2

The procedure outlined earlier generates the following matching: $(2,6)$,$(4,5)$,$(1,3)$. By Proposition 2, we know that agents $1,3,4,5$ must contribute 0.5 in any stable allocation. Also agent 6 must receives her peak, otherwise $(1,6)$ can block the allocation with $(0.25,0.75)$. So agent 2's contribution is 0.25. Then the pair $(2,3)$ can block with $(0.49,0.51)$. Thus no stable allocation can be obtained using this procedure. Observe that $(1,6,0.25)$, $(2,3,0.5)(4,5,0.5)$ is a stable allocation.

The following example shows that giving the peaks to either side of the market when the market is balanced (the number of high type agents is equal to the number of low type agents) may not generate a stable allocation. We consider a procedure where agents are partitioned into high and low type agents as before. An allocation is constructed by giving the peaks of the agents on one side of the market and matching them with agents of the other type.

Example **3.** Let $N=\{1,2,3,4,5,6\}$. Agents' preferences are symmetric and the peaks are summarized in Table 1.3.

The set of low and high type agents are $\{1,2,3\}$ and $\{4,5,6\}$ respectively. Assume w.l.o.g. that the pairs in the allocation are $(1,4)$, $(2,5)$ and $(3,6)$. The allocation where all high type agents get their peaks is not stable. Consider the allocation with the triples

p_1	p_2	p_3	p_4	p_5	p_6
0	0.45	0.45	0.65	0.65	0.65

Table 1.3: Peaks of agents in Example 3

$(1, 4, 0.35)$, $(2, 5, 0.35)$ and $(3, 6, 0.35)$. The pair $(2, 3)$ blocks with the contribution vector $(0.5, 0.5)$. Similarly the allocation where all low type agents get their peaks is not stable. For instance, the allocation $(1, 4, 0)$, $(2, 5, 0.45)$ and $(3, 6, 0.45)$ is blocked by the pair $(4, 5)$ using the contribution vector $(0.35, 0.65)$.

Stable allocations exist in this case as well. One such allocation is $(1, 4, 0.2)$, $(2, 5, 0.45)$ and $(3, 6, 0.45)$. □

The examples illustrate the important role of "same side" blocking in the model. Proposition 1 also seems to suggest a natural partitioning of agents into low and high types. However, doing so and naively following a Shapley and Shubik (1971) type procedure will not work because of the possibility of same-side blocking.

Another "obvious" procedure would be to arrange the agents in order of their peaks from the highest to the lowest. The highest agent would then be matched with the lowest, the second highest with the second lowest and so on. In Chapter 2 we show that this procedure works when all agents have symmetric preferences. However, the next example shows that this procedure may fail even when one agent has non-symmetric preferences.

Example **4.** Let $N = \{1, 2, 3, 4\}$. The peaks of the agents are summarized in Table 1.4. Agents 2, 3 and 4 have symmetric preferences while 1 has single-peaked but non-symmetric preferences with the following restriction: $0.35 \sim_1 0.51$.

p_1	p_2	p_3	p_4
0.39	0.4	0.4	0.9

Table 1.4: Peaks of agents in Example 4.

The pairs formed by the procedure are $(1, 4)$ and $(2, 3)$. Let (t_1, t_4) and (t_2, t_3) be their contribution vectors in a stable allocation.

By feasibility, one of the agents $i \in \{2, 3\}$ will have a contribution $t_i \geq 0.5$. Assume w.l.o.g. $i = 2$. We claim $t_1 \geq 0.35$. If $t_1 < 0.35$, then the pair $(1, 2)$ can block by proposing the contribution vector $(0.51, 0.49)$. Agent 2 strictly improves as $0.49 \succ_2 t_2$. For agent 1, single-peakedness implies $0.35 \succ_1 t_1$. Since $0.35 \sim_1 0.51$, we have $0.51 \succ_1 t_1$ and agent 1 strictly improves.

We also claim $t_4 \geq 0.68$. If $t_4 < 0.68$, then $(2, 4)$ can block by proposing $(0.31, 0.69)$. Agent 4 strictly improves by blocking as she moves closer to her peak. For agent 2, symmetry and $t_2 \geq 0.5$ implies $0.31 \succ_2 t_2$. Thus agent 2 also strictly improves.

We have argued that $t_1 \geq 0.35$ and $t_4 \geq 0.68$. Since 1 and 4 are paired together, we have a violation of feasibility. Hence there are no stable allocations with the pairs $(1, 4)$ and $(2, 3)$. □

The previous procedure first specified a way to match agents in pairs and then attempted to find suitable contributions. An alternative approach would be to first choose a rule for determining the contributions of agents and then finding a way to form pairs. The uniform rule characterized by Sprumont (1991) is a natural candidate for determining agents' contributions.

Let i and j be agents with peaks p_i and p_j respectively who are paired together. The uniform rule contribution vector (t_i^u, t_j^u) is defined as follows. Suppose $p_i + p_j \geq 1$. Then $t_i^u = \min\{p_i, \lambda\}$ and $t_j^u = \min\{p_j, \lambda\}$ where λ solves the equation $\min\{p_i, \lambda\} + \min\{p_j, \lambda\} = 1$. If $p_i + p_j < 1$, then $t_i^u = \max\{p_i, \lambda\}$ and $t_j^u = \max\{p_j, \lambda\}$ where λ solves the equation $\max\{p_i, \lambda\} + \max\{p_j, \lambda\} = 1$.

In Example 5 below, we show that the uniform rule cannot be used to determine the contribution vector irrespective of the pairing of agents.

Example **5.** Let $N = \{1, 2, \ldots, 6\}$. Agents' preferences are symmetric and Table 1.5 summarizes the peaks of the agents.

p_1	p_2	p_3	p_4	p_5	p_6
0.3	0.3	0.3	0.3	0.9	0.9

Table 1.5: Peaks of agents in Example 5.

Since the first four agents have the same preferences, as also agents 5 and 6, w.l.o.g. we consider only two types of pairings in an allocation: one where agents 5 and 6 are paired together and one where they are not.

Consider an allocation where agents 5 and 6 are paired together. The uniform rule assigns to each agent a contribution equal to 0.5.[8] The allocation is blocked by the coalition $(3, 6, 0.3)$. Consider an allocation where agents 5 and 6 are not paired together. Assume w.l.o.g. that the pairs $(1, 6)$, $(2, 5)$ and $(3, 4)$ belong to the allocation. The uniform rule assigns 0.3 to agents 1 and 2, 0.7 to agents 5 and 6, and 0.5 to agents 3 and 4. This allocation is blocked by the coalition $(3, 6, 0.29)$. □

[8]Note that 0.5 solves the equation $\max\{\lambda, 0.3\} + \max\{\lambda, 0.3\} = 1$ and the equation $\min\{\lambda, 0.9\} + \min\{\lambda, 0.9\} = 1$.

1.3 Improvement sets and stability

We introduce the notion of improvement sets, which plays a key role in our algorithm.

We partition agents into "high" type (H) and "low" type (L) agents depending upon whether their peaks are greater than or less than 0.5. Formally, $H = \{i \in N : p_i \geq 0.5\}$ and $L = \{i \in N : p_i < 0.5\}$. We represent the peaks and the contributions of agents in the interval $[0, 0.5]$. The peak of a low type agent will be measured from left to right starting at 0, while the peak of a high type agent will be measured from right to left starting at 0.5.[9]

Definition **4.** *Consider agent $i \in L$ with preference $\succsim_i$ (with peak p_i) and contribution t_i. We define the improvement set for i at t_i as follows:*

$$I_{i,t_i} = \begin{cases} \{x \in [0, 0.5] : x \succ_i t_i\} & \text{if } t_i \neq p_i, \\ \emptyset & \text{if } t_i = p_i. \end{cases}$$

Definition **5.** *Consider agent $i \in H$ with preference $\succsim_i$ (with peak p_i) and contribution t_i. We define the improvement set for i at t_i as follows:*

$$I_{i,t_i} = \begin{cases} \{x \in [0, 0.5] : 1 - x \succ_i t_i\} & \text{if } t_i \neq p_i, \\ \emptyset & \text{if } t_i = p_i. \end{cases}$$

We make a brief remark about the asymmetry in the definitions of improvement sets for low and high type agents. For an agent $i \in L$, the improvement set consists of contributions in $[0, 0.5]$ which she strictly prefers to t_i. For an agent $i \in H$, the improvement set consists of contributions made by a potential partner in $[0, 0.5]$ which would make agent i strictly better-off relative to t_i.

The assumptions of single-peakedness and continuity on $\succsim_i$ imposes structure on the improvement sets which we record below as an observation.

Observation **1.** The improvement set of an agent is a connected open subset of $[0, 0.5]$ or equivalently an open interval in $[0, 0.5]$.

Example 6 illustrates improvement sets for both low and high type agents.

Example **6.** Let $N = \{1, 2, 3, 4\}$. Agents' preferences are symmetric. Table 1.6 summarizes their peaks and contributions. Agents 1 and 2 are matched together as are 3 and 4.

[9] The interval $[0, 0.5]$ can be thought of as a truncated one-dimensional Edgeworth box. For a low type agent, we are not interested in representing contributions greater than 0.5. Similarly we do not need to represent contributions smaller than 0.5 for a high type agent.

Agent	1	2	3	4
p_i	0.6	0.75	0.10	0.75
t_i	0.3	0.7	0.25	0.75

Table 1.6: Peaks of agents in Example 6.

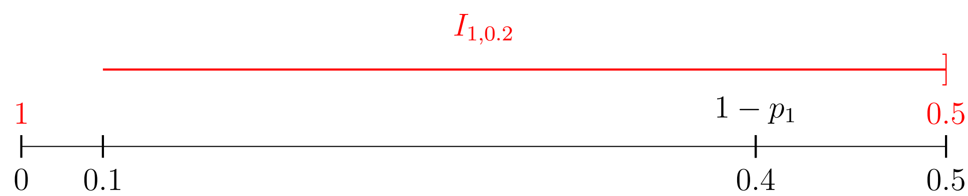

Figure 1.1: Improvement set for agent 1 in Example 6.

Figure 1 shows the improvement set of agent 1 while Figure 2 shows the improvement sets of agents 2 and 3. The improvement set of agent 4 is empty. Improvement sets of high types are marked in red while that of the low type is marked in blue. □

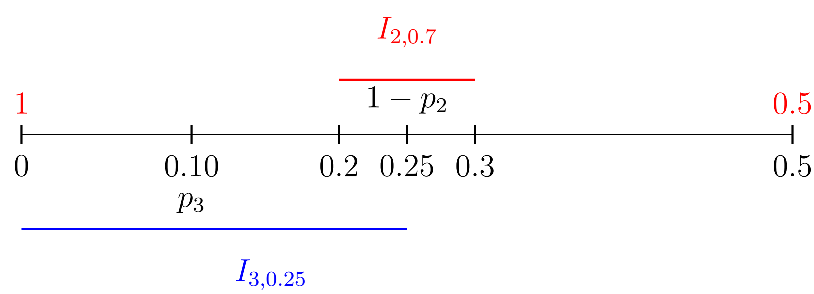

Figure 1.2: Improvement sets in Example 6.

It is useful to define the notion of an *equivalent* contribution. We denote the equivalent contribution for agent i at t_i by $e_i(t_i)$ when $i \in L$ and $e_i(1-t_i)$ when $i \in H$.[10]

Consider $i \in L$. If there exists a contribution $x \in [0, 0.5]$ such that $x \sim_i t_i$ and $x \neq t_i$, then $e_i(t_i) = x$. The equivalent $e_i(t_i)$ is an end-point of the improvement set, but it is not included in the latter. Here the improvement set is one of the following: $(t_i, e_i(t_i))$ if $t_i < p_i$, $(e_i(t_i), t_i)$ if $p_i < t_i \leq 0.5$ or $(e_i(t_i), 0.5]$ if $p_i < 0.5 < t_i$.
If such an x does not exist, then $e_i(t_i)$ is defined as follows:

[10]We suppress the dependence of the equivalent of agent i on $\succsim_i$ since we keep the latter constant throughout the analysis.

$$e_i(t_i) = \begin{cases} -\epsilon & \text{if } I_{i,t_i} = [0, t_i) \text{ or } I_{i,t_i} = [0, 0.5], \\ 0.5 + \epsilon & \text{if } I_{i,t_i} = (t_i, 0.5], \\ t_i & \text{if } I_{i,t_i} = \emptyset, \end{cases}$$

where ϵ is any small positive number.

Consider $i \in H$. If there exists a contribution $x \in [0, 0.5]$ such that $1-x \sim_i t_i$ and $1-x \neq t_i$, then $e_i(1-t_i) = x$. As before, the equivalent $e_i(1-t_i)$ is an end-point of the improvement set but not included in it. Here the improvement set is one of the following: $(1-t_i, e_i(t_i))$, $(e_i(t_i), 1-t_i)$ or $(e_i(t_i), 0.5]$ if $t_i < 0.5$.
If such an x does not exist, then $e_i(1-t_i)$ is defined as follows:

$$e_i(1-t_i) = \begin{cases} -\epsilon & \text{if } I_{i,t_i} = [0, 1-t_i) \text{ or } I_{i,t_i} = [0, 0.5], \\ 0.5 + \epsilon & \text{if } I_{i,t_i} = (1-t_i, 0.5], \\ 1-t_i & \text{if } I_{i,t_i} = \emptyset, \end{cases}$$

where ϵ is any small positive number.

Improvement sets provide a natural way to check for the existence of stable allocations. For instance, in Example 6 agents 2 and 3, who receive contributions 0.7 and 0.25, respectively, in an allocation, will block. This is evident from the fact that their improvement sets have a non-empty intersection.

Definition **6.** *An allocation satisfies Condition S if the associated improvement sets satisfy the following:*

1. *For every $h \in H$ and $l \in L$, $I_{h,t_h} \cap I_{l,t_l} = \emptyset$.*
2. *For all $i \in N$, $0.5 \notin I_{i,t_i}$.*

Proposition **3.** *If an allocation satisfies Condition S, then it is stable. Moreover, if an allocation is stable, then it satisfies Part* 1 *of Condition S.*

Proof: Consider an allocation that satisfies Condition S but is not stable, i.e. there exists a pair of agents who block. There are two cases to consider.

Case 1: The blocking pair is (l, h) where $l \in L$ and $h \in H$. Let t_l and t_h be the contributions of l and h respectively in the allocation. Suppose they block with the contribution vector (t'_l, t'_h). If $t'_l \geq 0.5$, single-peakedness implies $0.5 \in I_{l,t_l}$, which would contradict requirement 2 of Condition S. Therefore $t'_l < 0.5$, i.e. $1-t'_l = t'_h > 0.5$. Since $t'_l \succ_l t_l$ and $t'_l < 0.5$, we have $t'_l \in I_{l,t_l}$. Since $t'_h \succ_h t_h$ and $t'_h = 1-t'_l > 0.5$, we have $t'_l \in I_{h,t_h}$. Therefore $t'_l \in I_{l,t_l} \cap I_{h,t_h}$ contradicting requirement 1 of Condition S.
Case 2: Both agents in the blocking pair are of the same type. Suppose (l_1, l_2) is the blocking

pair where $l_1, l_2 \in L$. Let t_{l_1} and t_{l_2} be the contributions of agents l_1 and l_2 respectively in the allocation. In the contribution vector used to block, at least one of the agents in the pair, say l_1 has a contribution of at least 0.5. Single-peakedness implies $0.5 \in I_{l_1,t_{l_1}}$, contradicting requirement 2 of Condition S. The argument in the case where both agents are of the high type is virtually identical.
We now show that any stable allocation must satisfy requirement 1 of Condition S.
Consider a stable allocation. Assume for contradiction that there exist $i \in L$ and $j \in H$ such that $I_{i,t_i} \cap I_{j,t_j} \neq \emptyset$ where t_i and t_j are contributions of agents i and j respectively in the allocation. Consider $x \in I_{i,t_i} \cap I_{j,t_j}$. By the definition of improvement sets, we have $x \succ_i t_i$ and $1 - x \succ_j t_j$. Thus the pair (i, j) blocks the allocation with the contribution vector $(x, 1 - x)$. ■

Requirement 2 of Condition S is not necessary for the existence of stable allocations. For instance, consider the case where there are four agents l_1, l_2, h_1, h_2 with symmetric preferences. The peaks of l_1 and l_2 are 0.4 and 0.3 while the peaks of h_1 and h_2 are 0.9 and 0.7. The allocation with the triples $(l_1, h_1, 0.1)$ and $(l_2, h_2, 0.3)$ is stable because all agents except l_1 are satiated. However $0.5 \in I_{l_1,0.1}$.
In fact, there is another condition, that we call Condition S^+ and that we define below, with the property that every allocation that satisfies it is also stable. Moreover, every stable allocation satisfy either condition S or condition S^+, as we will show in Proposition 5.

Definition **7.** *An allocation satisfies Condition S^+ if there exists a special agent $s \in N$ such that the associated improvement sets satisfy the following:*

1. *For every $h \in H$ and $l \in L$, $I_{h,t_h} \cap I_{l,t_l} = \emptyset$.*
2. *For all $i \in N$, $0.5 \in I_{i,t_i} \Rightarrow i = s$.*
3. *For every $h \in H \setminus \{s\}$ $\{x \in [0, 0.5] | x \succ_s t_s\} \cap I_{h,t_h} = \emptyset$.*
4. *For every $l \in L \setminus \{s\}$, $\{x \in [0, 0.5] | 1 - x \succ_s t_s\} \cap I_{l,t_l} = \emptyset$.*

Notice that agent s is either a high type or a low type agent, thus her improvement set is either the set $\{x \in [0, 0.5] | x \succ_s t_s\}$ or the set $\{x \in [0, 0.5] | 1 - x \succ_s t_s\}$. However, agent s behaves as an agent of both kinds concerning the nature of her improvement set, in this sense she is a special agent. In particular, if $s \in L$ the requirement 3 of Condition S^+ is implied by requirement 1 of Condition S^+ while if $s \in H$ the requirement 4 of Condition S^+ is implied by requirement 1 of Condition S^+.

Proposition **4.** *If an allocation satisfies Condition S^+, then it is stable.*

Proof: Consider an allocation that satisfies Condition S but is not stable, i.e. there exists a pair of agents who block. There are three cases to consider.

Case 1: agent s does not belong to the blocking pair. In this case the proof is identical to the one presented for Proposition 3.
Case 2: the blocking pair is formed by a high type agent h and agent s. Let t_s and t_h be the contributions of s and h respectively in the allocation. Suppose they block with the contribution vector (t'_s, t'_h). If $t'_h \leq 0.5$, by single peakedness $0.5 \in I_{h,t_h}$, which would contradict requirement 2 of condition S^+ since $h \neq s$ and s is the only agent in N that has 0.5 in her improvement set. Therefore $t'_h > 0.5$ and $t'_s < 0.5$. Since $t'_h \succ_h t_h$ and $t'_h > 0.5$, we have $t'_s = 1 - t'_h \in I_{h,t_h}$. $t'_s \in \{x \in [0, 0.5] | x \succ_s t_s\}$ by hypothesis. Therefore $t'_s \in \{x \in [0, 0.5] | x \succ_s t_s\} \cap I_{h,t_h}$ contradicting requirement 3 of Condition S^+.
Case 3: the blocking pair is formed by a low type agent l and agent s. Let t_l and t_s be the contributions of s and h respectively in the allocation. Suppose they block with the contribution vector (t'_l, t'_s). If $t'_l \geq 0.5$, by single peakedness $0.5 \in I_{l,t_l}$, which would contradict requirement 2 of condition S^+ since $l \neq s$ and s is the only agent in N that has 0.5 in her improvement set. Therefore $t'_l < 0.5$ and $t'_s > 0.5$. Since $t'_l \succ_l t_l$ and $t'_l < 0.5$, we have $t'_l =\in I_{l,t_l}$. $t'_l = 1 - t'_s \in \{x \in [0, 0.5] | 1 - x \succ_s t_s\}$ by hypothesis. Therefore $t'_l \in \{x \in [0, 0.5] | 1 - x \succ_s t_s\} \cap I_{l,t_l}$ contradicting requirement 4 of Condition S^+.

■

PROPOSITION **5.** *If an allocation is stable, then it either satisfies Condition S or it satisfies condition S^+.*

Proof: Let σ be a stable allocation and $\{t_i\}_{i \in N}$ be the set of the contribution values in σ. Let us consider now the set $M := \{i \in N | 0.5 \succ_i t_i\} = \{i \in N | 0.5 \in I_{i,t_i}\}$.[11] Let us suppose by contradiction that $|M| \geq 2$. In this case there is a pair of agents (i, j) that are both in M: this means that $0.5 \succ_i t_i$ and $0.5 \succ_j t_j$, thus they can block σ with the contribution vector $(0.5, 0.5)$. Therefore $|M| \in \{0, 1\}$. Let us distinguish the two cases.

Case 1: $|M| = 0$. In this case σ satisfies condition S. In Proposition 3 we proved that requirement 1 of Property S is necessary for stability, and requirement 2 of property S is true by hypothesis.

Case 2: $|M| = 1$. In this case here is exactly one agent in M, we call this agent s and we want to prove that σ satisfies Property S^+.
Requirement 1 of Condition S^+ is identical to requirement 1 of Condition S, which is necessary for stability as proved in Proposition 3. Requirement 2 is satisfied by hypothesis.
Let us now suppose by contradiction that σ does not satisfy requirement 3 of Condition S^+. Thus, there exist an agent $h \in H$ and a value $y \in \{x \in [0, 0.5] | x \succ_s t_s\} \cap I_{h,t_h}$. In this

[11]Since $0.5 = 1 - 0.5$, 0.5 is the only point in $[0, 0.5]$ that belong to the improvement set of a high type agent if and only if it is preferred by that agent to her contribution.

case $y \succ_s t_s$ and $1-y \succ_h t_h$ by definition of I_{h,t_h}, thus the pair (s,h) blocks σ with the contribution vector $(y, 1-y)$.
Let us now suppose by contradiction that σ does not satisfy requirement 4 of Condition S^+. thus there exist an agent $l \in L$ and a value $y \in \{x \in [0, 0.5] | 1-x \succ_s t_s\} \cap I_{l,t_l}$. In this case $1-y \succ_s t_s$ and $y \succ_l t_l$ by definition of I_{l,t_l}, thus the pair (l,s) blocks σ with the contribution vector $(y, 1-y)$. ■

1.4 The Select-Allocate-Match (SAM) Algorithm

In this section, we provide a formal description of our algorithm and state our main result. In the rest of the chapter, we adopt the following convention: whenever we write a triple (i, j, t_i) in the description of an allocation, we assume $p_i \leq p_j$.[12]

Let $\succ^N$ be a linear ordering of the set N: this ordering will serve as a tie-breaking rule. Fix an arbitrary preference profile $\succsim$. The peaks of the agents at $\succsim$ are $p_1, p_2, \ldots, p_n$. We begin by partitioning the set of agents N into the sets H and L.

Step 0: We remove excess agents from either the set H or the set L to ensure that the cardinality of the adjusted H and L sets is equal. If $|H| > |L|$, we remove $|H| - |L|$ agents (chosen in a specific way) from the set H. We denote the set of agents removed from H by $\bar{H}$. Similarly if $|H| < |L|$, we remove $|L| - |H|$ agents from L. The set of agents removed from L is denoted by $\bar{L}$. In addition, we define two sets U_1 and D_1 with $U_1, D_1 \subseteq [0, 0.5]$ which are the union of the improvement sets of low type agents and of high type agents, respectively, matched in this step.

There are three possibilities to consider.

1. $|H| = |L|$. Here $\bar{H} = \emptyset$ and $\bar{L} = \emptyset$. Also $U_1 = \emptyset$ and $D_1 = \emptyset$.

2. $|H| > |L|$. Compute $e_i(0.5)$ for all $i \in H$. Pick the $|H| - |L|$ agents whose equivalents $e_i(0.5)$ are closest to 0.5. Ties are broken using the ordering $\succ^N$. The set of these agents is $\bar{H}$. Pair the agents in $\bar{H}$ arbitrarily and the contribution of all agents is 0.5. Here $\bar{L} = \emptyset$, $U_1 = \cup_{i \in \bar{H}} I_{i,0.5}$ and $D_1 = \emptyset$.

3. $|H| < |L|$. Compute $e_i(0.5)$ for all $i \in L$. Pick the $|L| - |H|$ agents whose equivalents $e_i(0.5)$ are closest to 0.5. Ties are broken using the ordering $\succ^N$. The set of these agents is $\bar{L}$. Pair the agents in $\bar{L}$ arbitrarily and the contribution for all agents is 0.5. Here $\bar{H} = \emptyset$, $D_1 = \cup_{i \in \bar{L}} I_{i,0.5}$ and $U_1 = \emptyset$.

[12] Suppose agents 1 and 2 are paired in an allocation. Let $p_1 = 0.4$, $p_2 = 0.7$ and their contributions in the allocation be $t_1 = 0.1$, $t_2 = 0.9$. We shall write the triple as $(1, 2, 0.1)$.

The adjusted partition of N is $H_1 = H \backslash \bar{H}$ and $L_1 = L \backslash \bar{L}$. By construction, $|H_1| = |L_1| = K$. The algorithm has $K + 1$ steps including Step 0. In each Step q (where $1 \leq q \leq K$) we form a pair consisting of a high type agent and a low type agent. We denote these agents by h_q and l_q respectively. At the start of that step the algorithm is provided three inputs: (i) Preferences of the agents in H_q and L_q (ii) U_q where $U_q \subseteq [0, 0.5]$ and (iii) D_q where $D_q \subseteq [0, 0.5]$. The set D_q is the union of the improvement sets of the L type agents who have been matched until Step q. A similar comment holds for U_q and H type agents.

Step q: Each step q is divided into four substeps, referred to as Substep $q.s$ where $s \in \{1, 2, 3, 4\}$. We will determine the agents $l_q \in L_q$ and $h_q \in H_q$ who will be matched to each other and their contribution vector (t_{l_q}, t_{h_q}). At the end of the step, we will determine L_{q+1}, H_{q+1}, D_{q+1}, and U_{q+1}.

Step $q.1$: Consider the set $\{h : 1 - p_h > \inf D_q\}$. If it is empty, proceed to Step $q.2$. Otherwise, choose h_q to be the agent with the lowest peak (or the highest $1 - p_h$) in this set. The agent h_q is the primary agent in this substep. The contribution of agent h_q is $t_{h_q} = \max\{p_{h_q}, 1 - \inf U_q\}$. Choose l_q to be the low type agent in L_q who has the highest $e(1 - t_{h_q})$ (using the tie-breaking ordering $\succ^N$ on agents if necessary). The agent l_q is the secondary agent in this substep. We add the triple $(l_q, h_q, 1 - t_{h_q})$ to the allocation and proceed to Step $q.4$.

Step $q.2$: Consider the set $\{l : p_l > \inf U_q\}$. If it is empty, proceed to Step $q.3$. Otherwise, choose l_q to be the agent with the highest peak in this set. The contribution of agent l_q is $t_{l_q} = \min\{p_{l_q}, \inf D_q\}$. Choose h_q to be the high type agent in H_q who has the highest $e(t_{l_q})$ (using the tie-breaking ordering $\succ^N$ on agents in case of ties). We add the triple (l_q, h_q, t_{l_q}) to the allocation and proceed to Step $q.4$. In this substep, l_q is the primary agent while h_q is the secondary agent.

Step $q.3$: If $1 - p_h \leq \inf D_q$ for all $h \in H_q$ and $p_l \leq \inf U_q$ for all $l \in L_q$, we identify the following agents.

1. The high type agent with the lowest peak in H_q. Denote this agent by $\tilde{h}_q$.

2. The low type agent with the highest peak in L_q. Denote this agent by $\tilde{l}_q$.

There are two possibilities leading to Steps $q.3.1$ and $q.3.2$.

Step $q.3.1$: If $p_{\tilde{l}_q} \leq 1 - p_{\tilde{h}_q}$, choose $h_q = \tilde{h}_q$. The contribution of agent h_q is $t_{h_q} = \max\{p_{h_q}, 1 - \inf U_q\}$. Choose l_q to be the low type agent in L_q who has the highest $e(1 - t_{h_q})$ (using the tie-breaking ordering $\succ^N$ on agents if necessary). We add the triple $(l_q, h_q, 1 - t_{h_q})$ to the allocation and proceed to Step $q.4$. In this substep, h_q is the primary agent while l_q is the secondary agent.

Agent	1	2	3	4	5	6	7	8
Peak	0	0.32	0.45	0.45	0.45	0.65	0.75	0.77

Table 1.7: Peaks of agents in Example 7.

Step q.3.2: If $p_{\tilde{l}_q} > 1 - p_{\tilde{h}_q}$, choose $l_q = \tilde{l}_q$. The contribution of agent l_q is $t_{l_q} = \min\{p_{l_q}, \inf D_q\}$. Choose h_q to be the high type agent in H_q who has the highest $e(t_{l_q})$ (using the tie-breaking ordering $\succ^N$ on agents in case of ties). We add the triple (l_q, h_q, t_{l_q}) to the allocation. Proceed to Step q.4. In this substep, l_q is the primary agent while h_q is the secondary agent.

Step q.4: Sets $D_{q+1} = D_q \cup I_{l_q, t_{l_q}}$ and $U_{q+1} = U_q \cup I_{h_q, t_{h_q}}$. Also sets $H_{q+1} = H_q \setminus \{h_q\}$ and $L_{q+1} = L_q \setminus \{l_q\}$. Proceed to Step $q+1$.

⋮

⋮

Step K: Note that $|L_K| = |H_K| = 1$. After the completion of this step, all agents in N have been matched and the algorithm terminates.

We illustrate the algorithm with an example.

Example **7.** Let $N = \{1, 2, \ldots, 8\}$. Table 1.7 summarizes the peaks of the agents. All agents except agent 6 have symmetric preferences. Agent 6 has non-symmetric preferences with the following equivalents; $e_6(0.4) = 0.08 < 0.10$ and $e_6(0.32) = 0.37$.

The priority order of agents is $1 \succ^N 2 \succ^N 3 \ldots \succ^N 8$. The partition of agents is $L = \{1, 2, 3, 4, 5\}$ and $H = \{6, 7, 8\}$.

Step 0: We remove two agents from L. Since all agents in L have symmetric preferences and $p_1 < p_2 < p_3 = p_4 = p_5$, we have $e_1(0.5) < e_2(0.5) < e_3(0.5) = e_4(0.5) = e_5(0.5)$. Agents 3, 4 and 5 have the highest equivalent and ties are broken using $\succ^N$. Thus $\bar{L} = \{3, 4\}$. Note that $\bar{H} = \emptyset$.
Agents 3 and 4 are paired with the contribution vector $(0.5, 0.5)$. We add the triple $(3, 4, 0.5)$ to the allocation. The improvement set for agent $i \in \{3, 4\}$ is $I_{i,0.5} = (0.4, 0.5)$. Thus $D_1 = (0.4, 0.5)$. Since $\bar{H} = \emptyset$, we have $U_1 = \emptyset$. Table 1.8 summarizes these facts. The remaining high and low type agents are $H_1 = \{6, 7, 8\}$ and $L_1 = \{1, 2, 5\}$ respectively.

Step 1: The sets H_1, L_1, D_1 and U_1 are the inputs in this step.

Substep 1.1 is not applicable as there does not exist an $h \in H_1$ such that $1 - p_h > \inf D_1 = 0.4$. Substep 1.2 obviously does not apply.

Step q	Triple (i, j, t_i)	I_{i,t_i}	I_{j,t_j}	D_{q+1}	U_{q+1}
0	$(3, 4, 0.5)$	$(0.4, 0.5)$	$(0.4, 0.5)$	$(0.4, 0.5)$	$\emptyset$

Table 1.8: Output of Step 0.

Step q	Triple (i, j, t_i)	I_{i,t_i}	I_{j,t_j}	D_{q+1}	U_{q+1}
1	$(5, 7, 0.4)$	$(0.4, 0.5)$	$(0.1, 0.4)$	$(0.4, 0.5)$	$(0.1, 0.4)$

Table 1.9: Output of Step 1.

In Substep 1.3, we have $\tilde{h}_1 = 6$ and $\tilde{l}_1 = 5$. Since $p_{\tilde{l}_1} > 1 - p_{\tilde{h}_1}$, Substep 1.3.2 applies and agent 5 is the primary agent. The contribution of agent 5 is $t_5 = \min\{p_5, \inf D_1\} = 0.4$. In order to choose the secondary agent, we compute $e_i(0.4)$ for all $i \in H_1$. Recall $e_6(0.4) = 0.08$. Since agents 7 and 8 have symmetric preferences, we have $e_7(0.4) = 0.1$ and $e_8(0.4) = 0.06$. Agent 7 is therefore selected as the secondary agent and is paired with agent 5. The contribution vector is $(0.4, 0.6)$ and the triple $(5, 7, 0.4)$ is added to the allocation.
The improvement sets for the agents paired in this step are $I_{5,0.4} = (0.4, 0.5)$ and $I_{7,0.6} = (0.1, 0.4)$. Thus $D_2 = D_1 \cup I_{5,0.4} = (0.4, 0.5)$ and $U_2 = U_1 \cup I_{7,0.6} = (0.1, 0.4)$. Table 1.9 summarizes these facts. Also $H_2 = \{6, 8\}$ and $L_2 = \{1, 2\}$.

Step 2: The sets H_2, L_2, D_2 and U_2 are the inputs to this step.
Substep 2.1 does not apply. Substep 2.2 is applicable as $p_2 = 0.32 > \inf U_2 = 0.1$. Agent 2 is the only low type agent whose peak is greater than $\inf U_2$. Agent 2 is the primary agent. The contribution of agent 2 is $t_2 = \min\{p_2, \inf D_2\} = 0.32$. In order to choose the secondary agent, we compute $e_i(0.32)$ for all $i \in H_2$. Since $1 - p_6 = 0.35 > 0.32$, we have $e_6(0.32) > 0.35$. Since $1 - p_8 = 0.23 < 0.32$, we have $e_8(0.32) < 0.23$. So agent 6 has the highest equivalent and is chosen as the secondary agent. Agent 6 is paired with agent 2 and the contribution vector is $(0.32, 0.68)$.
The triple $(2, 6, 0.32)$ is added to the allocation. The improvement sets of the agents matched in this step are $I_{2,0.32} = \emptyset$ and $I_{6,0.68} = (0.32, 0.37)$ (recall $e_6(0.32) = 0.37$). Thus $D_3 = (0.4, 0.5)$ and $U_3 = (0.1, 0.4) \cup (0.32, 0.37) = (0.1, 0.4)$. Table 1.10 summarizes these facts. Also $H_3 = \{8\}$ and $L_3 = \{1\}$.

Step q	Triple (i, j, t_i)	I_{i,t_i}	I_{j,t_j}	D_{q+1}	U_{q+1}
2	$(2, 6, 0.32)$	$\emptyset$	$(0.32, 0.37)$	$(0.4, 0.5)$	$(0.1, 0.4)$

Table 1.10: Output of Step 2.

Step q	Triple (i,j,t_i)	I_{i,t_i}	I_{j,t_j}
0	$(3,4,0.5)$	$(0.4,0.5)$	$(0.4,0.5)$
1	$(5,7,0.4)$	$(0.4,0.5)$	$(0.1,0.4)$
2	$(2,6,0.32)$	$\emptyset$	$(0.32,0.37)$
3	$(1,8,0.1)$	$[0,0.1)$	$(0.1,0.36)$

Table 1.11: The allocation and improvement sets in Example 7

Step 3: The sets H_3, L_3, D_3 and U_3 are inputs to this step.
Clearly Substeps 3.1 and 3.2 are not applicable. Since $1 - p_8 = 0.23 > p_1 = 0$, Substep 3.3.1 is applicable. Agent 8 is the primary agent. The contribution of agent 8 is $t_8 = \max\{p_8, 1 - \inf U_3\} = 0.9$. Thus $1 - t_8 = 0.1$.
Agent 1 is the secondary agent. The triple $(1,8,0.1)$ is added to the allocation. The improvement sets for the agents matched in this step are $I_{1,0.1} = [0,0.1)$ and $I_{8,0.9} = (0.1,0.36)$. Thus $D_4 = [0,0.1) \cup (0.4,0.0.5)$ and $U_3 = (0.1,0.4)$.

This is the termination step of the algorithm. Table 1.11 summarizes the allocation generated by the algorithm and the improvement sets of the agents. □

We state our result below.

THEOREM **1.** *The SAM algorithm generates a stable and Pareto efficient allocation.*

The proof of Theorem 1 is in the Appendix. The allocation generated by the SAM algorithm is stable since it satisfies Condition S. The key step in the proof is to show that the sets D_q and U_q do not intersect and do not contain 0.5 at any step q of the algorithm. The proof of Pareto efficiency requires several steps.

1.5 Discussion

In this section, we discuss various aspects of our model.

1.5.1 Single-peaked preferences

In Example 8 we show that the single-peakedness assumption on preferences is vital for the existence of stable allocations.

EXAMPLE **8.** Let $N = \{1,2,\ldots,6\}$. Table 1.12 summarizes the peaks of agents 1 to 5 and the dip of agent 6. All agents in $\{1,\ldots,5\}$ have symmetric single-peaked preferences. Agent 6 has symmetric single-dipped preferences with 0.5 as the dip. This means that 0.5 is her worst contribution and she is progressively better-off as she moves farther away from 0.5. As a result, 0 and 1 are her most preferred contributions.

Agent	1	2	3	4	5	6
Peak	0.49	0.49	0.49	0.01	0.98	-
Dip	-	-	-	-	-	0.5

Table 1.12: Peaks/dip of agents in Example 8.

We argue that there are no stable allocations. Notice that one of the agents in $\{1,2,3\}$ must be paired with an agent from $\{4,5,6\}$. Assume w.l.o.g. that agent 3 is paired with an agent from $\{4,5,6\}$. We consider each case in turn.

Case *A*: Agent 3 is paired with agent 4. Let their contribution vector be (t_3, t_4). If the allocation is stable, it must be the case that $t_3 \geq 0.49$ and $t_4 \geq 0.01$.[13]
One of the agents 3, 4 must be at a distance of $\max\{t_3 - 0.49, t_4 - 0.01\}$ for any (t_3, t_4). Minimising over (t_3, t_4), we infer that one of the agents must be a distance of at least 0.25 from her peak. Suppose this agent is 3. An immediate consequence is that agents 1 and 2 must be receiving their peaks in the allocation. Otherwise agent 3 can block with the non-satiated agent by offering her 0.49 and being only 0.02 away from her own peak. This implies that agents 1 and 2 are not paired together but are paired with 5 and 6. Moreover agents 5 and 6 will each get a contribution of 0.51. Then the pair $(5,6)$ blocks with the contribution vector $(0.98, 0.02)$.
Suppose agent 4 is the agent who is at a distance of at least 0.25 from her peak. Then agent 5 must get her peak and agent 6 must be getting either 0 or 1. If not, agent 4 can block with 5 by offering her 0.98 and being 0.01 away from her peak. Agent 4 can block with agent 6 by offering her 1 and being 0.01 away from her own peak. It follows that 5 and 6 cannot be paired together but are paired with 1 and 2. The agent paired with 5, say agent 1, gets 0.02, while agent 2 (paired with 6) gets either 0 or 1. In either case, the pair $(1,2)$ blocks with the vector $(0.5, 0.5)$. These arguments establish that Case *A* cannot occur.

Case *B*: Agent 3 is paired with agent 5. Let (t_3, t_5) be the contribution vector. Using arguments similar to those in Case *A*, we can argue that $t_3 \leq 0.49$ and $t_5 \geq 0.98$. One of the agents 3, 5 must be at a distance of $\max\{0.49 - t_3, 0.98 - t_5\}$ for any (t_3, t_5). Therefore either 3 or 5 must be at a distance of at least 0.235 from her peak.
Suppose agent 3 is this agent. Like in Case *A*, agents 1 and 2 must receive their peaks. Thus they are not paired together and are paired with 4 and 6. Moreover 4 and 6 are each

[13]If $t_3 > 0.49$ and $t_4 < 0.01$, then the pair $(3,4)$ blocks by proposing a contribution vector (t'_3, t'_4) such that $t'_3 < t_3$ and $t'_4 > t_4$. Similarly the case $t_3 < 0.49$ and $t_4 > 0.01$ is ruled out. Of course $t_3 < 0.49$ and $t_4 < 0.01$ is infeasible.

receiving 0.51. The pair $(4,6)$ blocks with $(0,1)$.
Suppose agent 5 is at a distance of at least of 0.235 from her peak. Like in Case A, agents 4 and 6 must get their peaks. So they cannot be paired together and are paired with 1 and 2. Once again, 1 and 2 will form a blocking coalition.

Case C: Agent 3 is paired with agent 6. Since 1, 2 and 3 have the same preferences, we can apply Cases A and B to argue that neither 1 nor 2 can be matched to an agent in $\{4,5\}$. Consequently the pairs in this allocation are $(1,2)$ and $(4,5)$. Note that there must be an agent in each pair who does not get her peak. Assume w.l.o.g. that 1 is not getting her peak. To ensure that $(1,3)$ does not block with $(0.49, 0.51)$, it must be the case that $0.47 \leq t_3 \leq 0.51$. Thus $0.49 \leq t_6 \leq 0.53$. If 4 is the agent in the pair $(4,5)$ who is not getting her peak, then $(4,6)$ blocks with $(0.01, 0.99)$. If 5 is the agent not getting her peak, then $(5,6)$ blocks with $(0.98, 0.02)$.
Therefore, Case C cannot occur and there are no stable allocations. □

Example 8 illustrates the key role played by the "complementarity of preferences" in the existence of stable allocations in our model. For simplicity, suppose all agents have symmetric (single-peaked) preferences. Consider two agents of very high type (with peaks close to one) and one of a very low type (with a peak close to zero). Each of the two high type agents are a "good fit" for the low type agent but are not well-suited to be paired together. This prevents the cyclical pattern of blocking which typically underlies the non-existence of stable allocations. This is exactly what occurs in Example 8 - agents 4, 5 and 6 are mutually "good fits" for each other.

1.5.2 Continuity of preferences

The following example shows that stable allocations may not exist if preferences are single-peaked but not continuous.

Example **9.** Let $N = \{1, 2, \ldots, 6\}$. Table 9 summarizes the peaks of the agents. Preferences of agents 1 and 2 are symmetric and continuous. For any agent $i \in \{3,4,5,6\}$, $\succsim_i$ is single-peaked but not continuous at 0.3. In particular, $\succsim_i$ satisfies: (i) for any z such that $0.3 < z \leq 0.41$, $z \succ_i 0.5$ and (ii) $\exists\, \bar{\epsilon} > 0$ such that $0.5 + \bar{\epsilon} \succ_i 0.3$. Continuity of $\succsim_i$ will imply $0.3 \succsim_i 0.5$. Single-peakedness implies $0.5 \succ_i 0.5 + \bar{\epsilon}$. Thus $0.3 \succ_i 0.5 + \bar{\epsilon}$ contradicting (ii).

Agent	1	2	3	4	5	6
Peak	0.7	0.7	0.41	0.41	0.41	0.41

Table 1.13: Peaks of agents in Example 9.

We claim that stable allocations do not exist.
Consider an arbitrary allocation. If it is stable, agents 1 and 2 are not paired together. If they are, one of them, say 1, gets a contribution of at most 0.5. One of the agents in $\{3,4,5,6\}$, say 3, has a contribution of at least 0.5. Then the pair $(1,3)$ blocks with $(0.51, 0.49)$.
We can therefore assume w.l.o.g. that the pairs $(1,3)$, $(2,4)$ and $(5,6)$ belong to the allocation. By feasibility, one of the agents in $\{5,6\}$, say 5 has a contribution $t_5 \geq 0.5$.
Let (t_1, t_3) be the contribution vector for the pair $(1,3)$. In order for $(1,3)$ not to block, we must have $0.59 \leq t_1 \leq 0.7$ and $0.3 \leq t_3 \leq 0.41$. There are two cases to consider.
The first is when $t_3 > 0.3$. Since $t_3 \leq 0.41$, we can find $\delta > 0$ small enough such that $t_3 - \delta \in (0.3, 0.41)$ and $t_1 + \delta < 0.7$. By assumption (i), $t_3 - \delta \succ_5 t_5$ as $t_5 \geq 0.5$. Therefore the pair $(1,5)$ can block with $(t_1 + \delta, t_3 - \delta)$.
The remaining case is $t_3 = 0.3$. If $t_5 > 0.5$, the pair $(3,5)$ blocks with $(0.5, 0.5)$. Agent 3 strictly improves as $0.5 \succ_3 0.3$ (Assumption (ii) and single-peakedness). Agent 5 strictly improves as she moves closer to her peak. Suppose $t_5 = 0.5$. Pick $0 < \epsilon < \bar{\epsilon}$ where $\bar{\epsilon}$ is specified in Assumption (ii). By single-peakedness and Assumption (ii), $0.5 + \epsilon \succ_3 0.5 + \bar{\epsilon} \succ_3 0.3$. Hence $(3,5)$ blocks with $(0.5 + \epsilon, 0.5 - \epsilon)$. □

1.5.3 Coalitions of arbitrary size

The example below shows that stable allocations may not exist if coalitions of arbitrary size are permitted. Agents in a coalition have to make an aggregate contribution of 1.

Example **10.** Let $N = \{1,2,3,4\}$. Agents' preferences are symmetric. Table 1.14 summarizes the peaks of the agents.

p_1	p_2	p_3	p_4
0.55	0.55	0.55	1

Table 1.14: Peaks of agents in Example 10.

In any stable allocation, agent 4 must have a contribution of 1. Suppose agent 4 belongs to a coalition C with some other agents. All these agents will have a contribution of 0. If $|C| = 4$ or $|C| = 3$, then any two agents from $C \setminus \{4\}$ will block with the contribution vector $(0.5, 0.5)$. Assume w.l.o.g. $C = \{1,4\}$. One of the agents in the set $\{2,3\}$ (say 2) does not get her peak. The pair $(1,2)$ blocks with $(0.45, 0.55)$. Finally, consider the case where $C = \{4\}$. If 1, 2 and 3 belong to the same coalition, there exists an agent $i \in \{1,2,3\}$ with $t_i \leq \frac{1}{3}$. Also, there is at most one agent who receives her peak, i.e there exists $j \neq i$ with $t_j \neq 0.55$. The pair (i,j) blocks with $(0.45, 0.55)$. In all other remaining cases, there exists an agent i who is on her own (her contribution is 1) and another agent j who does not get her peak. Then (i,j) can block with $(0.45, 0.55)$. □

Our negative result in the case of arbitrary coalitions bears a resemblance to some earlier results on stability in division problems. Gensemer et al. (1996) consider the problem of allocating agents with single-peaked preferences across a set of islands. Each island has a unit amount of resource and operates with a fixed division rule. It can also accommodate an arbitrary number of agents. The paper formulates a notion of migration equilibrium according to which no agent can benefit by migrating to another island. No island has the right to refuse an entrant. The paper provides a number of negative results about the existence of migration equilibria.

Bergantiños et al. (2015) consider a related model where all islands use the same division rule. They also weaken the equilibrium condition to a stability notion - in order for successful blocking to take place, the migrant's well-being must strictly improve while no member of the receiving island is made strictly worse-off by the move. The paper shows that stable allocations exist for some special division rules such as the proportional rule and the sequential dictatorship rule, provided agents' preferences are symmetric.

1.5.4 Strategy-proofness

An allocation rule is strategy-proof if no agent can strictly improve by misrepresenting her preferences. This property ensures that the mechanism designer can achieve the allocation specified at a preference profile by relying on the reports of the agents themselves.

We show by an example that the SAM algorithm is not strategy-proof.[14]

Example **11.** Let $N = \{1,2,3,4\}$. Agents' preferences $\succsim_i$ are symmetric and Table 1.15 summarizes their peaks. The ordering of the agents is $1 \succ^N 2 \succ^N 3 \succ^N 4$.

p_1	p_2	p_3	p_4
0.41	0.42	0.43	0.6

Table 1.15: Peaks of agents in Example 11.

In Step 0, agents 2 and 3 are removed from L and paired together. This is because $e_3(0.5) > e_2(0.5) > e_1(0.5)$. The triple $(2,3,0.5)$ is added to the allocation. The set $D_1 = (0.34, 0.5)$ and $U_1 = \emptyset$. Since $1 - p_4 = 0.4 > \inf D_1 = 0.34$, Substep 1.1 applies. The pair $(1,4)$ is formed and $t_1 = \min\{1 - p_4, \inf U_1\} = 0.4$. The triple $(1,4,0.4)$ belongs to the allocation. The allocation generated by the SAM algorithm is $(2,3,0.5)$, $(1,4,0.4)$.

Suppose agent 2 reports a peak of 0.4 and symmetric preferences $\succsim_2'$. Now in Step 0, the triple $(1,3,0.5)$ is formed and $D_1 = (0.32, 0.5)$. Substep 1.1 applies as $1 - p_4 > \inf D_1 =$

[14] According to the definition, the algorithm specifies an allocation at a preference profile. We are slightly abusing terms here by regarding the algorithm as an allocation rule.

0.32. The triple $(2,4,0.4)$ is formed. Agent 2 strictly improves at $\succsim_2$ by misreporting since $0.4 \succ_2 0.5$. □

The existence of a strategy-proof, stable and Pareto efficient rule in our model remains an open question. However, in Chapter we will prove the non-existence of a strategy-proof rule that maximizes the sum of agents' utilitarian welfare in the case of symmetric preferences.

1.5.5 Non-convexity of the set of stable allocations

The next example shows that a convex combination of two stable allocations with the same set of matched pairs may not be stable.

Example **12.** Let $N = \{1,2,3,4\}$. The peaks of the agents are summarized in Table 1.16. All agents have symmetric preferences.

p_1	p_2	p_3	p_4
0.3	0.3	0.8	0.9

Table 1.16: Peaks of agents in Example 12.

It is easy to verify that the allocations $\sigma^1 = \{(1,4,0.3),(2,3,0.3)\}$ and $\sigma^2 = \{(1,4,0.2),(2,3,0.1)\}$ are both stable. However the allocation $\sigma^3 = \{(1,4,0.25),(2,3,0.2)\}$ is not stable because the pair $(2,4)$ can block with $(0.21,0.79)$. Note that σ^3 has the same matched pairs as σ^1 and σ^2 but the contribution vector of each pair is a convex combination of the respective contributions in σ^1 and σ^2 with weights $(0.5,0.5)$. □

1.6 Appendix

In this section, we provide a proof of Theorem 1. The proof is divided into two parts - Subsection 1.6.1 contains the proof of stability and Subsection 1.6.2 contains the proof of Pareto efficiency.

1.6.1 Stability

We show that the SAM algorithm generates a stable allocation. We begin with a few key observations.

Recall that $D_{q+1} = D_q \cup I_{l_q,t_{l_q}}$ and $U_{q+1} = U_q \cup I_{h_q,t_{h_q}}$ for $q \in \{0,1,\ldots,K\}$. For every step q of the algorithm where $q \in \{1,\ldots,K\}$, we have $D_q = \cup_{r<q} I_{l_r,t_{l_r}}$ and $U_q = \cup_{r<q} I_{h_r,t_{h_r}}$. Since the improvement sets are open (see Observation 1), it follows that D_q and U_q are also open in $[0,0.5]$.

The sets D_q and U_q can be written as the disjoint union of their connected components. Since D_q and U_q are open sets, none of their connected components are singletons - thus each connected component of D_q and U_q is an interval in $[0, 0.5]$. Moreover the connected components of D_q and U_q can be ordered from "left" to "right". Let D_q^1 and U_q^1 denote the "leftmost" connected components of D_q and U_q respectively. By definition, $\inf D_q = \inf D_q^1$ and $\sup D_q^1 \leq \inf D_q^r$ for any component D_q^r other than D_q^1. Similar inequalities hold for U_q^1. Incase D_q or U_q is empty (then D_q^1 or U_q^1 do not exist), we adopt the convention that the infimum and supremum of D_q^1 and U_q^1 is $+\infty$.

Observation **2.** Consider step q where $q \in \{1, \dots, K\}$. Recall that the triple formed in this step is (l_q, h_q, t_{l_q}). If $p_{l_q} \geq \inf D_q^1$ then $\sup D_q^1 \leq \sup D_{q+1}^1$. Similarly, if $1 - p_{h_q} \geq \inf U_q^1$ then $\sup U_q^1 \leq \sup U_{q+1}^1$. This is an immediate consequence of the definition of improvement sets.

We now establish a series of results that are loop invariants of the algorithm.

Lemma **1.** *Fix $q \in \{1, \dots, K\}$ and assume $D_q \cap U_q = \emptyset$. Then for all $q \in \{1, \dots, K\}$, we have*

$$[\forall h \in H_q,\ 1 - p_h < \sup U_q^1] \textit{ and } [\forall l \in L_q,\ p_l < \sup D_q^1].$$

Proof: We will prove the lemma by induction on q.

- Base Case ($q = 1$): There are two cases to consider - $U_1 = \emptyset$ and $U_1 \neq \emptyset$. If the former holds, then $[\forall h \in H_1,\ 1 - p_h < \sup U_1^1]$ is true since $\sup U_1^1 = +\infty$. Suppose $U_1 \neq \emptyset$. All agents allocated in Step 0 have a contribution of 0.5. Hence $\sup U_1^1 = 0.5$ and $1 - p_h \leq 0.5 = \sup U_1^1$ for all $h \in H$. Suppose there exists an agent $h' \in H_1$ and $1 - p_{h'} = 0.5$, i.e. $e(p_{h'}) = 0.5$. Since $U_1 \neq \emptyset$, there exists an agent $\bar{h}$ allocated in Step 0 for whom $p_{\bar{h}} > 0.5$, i.e. $e_{\bar{h}}(0.5) < 0.5$. But then h' has higher priority than $\bar{h}$ in H and should have been allocated in Step 0. Therefore $[\forall h \in H_1,\ 1 - p_h < \sup U_1^1]$ holds. The argument for L_1 is identical and omitted.

- Inductive step: Consider $q \in \{1, \dots, K\}$. Assume $D_q \cap U_q = \emptyset$, $[\forall h \in H_q,\ 1 - p_h < \sup U_q^1]$ and $[\forall l \in L_q,\ p_l < \sup D_q^1]$. We have to show

$$[\forall h \in H_{q+1},\ 1 - p_h < \sup U_{q+1}^1] \text{ and } [\forall l \in L_{q+1},\ p_l < \sup D_{q+1}^1].$$

We refer to $[\forall h \in H_{q+1},\ 1 - p_h < \sup U_{q+1}^1]$ and $[\forall l \in L_{q+1},\ p_l < \sup D_{q+1}^1]$ as Statements A and B respectively. There are two cases to consider depending on whether D_q^1 lies to the left or to the right of U_q^1.

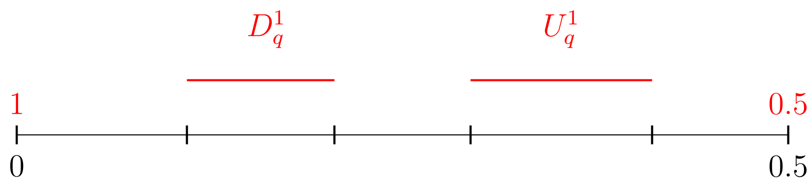

Figure 1.3: Case I in the proof of Lemma 1.

Case I: $\sup D_q^1 \leq \inf U_q^1$ (see Figure 1.3). There are two sub-cases to consider depending on the selection of the primary agent.

Case I.a: The primary agent is h_q. If $1 - p_{h_q} > \inf U_q^1$, Substep $q.1$ applies and the contribution of h_q is $1 - \inf U_q^1$. By Observation 2, $\sup U_q^1 \leq \sup U_{q+1}^1$. Statement A follows from the induction hypothesis and $H_{q+1} \subset H_q$. If $1 - p_{h_q} \leq \inf U_q^1$, then the contribution of h_q is p_{h_q}. Consequently the improvement set of h_q is empty and $\sup U_q^1 = \sup U_{q+1}^1$. Statement A once again follows from the induction hypothesis and $H_{q+1} \subset H_q$.

We now prove Statement B for the secondary agent l_q. By the induction hypothesis, $p_{l_q} < \sup D_q^1$. If $p_{l_q} \geq \inf D_q^1$, then $p_{l_q} \in D_q^1$. Observation 2 implies $\sup D_q^1 \leq \sup D_{q+1}^1$ and Statement B holds following the earlier argument. Suppose $p_{l_q} < \inf D_q$. There are two possibilities depending on the location of the peak of h_q.

(i) If $1 - p_{h_q} > \inf D_q^1$, then Substep $q.1$ is applicable. Here $t_{l_q} \geq \inf D_q^1$ as $t_{l_q} = \min\{1 - p_{h_q}, \inf U_q\}$. This implies $\sup D_q^1 \leq \sup D_{q+1}^1$ and Statement B follows from the induction hypothesis.

(ii) The remaining case is when $1 - p_{h_q} \leq \inf D_q^1$. This can happen only if Substep $q.3.1$ occurs. In particular, we have $p_l \leq 1 - p_{h_q}$ for all $l \in L_q$. So $p_{l_q} \leq 1 - p_{h_q}$. The contribution of agent l_q is $t_{l_q} = 1 - p_{h_q}$. If $p_{l_q} = 1 - p_{h_q}$, then the improvement set of l_q is empty and Statement B follows from the induction hypothesis and the fact that $D_{q+1}^1 = D_q^1$. Suppose $p_{l_q} < 1 - p_{h_q} = t_{l_q}$. Then $\sup D_{q+1}^1 = t_{l_q}$ and $p_l \leq \sup D_{q+1}^1$ for all $l \in L_q$. Suppose there exists $\bar{l} \in L_q$ with $p_{\bar{l}} = \sup D_{q+1}^1 = t_{l_q}$. Then $e_{\bar{l}}(t_{l_q}) = t_{l_q}$. However $e_{l_q}(t_{l_q}) < t_{l_q}$ as $p_{l_q} < t_{l_q}$. Then $\bar{l}$ should have been matched with h_q in Step q instead of l_q. This establishes Statement B in this case.

Case I.b: The primary agent is l_q. This case is symmetric to the case when h_q is the primary agent. If $p_{l_q} \geq \inf D_q^1$, then the result follows from Observation 2. If $p_{l_q} < \inf D_q^1$, then l_q receives her peak and the improvement set is empty. Statement B again follows immediately.

Here h_q is the secondary agent. This can happen only if Step $q.3.2$ occurs. In particular, $1 - p_h < p_{l_q}$ and $1 - p_h \leq \inf D_q$ for all $h \in H_q$. Note that $t_{l_q} = \min\{p_{l_q}, \inf D_q\}$. Thus

$1 - p_h \leq t_{l_q}$ for all $h \in H_q$ and $\sup U_{q+1}^1 = t_{l_q}$. Using arguments like those in Case I.a.(ii) above, we can show $1 - p_h < t_{l_q}$ for all $h \in H_q$. This establishes Statement A.

Case II: $\sup U_q^1 \leq \inf D_q^1$ (see Figure 1.4). Once again there are two cases, depending upon the selection of the primary agent.

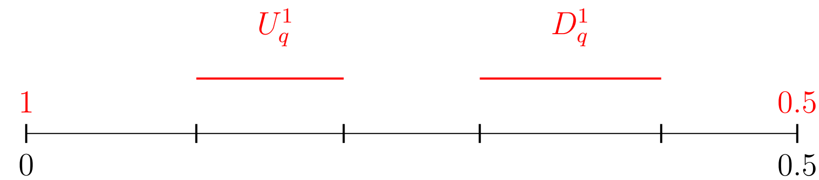

Figure 1.4: Case II in the proof of Lemma 1.

The arguments for the primary agent are symmetric to those for the primary agent in Case I. This is also true for the secondary agent. For instance, consider the case where l_q is the secondary agent. This occurs only if Step q.3.1 occurs. In particular, $p_l \leq 1 - p_{h_q}$ and $p_l \leq \inf U_q$ for all $l \in L_q$. So $p_l \leq t_{l_q}$ for all $l \in L_q$. Hence $\sup D_{q+1}^1 = t_{l_q}$. Using arguments similar to those in Case I.b.(ii), we can verify Statement B.

We omit the other arguments.

■

Lemma **2.** *Consider Step q where $q \in \{1, \ldots, K\}$. Assume $D_q \cap U_q = \emptyset$. Then the triple formed in Step q satisfies the following:*

1. *if it is formed in Substep q.1, then either $p_{l_q} \leq t_{l_q} \leq 1 - p_{h_q}$ or $t_{l_q} = 1 - p_{h_q} < p_{l_q}$.*
2. *if it is formed in Substep q.2, then either $1 - p_{h_q} \leq t_{l_q} \leq p_{l_q}$ or $t_{l_q} = p_{l_q} < 1 - p_{h_q}$.*
3. *if it is formed in Substep q.3.1, then $p_{l_q} \leq t_{l_q} \leq 1 - p_{h_q}$.*
4. *if it is formed in Substep q.3.2, then $1 - p_{h_q} \leq t_{l_q} \leq p_{l_q}$.*

Proof: Since $D_q \cap U_q = \emptyset$, it must be the case that D_q^1 either lies entirely to the "left" of U_q^1 or entirely to the "right" of U_q^1. We now consider each of the four cases in turn.

1. Suppose the triple (l_q, h_q, t_{l_q}) is formed in Substep q.1. Lemma 1 rules out the case where D_q^1 lies entirely to the right of U_q^1.

There are now two possibilities. The first is $1 - p_{h_q} < p_{l_q}$. By Lemma 1, we know $p_{l_q} < \sup D_q^1$. Since $t_{l_q} = \min\{1 - p_{h_q}, \inf U_q\}$ and $\sup D_q^1 \leq \inf U_q$, we have $t_{l_q} = 1 - p_{h_q}$. Thus $t_{l_q} = 1 - p_{h_q} < p_{l_q}$.

The second possibility is $1 - p_{h_q} \geq p_{l_q}$. If $1 - p_{h_q} \geq \inf U_q$, we have $t_{l_q} = \inf U_q$. By Lemma 1, $p_{l_q} < \sup D_q^1$. Thus $p_{l_q} \leq t_{l_q} \leq 1 - p_{h_q}$. If $1 - p_{h_q} < \inf U_q$, we have $t_{l_q} = 1 - p_{h_q}$ and once again $p_{l_q} \leq t_{l_q} = 1 - p_{h_q}$.

2. Suppose the triple (l_q, h_q, t_{l_q}) is formed in Substep $q.2$. In this case, Lemma 1 implies D_q^1 must lie to the right of U_q^1. We can use the symmetric counterparts of the arguments in Case 1 to derive the necessary conclusion.

3. Suppose the triple (l_q, h_q, t_{l_q}) is formed in Substep $q.3.1$. Here $t_{l_q} = \min\{\inf U_q, 1-p_{h_q}\}$. By the hypothesis of Substep $q.3.1$, we know $p_l \leq \inf U_q$ for all $l \in L_q$. Also $p_l \leq 1-p_{\tilde{h}_q} = 1-p_{h_q}$ for all $l \in L_q$.[15] In particular, $p_{l_q} \leq \inf U_q$ and $p_{l_q} \leq 1-p_{h_q}$. Thus $p_{l_q} \leq \min\{\inf U_q, 1-p_{h_q}\} = t_{l_q} \leq 1-p_{h_q}$.

4. Suppose the triple (l_q, h_q, t_{l_q}) is formed in Substep $q.3.2$. Here $t_{l_q} = \min\{\inf D_q, p_{l_q}\}$. We can use the symmetric counterparts of the arguments in Case 3 to derive the necessary conclusion. ■

Lemma 2 immediately leads to the following corollary.

Corollary **1.** *Consider Step q where $q \in \{1, \ldots, K\}$. Assume $D_q \cap U_q = \emptyset$. Then the contribution of agent l_q, t_{l_q} lies in the closed interval with the end points p_{l_q} and $1-p_{h_q}$.*

Lemma **3.** *Consider Step q where $q \in \{1, \ldots, K\}$. Assume $D_q \cap U_q = \emptyset$.*

1. If $t_{l_q} \leq p_{l_q}$, then $I_{l_q, t_{l_q}} \subseteq D_q$.

2. If $t_{l_q} \leq 1-p_{h_q}$, then $I_{h_q, t_{h_q}} \subseteq U_q$.

Proof: We only consider Part 1 - a symmetric argument applies for Part 2. Consider Step q where the triple (l_q, h_q, t_{l_q}) is formed.

If $p_{l_q} = t_{l_q}$, the improvement set of l_q is empty and the result follows immediately. Assume therefore that $t_{l_q} < p_{l_q}$. By Corollary 1, we have $1-p_{h_q} \leq t_{l_q} < p_{l_q}$.

We will argue that D_q is non-empty when $t_{l_q} < p_{l_q}$. By Lemma 2, we know Substep $q.3.1$ cannot occur as $t_{l_q} < p_{l_q}$. We establish the claim for Substeps $q.1$, $q.2$ and $q.3.2$. If Step $q.1$ occurs, we know $1-p_{h_q} > \inf D_q$. Thus $\inf D_q$ is not $+\infty$ and D_q is non-empty. If Step $q.2$ or Step $q.3.2$ occurs, we know $t_{l_q} = \min\{p_{l_q}, \inf D_q\}$. Since $t_{l_q} < p_{l_q}$, we have $t_{l_q} = \inf D_q$ and thus D_q is non-empty.

Claim 1: If $t_{l_q} < p_{l_q}$, then $t_{l_q} \geq \inf D_q$.

Proof: Lemma 2 implies that the triple cannot be formed in Substep $q.3.1$. We establish the claim for Substeps $q.1$, $q.2$ and $q.3.2$.
Suppose the triple is formed in Substep $q.1$. Since $D_q \cap U_q = \emptyset$ by assumption, Lemma 1 implies D_q^1 lies entirely to the left of U_q^1 and $\sup D_q^1 \leq \inf U_q^1$. In Step $q.1$, we know $1-p_{h_q} > \inf D_q$ and $t_{l_q} = \min\{1-p_{h_q}, \inf U_q\}$. Thus $t_{l_q} > \inf D_q$.
Suppose the triple is formed in Substep $q.2$. Here $t_{l_q} = \min\{p_{l_q}, \inf D_q\}$. Since $t_{l_q} < p_{l_q}$ by assumption, it must be the case that $t_{l_q} = \inf D_q$.

[15] Recall $\tilde{h}_q$ is the high type agent with the lowest peak in h_q and $h_q = \tilde{h}_q$ in Substep $q.3.1$.

Suppose the triple is formed in Substep $q.3.2$. Here $t_{l_q} = \min\{p_{l_q}, \inf D_q\}$. Since $t_{l_q} < p_{l_q}$ by assumption, it follows that $t_{l_q} = \inf D_q$. This completes the proof of the claim.

■

Since D_q is a finite union of intervals, $\inf D_q$ is the infimum of at least one of these intervals. Thus $\inf D_q$ is attained in a step before q. Let i be the smallest integer such that $\inf I_{l_i,t_{l_i}} = \inf D_q$ where l_i is a low type agent matched in Step i.[16] Note that $\inf D_j > \inf D_q$ for $j \in \{1, \ldots, i\}$ and $\inf D_{i+1} = \inf D_q$ when $i \geq 1$. If $i = 0$, then $\inf D_1 = \inf D_q$.
Consider Step i where agent l_i is matched and her contribution is t_{l_i}. We claim that $t_{l_i} > p_{l_i}$. Suppose not. If $t_{l_i} = p_{l_i}$, the improvement set of l_i is empty, contradicting the assumption that $\inf I_{l_i,t_{l_i}} = \inf D_q$. Suppose $t_{l_i} < p_{l_i}$. Here $\inf I_{l_i,t_{l_i}} = t_{l_i}$. Arguing as we did to establish Claim 1, it follows $t_{l_i} \geq \inf D_i$ in Step i. Therefore $t_{l_i} = \inf D_q \geq \inf D_i$. This leads to a contradiction since $\inf D_i > \inf D_q$.
Since $t_{l_i} > p_{l_i}$, we have $e_{l_i}(t_{l_i}) \leq \inf I_{l_i,t_{l_i}} = \inf D_q$.[17]

Claim 2: For agents l_i and l_q, we have (i) $e_{l_q}(t_{l_i}) \leq e_{l_i}(t_{l_i}) \leq \inf D_q$ and (ii) $p_{l_q} < t_{l_i}$.

Proof: Recall that agent l_i is matched in Step i.
Suppose $i = 0$. Then it must be the case that $|L| > |H|$ and $D_1 \neq \emptyset$. Every agent matched in this step has a contribution of 0.5. There exists $l_0 \in \bar{L}$ such that $\inf I_{l_0,0.5} = \inf D_1$. Since the improvement set of l_0 is non-empty, $p_{l_0} < 0.5$ and $e_{l_0}(0.5) < 0.5$. As l_q is not matched in Step 0, we have $e_{l_q}(0.5) \leq e_{l_0}(0.5) < 0.5$. Hence $p_{l_q} < 0.5$. This establishes Claim 2 for $i = 0$.
Suppose $i \geq 1$. Recall that $t_{l_i} > p_{l_i}$. By Lemma 2, we know that l_i is matched in Substep $i.1$ or Substep $i.3.1$. In both cases, l_i is the secondary agent. Also $l_q \in L_i$ as $i < q$. Thus $e_{l_q}(t_{l_i}) \leq e_{l_i}(t_{l_i})$. This establishes Part (i).
Now consider Part (ii). If $p_{l_q} = t_{l_i}$, then $e_{l_q}(t_{l_i}) = t_{l_i}$. Since $p_{l_i} < t_{l_i}$, then $e_{l_i}(t_{l_i}) < t_{l_i}$ and l_i should not have been chosen as the secondary agent in Step i. If $p_{l_q} > t_{l_i}$, we know $e_{l_q}(t_{l_i}) > t_{l_i}$. Once again l_i should not have been chosen as the secondary agent in Step i. This establishes Part (ii). ■

We now return to the proof of the lemma. Claims 1, 2 and the assumption $t_{l_q} < p_{l_q}$ imply

$$e_{l_q}(t_{l_i}) \leq e_{l_i}(t_{l_i}) \leq \inf D_q \leq t_{l_q} < p_{l_q} < t_{l_i}.$$

Since $\succsim_{l_q}$ is single-peaked, we have $t_{l_q} \succsim_{l_q} \max\{0, e_{l_q}(t_{l_i})\}$. Also $\max\{0, e_{l_q}(t_{l_i})\} \succsim_{l_q} t_{l_i}$.[18] Thus $t_{l_q} \succsim_{l_q} t_{l_i}$ and $t_{l_i} \notin I_{l_q,t_{l_q}}$. Consequently $\sup I_{l_q,t_{l_q}} \leq t_{l_i} = \sup I_{l_i,t_{l_i}}$. We have already shown $\inf I_{l_i,t_{l_i}} = \inf D_q \leq t_{l_q} = \inf I_{l_q,t_{l_q}}$. Therefore $I_{l_q,t_{l_q}} \subseteq I_{l_i,t_{l_i}} \subseteq D_q$.

[16] If $i = 0$, note that l_i may not be unique. If $i \geq 1$, then l_i is unique.
[17] Note that $e_{l_i}(t_{l_i}) < \inf I_{l_i,t_{l_i}}$ if and only if $e_{l_i}(t_{l_i}) = -\epsilon$. Here $I_{l_i,t_{l_i}} = [0, t_{l_i})$.
[18] If $\max\{0, e_{l_q}(t_{l_i})\} = 0$, then $e_{l_q}(t_{l_i}) = -\epsilon$ and $I_{l_i,t_{l_i}} = [0, t_{l_i})$. Here $0 \succ_{l_q} t_{l_i}$.

■

We complete the proof of the theorem by showing that the SAM algorithm generates an allocation which satisfies Condition S (Proposition 3).

Proof: Recall that the algorithm terminates in K steps, generating the sets D_{K+1} and U_{K+1}. We will show that $D_{K+1} \cap U_{K+1} = \emptyset$ and $0.5 \notin D_{K+1} \cup U_{K+1}$. This guarantees that Condition S is satisfied by the allocation generated.[19]
In fact, we will show that $D_q \cap U_q = \emptyset$ and $0.5 \notin D_q \cup U_q$ holds for all $q \in \{1, \ldots, K+1\}$. We will use induction on q for this purpose.
We claim that $D_1 \cap U_1 = \emptyset$ and $0.5 \notin D_1 \cup U_1$. By construction, at least one of the sets D_1 and U_1 is empty. Also the contribution of any agent matched in Step 0 is 0.5. Thus the improvement sets of agents matched in this step do not contain 0.5.

Induction Hypothesis (IH): Fix $q \in \{1, \ldots, K\}$ and assume (i) $D_q \cap U_q = \emptyset$ and (ii) $0.5 \notin D_q \cup U_q$.

The IH implies that the antecedents of Lemmas 1, 2, 3 and Corollary 1 are satisfied. We show that $D_{q+1} \cap U_{q+1} = \emptyset$ and $0.5 \notin D_{q+1} \cup U_{q+1}$. We have

$$\begin{aligned} D_{q+1} \cap U_{q+1} &= (D_q \cup I_{l_q,t_{l_q}}) \cap (U_q \cup I_{h_q,t_{h_q}}) \\ &= (D_q \cap U_q) \cup (D_q \cap I_{h_q,t_{h_q}}) \cup (I_{l_q,t_{l_q}} \cap U_q) \cup (I_{l_q,t_{l_q}} \cap I_{h_q,t_{h_q}}) \end{aligned}$$

From IH, it follows $D_q \cap U_q = \emptyset$. By Corollary 1 and the fact that the improvement sets are open intervals, we have $I_{l_q,t_{l_q}} \cap I_{h_q,t_{h_q}} = \emptyset$. We will show (A) $I_{l_q,t_{l_q}} \cap U_q = \emptyset$, $0.5 \notin I_{l_q,t_{l_q}}$ and (B) $I_{h_q,t_{h_q}} \cap D_q = \emptyset$, $0.5 \notin I_{h_q,t_{h_q}}$.
We first prove (A). There are two cases to consider depending on the contribution of agent l_q. The first case is when $t_{l_q} \leq p_{l_q}$. Part 1 of Lemma 3 implies $I_{l_q,t_{l_q}} \subseteq D_q$. Thus $I_{l_q,t_{l_q}} \cap U_q = \emptyset$ and $0.5 \notin I_{l_q,t_{l_q}}$ since $D_q \cap U_q = \emptyset$ and $0.5 \notin D_q$ by IH.
In the second case, $p_{l_q} < t_{l_q}$. Here $t_{l_q} = \sup I_{l_q,t_{l_q}}$. Lemma 2 implies only Substeps q.1 and q.3.1 can occur. In both steps, we have $t_{l_q} = \min\{1 - p_{h_q}, \inf U_q\}$. Since $1 - p_{h_q} \leq 0.5$, it follows that $t_{l_q} \leq \min\{\inf U_q, 0.5\}$. Furthermore, $t_{l_q} = \sup I_{l_q,t_{l_q}}$ and $I_{l_q,t_{l_q}}$ is an open set. Therefore $x < t_{l_q} \leq \min\{\inf U_q, 0.5\}$ for all $x \in I_{l_q,t_{l_q}}$. Thus $I_{l_q,t_{l_q}} \cap U_q = \emptyset$ and $0.5 \notin I_{l_q,t_{l_q}}$.
The proof of (B) is virtually identical to the arguments for (A), but uses Part 2 of Lemma 2. We omit the details. This completes the proof of the theorem. ■

We have shown above that the sets D_q and U_q do not intersect for any $q \in \{1, \ldots, K\}$. Thus the antecedents of Lemmas 1, 2, 3 and Corollary 1 are true. We will use these facts in the proof of Pareto efficiency in the next subsection.

[19]Recall D_{K+1} and U_{K+1} are the unions of the improvement sets of all L and H type agents respectively.

1.6.2 Pareto Efficiency

We define some notation which will be used in the proof of Pareto efficiency. For any allocation σ and agent $i \in N$, we shall denote the contribution of agent i in σ by t_i^σ. The improvement set for agent i at t_i^σ is I_{i,t_i^σ} and it's closure is $\overline{I_{i,t_i^\sigma}}$. Note that $\overline{I_{i,t_i^\sigma}} = \{x \in [0, 0.5] : x \succsim_i t_i^\sigma\}$ when $i \in L$ and $\overline{I_{i,t_i^\sigma}} = \{x \in [0, 0.5] : 1 - x \succsim_i t_i^\sigma\}$ when $i \in H$. For the allocation σ, we define sets $D^\sigma = \cup_{i \in L} I_{l,t_l^\sigma}$ and $U^\sigma = \cup_{i \in H} I_{l,t_h^\sigma}$.

We first establish several key observations and lemmas.

Observation **3.** In any stable allocation, agents who are paired together must be given contributions on the same side of the peak. We refer to this property as *internal stability* for the pair of agents who are matched together in the allocation. Internal stability is a necessary condition for both stability and Pareto efficiency.

Our next step is to establish a monotonicity lemma.

Lemma **4.** *For any Step q where $q \in \{1, \dots, K-1\}$, we have (i) $t_{l_{q+1}} \leq t_{l_q} \leq 0.5$ and (ii) $0.5 \leq t_{h_q} \leq t_{h_{q+1}}$.*

Proof: We first prove Part (i). Suppose $q = 1$. The allocation to agent l_1 in Step 1 is either $\min\{\inf D_1, p_{l_1}\}$ or $\min\{\inf U_1, 1 - p_{h_1}\}$. Both these values are smaller or equal to 0.5.
Suppose $q \geq 1$. The triple (l_q, h_q, t_{l_q}) is formed in Step q. Next we show that $t_{l_q+1} \leq t_{l_q}$. We have to consider four cases based on which substep occurs in Step q.

Case 1: Suppose the allocation is made in Substep $q.1$.

Here $t_{l_q} = \min\{\inf U_q, 1 - p_{h_q}\} \leq 1 - p_{h_q}$. By Lemma 3, we know $U_{q+1} = U_q$ and $\inf U_{q+1} = \inf U_q$.
Lemma 1 and the hypothesis of Substep $q.1$ imply $\inf D_q < 1 - p_{h_q} < \sup U_q^1$. Since $D_q \cap U_q = \emptyset$, it must be the case that D_q^1 lies to the left of U_q^1, i.e. $\inf D_q < \inf U_q$. Thus $\inf D_q < \min\{\inf U_q, 1 - p_{h_q}\} = t_{l_q}$.
Now we consider the triple $(l_{q+1}, h_{q+1}, t_{l_{q+1}})$ formed in Step $q+1$. There are four possibilities to consider.

1. The allocation is made in Substep $q + 1.1$. By hypothesis of the substep, we have $\inf D_{q+1} < 1 - p_{h_{q+1}}$. If $1 - p_{h_{q+1}} \leq \inf D_q$, then $1 - p_{h_q} < \inf U_q = \inf U_{q+1}$ and $t_{l_{q+1}} = 1 - p_{h_{q+1}}$. Thus $t_{l_{q+1}} = 1 - p_{h_{q+1}} \leq \inf D_q < t_{l_q}$.

   If $1 - p_{h_{q+1}} > \inf D_q$, then $h_{q+1} \in \{h \in H_q : 1 - p_h > \inf D_q\}$. Since h_q is matched in Step q, it must be the case that $1 - p_{h_{q+1}} \leq 1 - p_{h_q}$. Recall $\inf U_{q+1} = \inf U_q$. Thus $t_{l_{q+1}} = \min\{\inf U_{q+1}, 1 - p_{h_{q+1}}\} \leq \min\{\inf U_q, 1 - p_{h_q}\} = t_{l_q}$.

2. The allocation is made in Substep $q+1.2$. Note than in Step q, D_q^1 lies to the left of U_q^1. Since $D_q \cap U_q = \emptyset$, we have $\sup D_q^1 \leq \inf U_q^1$. Thus $p_l < \sup D_q^1 \leq \inf U_q^1$ for all $l \in L_q$ (Lemma 1). Since $U_{q+1} = U_q$ and $L_{q+1} \subset L_q$, there does not exist $l \in L_{q+1}$ such that $p_l > \inf U_{q+1}$. So Substep $q+1.2$ is not possible.

3. The allocation is made in Substep $q+1.3.1$. By the hypothesis of the substep, we know $1 - p_{h_{q+1}} \leq \inf D_{q+1}$. Note that $\inf D_{q+1} \leq \inf D_q$.[20] Thus

$$t_{l_q+1} = \min\{\inf U_{q+1}, 1 - p_{h_{q+1}}\} \leq \inf D_{q+1} \leq \inf D_q < t_{l_q}.$$

4. The allocation is made in Substep $q+1.3.2$. Here $t_{l_{q+1}} = \min\{\inf D_{q+1}, p_{l_{q+1}}\}$. Also $\inf D_{q+1} \leq \inf D_q$. Thus

$$t_{l_{q+1}} \leq \inf D_{q+1} \leq \inf D_q < t_{l_q}.$$

Case 2: The allocation is made in Substep $q.3.1$.
Here $t_{l_q} = \min\{\inf U_q, 1 - p_{h_q}\} \leq 1 - p_{h_q}$. By Lemma 3, we have $U_{q+1} = U_q$ and $\inf U_{q+1} = \inf U_q$.
Consider the allocation is made in Step $q+1$. There are four possibilities.

1. The allocation is done in Substep $q+1.1$. Here $t_{l_{q+1}} = \min\{\inf U_{q+1}, 1 - p_{h_{q+1}}\} \leq 1 - p_{h_{q+1}}$. Since agent h_q is matched in Step q and $h_{q+1} \in H_q$, it must be the case that $1 - p_{h_{q+1}} \leq 1 - p_{h_q}$. Thus

$$t_{l_{q+1}} = \min\{\inf U_{q+1}, 1 - p_{h_{q+1}}\} \leq \{\inf U_q, 1 - p_{h_q}\} = t_{l_q}.$$

2. The allocation is made in Subtep $q+1.2$. We will show that this is not possible. Since the first allocation is done in Substep $q.3.1$, we know $p_l \leq \inf U_q$ for all $l \in L_q$. Note that $\inf U_{q+1} = \inf U_q$ and $L_{q+1} \subset L_q$. Thus there does not exist $l \in L_{q+1}$ such that $p_l > \inf U_{q+1}$.

3. The allocation is made in Substep $q+1.3.1$. Here $t_{l_{q+1}} = \min\{\inf U_{q+1}, 1 - p_{h_{q+1}}\}$. Since $h_{q+1} \in H_q$ and agent h_q is matched in Substep $q.3.1$, we have $1 - p_{h_{q+1}} \leq 1 - p_{h_q}$. Thus

$$t_{l_{q+1}} = \min\{\inf U_{q+1}, 1 - p_{h_{q+1}}\} \leq \{\inf U_q, 1 - p_{h_q}\} = t_{l_q}.$$

4. The allocation is made in Substep $q+1.3.2$. Here $t_{l_{q+1}} = \min\{\inf D_{q+1}, p_{l_{q+1}}\}$. Since the first allocation was done in Substep $q.3.1$ and $l_{q+1} \in L_q$, we know $p_{l_{q+1}} \leq p_{\tilde{l_q}} \leq 1 - p_{h_q}$ and $p_{l_{q+1}} \leq \inf U_q$.[21] Thus

$$t_{l_{q+1}} = \min\{\inf D_{q+1}, p_{l_{q+1}}\} \leq p_{l_{q+1}} \leq \min\{\inf U_q, 1 - p_{h_q}\} = t_{l_q}.$$

[20] Recall $D_q \subseteq D_{q+1}$. Thus $\inf D_{q+1} \leq D_q$.
[21] If $p_{l_{q+1}} > \inf U_q$, then the allocation is Step q would be done in Substep $q.2$.

Case 3 occurs when the allocation is made in Substep $q.2$. Case 4 occurs when the allocation is made in Substep $q.3.2$. These cases can be argued similarly by making the appropriate changes. We omit the details. This completes the proof of Part (i) of the lemma.
Note that $t_{h_q} = 1 - t_{l_q}$ for any $q \in \{1, \ldots, K-1\}$. Thus Part (i) of the lemma implies Part (ii). ■

OBSERVATION **4.** Consider an allocation σ and an $x \in [0,1]$. Suppose there exists $i \in N$ with $t_i^\sigma = x$. Then agent i's partner in σ, say agent j has contribution $t_j^\sigma = 1 - x$. Thus $|\{i \in N : t_i^\sigma = x\}| = |\{j \in N : t_j^\sigma = 1 - x\}|$.

LEMMA **5.** *Consider any preference profile $\succsim$. Let σ and τ be stable allocations at the preference profile $\succsim$. For all $x \in [0, 0.5]$ such that σ and τ satisfy*

(1) $t_i^\sigma \succsim_i x$ and $t_i^\sigma \succsim_i 1 - x$ for all $i \in N$ and

(2) $t_i^\tau \succsim_i x$ and $t_i^\tau \succsim_i 1 - x$ for all $i \in N$

we have,

(a) $|\{i \in N : t_i^\sigma = x\}| = |\{i \in N : t_i^\tau = x\}|$.

(b) Consider agents $i, j \in N$ such that $p_i < 0.5$ and $p_j > 0.5$. If 0.5 satisfies (1) and (2), then $t_i^\sigma = t_j^\tau = 0.5$ is not possible.

Proof: Consider an arbitrary preference profile $\succsim$. Let σ and τ be stable allocations at $\succsim$. Also consider an $x \in [0, 0.5]$ such that σ satisfies Condition (1) and τ satisfies Condition (2) in the antecedent of the lemma.
We partition the agents in N into three groups depending on where their peaks lie with respect to the points x and $1 - x$.[22] Define

1. $S_x^1 = \{i \in N | p_i < x\}$,
2. $S_x^2 = \{i \in N | x \leq p_i \leq 1 - x\}$,
3. $S_x^3 = \{i \in N | p_i > 1 - x\}$.

We will prove several claims about the allocation σ.

Claim 1: For the allocation σ, we have

1. $t_i^\sigma \leq x$ for all $i \in S_x^1$,

[22]Note that it is possible that $x = 1 - x = 0.5$. Then S_x^2 is the set of agents whose peaks are exactly 0.5.

2. $x \leq t_i^\sigma \leq 1 - x$ for all $i \in S_x^2$,

3. $1 - x \leq t_i^\sigma$ for all $i \in S_x^3$.

Proof: We will prove the claim by contradiction. We first prove Part 1. Consider an agent $i \in S_x^1$ such that $t_i^\sigma > x$. Here $p_i < x < t_i^\sigma$ and single-peakedness implies $x \succ_i t_i^\sigma$. This contradicts the fact that σ satisfies Condition (1) in the antecedent of the lemma.
For Part 2, consider $i \in S_x^2$ such that $t_i^\sigma < x$ or $t_i^\sigma > 1 - x$. If $t_i^\sigma < x$, we have $t_i^\sigma < x \leq p_i$. Thus $x \succ_i t_i^\sigma$ by single-peakedness and we have a contradiction. If $t_i^\sigma > 1 - x$, we have $p_i \leq 1 - x < t_i^\sigma$ and $1 - x \succ_i t_i^\sigma$. Once again we have a contradiction.
Part 3 can be proved using similar arguments. ■

Claim 2: For the allocation σ, we have

(a) Consider $i \in S_x^1$. Let (i, j, t_i^σ) be the triple that agent i belongs to in σ. If $t_i^\sigma \neq x$, then $j \in S_x^3$.

(b) Consider agents $i \in S_x^1$, $j \in S_x^3$ such that $(i, j, t_i^\sigma) \in \sigma$. Then $t_i^\sigma < x$.

Proof: (a) Consider $i \in S_x^1$. By assumption, $t_i^\sigma \neq x$. By Part 1 of Claim 1, we have $t_i^\sigma < x$. Thus $t_j^\sigma = 1 - t_i^\sigma > 1 - x$. From Claim 1, we know that all agents in S_x^1 receive a contribution of contribution of at most x in σ (Part 1). Similarly, all agents in S_x^2 have a contribution of at most $1 - x$ (Part 2). Finally all agents in S_x^3 have a contribution of at least $1 - x$ in σ (Part 3). Thus $j \in S_x^3$.

(b) Consider a triple $(i, j, t_i^\sigma) \in \sigma$ where $i \in S_x^1$ and $j \in S_x^3$. By internal stability for the pair (i, j), we have

$$\min\{p_i, 1 - p_j\} \leq t_i^\sigma \leq \max\{p_i, 1 - p_j\}.$$

Since $\max\{p_i, 1 - p_j\} < x$ (follows from the definition of S_x^1 and S_x^3), we have $t_i^\sigma < x$.

■

Claim 3: For the allocation σ, we have

(a) Consider $i \in S_x^3$. Let (i, j, t_i^σ) be the triple that agent i belongs to in σ. If $t_i^\sigma \neq 1 - x$, then $j \in S_x^1$.

(b) Consider agents $i \in S_x^3$, $j \in S_x^1$ such that $(i, j, t_i^\sigma) \in \sigma$. Then $t_i^\sigma > 1 - x$.

Proof: (a) Consider $i \in S_x^3$. By assumption, $t_i^\sigma \neq 1 - x$. By Part 3 of Claim 1, we have $t_i^\sigma > 1 - x$. Thus $t_j^\sigma = 1 - t_i^\sigma < x$. From Claim 1, all agents in S_x^2 must have a contribution of at least x in σ (Part 2). Also all agents in S_x^3 have a contribution of at least $1 - x$ in σ (Part3). Finally all agents in S_x^1 have a contribution of at most x in σ (Part 1). Thus $j \in S_x^1$.

(b) This follows from Part (b) of Claim 2. ■

Claim 4: For the allocation σ, there does not exist a pair of agents $i \in S_x^1$, $j \in S_x^3$ such that $t_i^\sigma = x$ and $t_j^\sigma = 1 - x$.

Proof: Suppose not. Then there exist $i \in S_x^1$, $j \in S_x^3$ such that $t_i^\sigma = x$ and $t_j^\sigma = 1 - x$. By the definition of S_x^1 and S_x^3, we know $\max\{p_i, 1 - p_j\} < x$. Thus there exists $\epsilon > 0$ such that $x > x - \epsilon \geq \max\{p_i, 1 - p_j\}$. The pair (i, j) blocks σ with the contribution vector $(x - \epsilon, 1 - x + \epsilon)$. This results in a contradiction as σ is stable by assumption. ■

Claim 5: For the allocation σ,

1. For any $i \in S_x^1$ and $j \in N \setminus S_x^3$ such that $(i, j, t_i^\sigma) \in \sigma$, we have $t_i^\sigma = x$.

2. For any $i \in N \setminus S_x^1$ and $j \in S_x^3$ such that $(i, j, t_i^\sigma) \in \sigma$, we have $t_i^\sigma = x$ and $t_j^\sigma = 1 - x$.

Proof: (1) Consider agents $i \in S_x^1$ and $j \in N \setminus S_x^3$ such that $(i, j, t_i^\sigma) \in \sigma$. By Claim 1 and $i \in S_x^1$, we know $t_i^\sigma \leq x$. Agent j belongs to S_x^1 or S_x^2. If $j \in S_x^2$, Claim 1 implies $x \leq t_j^\sigma \leq 1 - x$. These together imply $t_i^\sigma = x$. If $j \in S_x^1$, we have $t_j^\sigma \leq x$. By feasibility, it must be the case that $x = 0.5$ and $t_i^\sigma = x$.
(2) Consider agents $i \in N \setminus S_x^1$ and $j \in S_x^3$ such that $(i, j, t_i^\sigma) \in \sigma$. By Claim 1 and $j \in S_x^3$, we know $t_j^\sigma \geq 1 - x$. Agent i either belongs to S_x^3 or S_x^2. If $i \in S_x^3$, then $t_i^\sigma \geq 1 - x$ (by Claim 1). Also since $j \in S_x^3$, $t_j^\sigma \geq 1 - x$. Feasibility implies $1 - x = 0.5$. Thus $t_i^\sigma = 0.5 = x$ and $t_j^\sigma = 1 - x$. If $i \in S_x^2$, Claim 1 implies $x \leq t_i^\sigma \leq 1 - x$. We know $t_j^\sigma \geq 1 - x$. Feasibility implies $t_i^\sigma = x$ and $t_j^\sigma = 1 - x$. ■

Claim 6: Consider the allocation σ. If $x \neq 0.5$, we have

1. There does not exist a triple $(i, j, t_i^\sigma) \in \sigma$ such that $i, j \in S_x^1$.

2. There does not exist a triple $(i, j, t_i^\sigma) \in \sigma$ such that $i, j \in S_x^3$.

Proof: Part 1 follows immediately from Claim 1 and the definition of S_x^1. Similarly Part 2 follows from Claim 1 and the definition of S_x^3. ■

Claim 7: Consider agents $i, j \in S_x^2$ such that $p_i, p_j \in (x, 1 - x)$. Then in the allocation σ, we have $\neg[t_i^\sigma = x \text{ and } t_j^\sigma = 1 - x]$.

Proof: We assume for contradiction that there exist $i, j \in S_x^2$ such that $p_i, p_j \in (x, 1 - x)$, $t_i^\sigma = x$ and $t_j^\sigma = 1 - x$. Here $x < \min\{p_i, 1 - p_j\}$. Thus there exists $\epsilon > 0$ such that $x + \epsilon \succ_i t_i^\sigma$ and $1 - x - \epsilon \succ_j t_j^\sigma$. The pair (i, j) blocks σ. This is a contradiction as σ is stable by assumption. ■

We will first prove Part (a) of the lemma. Our aim is to calculate the cardinality of the set $\{i \in N : t_i^\sigma = x\}$. In order to do this, we will first deduce how agents are matched across the three groups. There are two cases to consider depending on the cardinalities of the sets S_x^1 and S_x^3: (I) $|S_x^1| \geq |S_x^3|$ and (II) $|S_x^1| < |S_x^3|$.

(I) Consider the first case where $|S_x^1| \geq |S_x^3|$. We claim that in this case, all agents in S_x^3 are matched to agents in S_x^1 in σ. We assume for contradiction that there exists $i \in S_x^3$ who is not matched to an agent in S_x^1 in σ. By Claim 5 (Part 2), we know $t_i^\sigma = 1 - x$. Also there exists $j \in S_x^1$ such that j is not matched to an agent in S_x^3. This is because $|S_x^1| \geq |S_x^3|$ and the assumption that an agent in S_x^3 is not matched to an agent in S_x^1. By Claim 5 (Part 1), we know $t_j^\sigma = x$. We have shown that there exists $i \in S_x^3$ with $t_i^\sigma = 1 - x$ and $j \in S_x^1$ with $t_j^\sigma = x$. This is not possible by Claim 4.
Also for any $i \in S_x^3$, we have $t_i^\sigma \neq 1 - x$. Suppose not, i.e. $t_i^\sigma = 1 - x$. We have shown above that agent i is matched with some agent $j \in S_x^1$. Thus $t_j^\sigma = x$. This is not possible by Claim 4.

Now we will calculate the cardinality of the set $\{i \in N : t_i^\sigma = x\}$. There are two possibilities to consider.

1. If $x = 0.5$, then
$$|\{i \in N : t_i^\sigma = x\}| = |S_x^1| + |S_x^2| - |S_x^3|.$$
We have shown that all agents in S_x^3 are matched to agents in S_x^1. Thus $|S_x^3|$ agents in S_x^1 are matched to agents in S_x^3. Also for all $i \in S_x^3$, we have $t_i^\sigma \neq 1-x$. Thus the agents in S_x^1 matched to agents in S_x^3 do not get a contribution of x. The remaining agents in S_x^1 (the cardinality of this set is $|S_x^1| - |S_x^3|$) are matched to agents in $N \setminus S_x^3$. By Claim 5 (Part 1), we know all such agents get a contribution of x. Thus $|S_x^1| - |S_x^3|$ agents in S_x^1 have a contribution of x in σ. Since $x = 0.5$, we know $p_i = 0.5$ for all $i \in S_x^2$. By Claim 1, we have $t_i^\sigma = 0.5$ for all $i \in S_x^2$. So all agents in S_x^2 get a contribution of x in σ. Thus $|S_x^1| - |S_x^3|$ agents in S_x^1, all agents in S_x^2 and none of the agents in S_x^3 receive x in σ.

2. If $x \neq 0.5$, then
$$|\{i \in N | t_i^\sigma = x\}| = \max\{|S_x^1| - |S_x^3| + |\{i \in S_x^2 : p_i = x\}|, |\{j \in S_x^2 : p_j = 1 - x\}|\}.$$
Note that $|S_x^1| - |S_x^3|$ agents in S_x^1 receive x in σ. Consider the set S_x^2. For all $i \in S_x^2$ with $p_i = x$, we have $t_i^\sigma = x$. This is because $t_i^\sigma \succsim_i x$ (recall σ satisfies Condition (1) in the antecedent of the lemma). Similarly for all $i \in S_x^2$ with $p_i = 1 - x$, we have $t_i^\sigma = 1 - x$. This is because $t_i^\sigma \succsim_i 1 - x$ by Condition (1) of the lemma. Note that their partners in σ get x (recall Observation 4). Also none of the agents in S_x^3 receive x in σ. This follows from Claim 1 (all agents in S_x^3 get at least $1 - x$) and the fact that $x < 0.5$. Thus $\max\{|S_x^1| - |S_x^3| + |\{i \in S_x^2 : p_i = x\}|, |\{j \in S_x^2 : p_j = 1 - x\}|\}$ is the minimum number of agents in σ who receive x. We have,
$$|\{i \in N : t_i^\sigma = x\}| \geq \max\{|S_x^1| - |S_x^3| + |\{i \in S_x^2 : p_i = x\}|, |\{j \in S_x^2 : p_j = 1 - x\}|\}.$$

Suppose the above condition holds with strict inequality. We know that none of the agents in S_x^3 get x. Also exactly $|S_x^1| - |S_x^3|$ in S_x^1 receive x. Thus in the case of strict inequality, there exists an agent $i \in S_x^2$ with $p_i \in (x, 1-x)$ and $t_i^\sigma = x$.

Note that $|\{i \in N : t_i^\sigma = x\}| = \{j \in N : t_j^\sigma = 1-x\}$. Since the condition holds with strict inequality, we have

$$|\{j \in N : t_i^\sigma = 1-x\}| > |\{j \in S_x^2 : p_j = 1-x\}|.$$

Thus there exists $j \in N$ such that $p_j \neq 1-x$ and $t_j^\sigma = 1-x$. Note that j cannot belong to S_x^3 (recall we have shown that all agents in S_x^3 do not receive $1-x$). Also all agents in S_x^1 get at most x. So $j \in S_x^2$.[23]

So there exists $i, j \in S_x^2$ such that $p_i, p_j \in (x, 1-x)$, $t_i^\sigma = x$ and $t_j^\sigma = 1-x$. This is not possible by Claim 7.

Thus,

$$|\{i \in N : t_i^\sigma = x\}| = \max\{|S_x^1| - |S_x^3| + |\{i \in S_x^2 : p_i = x\}|, |\{j \in S_x^2 : p_j = 1-x\}|\}.$$

(II) The second case is $|S_x^1| < |S_x^3|$. We claim that all agents in S_x^1 are matched to agents in S_x^3 in σ. Also for all $i \in S_x^1$, we have $t_i^\sigma \neq x$.
In order to compute the cardinality of the set $\{i \in N | t_i^\sigma = x\}$, we compute the minimum number of agents that must receive $1-x$ in σ.[24] There are two possibilities.

1. If $x = 0.5$, then
$$\{i \in N : t_i^\sigma = x\} = |S_x^3| - |S_x^1| + |S_x^2|.$$

   All agents in S_x^1 are matched to agents in S_x^3. Also none of the agents in S_x^1 get $1-x$ in σ.[25] So $|S_x^1|$ agents in S_x^3 do not receive x in σ. By Claim 5 (Part 2), the remaining agents in S_x^3 (who are matched to agents in $N \setminus S_x^1$) must get $1-x$ in σ. Thus $|S_x^3| - |S_x^1|$ agents in S_x^3 have a contribution of $1-x$ in σ.

   Since $x = 0.5$, we have $p_i = 0.5$ for all $i \in S_x^2$. By Claim 2, we know that all agents in S_x^2 have a contribution of $x = 1-x = 0.5$ in σ.

   Thus $|S_x^3| - |S_x^1|$ agents in S_x^1, all agents in S_x^2 and none of the agents in S_x^1 receive $1-x$ in σ.

[23]Note that $p_j \neq x$. We have shown above that if $j \in S_x^2$ and $p_j = x$, then $t_j^\sigma = x$.
[24]This number is useful as for each agent who gets $1-x$ in σ, her partner gets x in σ (Observation 4).
[25]This is because none of the agents in S_x^1 get $x = 0.5$.

2. If $x \neq 0.5$, then

$$|\{i \in N | t_i^\sigma = x\}| = \max\{|S_x^3| - |S_x^1| + |\{i \in S_x^2 : p_i = 1 - x\}|, |\{i \in S_x^2 : p_i = x\}|\}.$$

We can prove this case using similar arguments to Part 2 in Case I.

We have shown that $|\{i \in N | t_i^\sigma = x\}|$ only depends on the preference profile $\succ$ and x. In particular, it does not depend on the allocation σ.
Note that the cardinality of the set $\{i \in N : t_i^\tau = x\}$ can be computed in exactly the same manner as we did for the allocation σ.[26] Thus we conclude $|\{i \in N : t_i^\sigma = x\}| = |\{i \in N : t_i^\tau = x\}|$.

We now prove Part (b) of the lemma. Consider agents $i, j \in N$ such that $p_i < 0.5$ and $p_j > 0.5$. There are two possibilities based on the cardinalities of $S_{0.5}^1|$ and $S_{0.5}^3$.
In Case I where $|S_{0.5}^1| \geq |S_{0.5}^3|$, we have shown above that all agents in $S_{0.5}^3$ are matched to agents in $S_{0.5}^1$. Also each agent in $S_{0.5}^3$ has a contribution strictly greater than 0.5 (by Claim 2 (b)) in both σ and τ. Since $p_j > 0.5$, we know $j \in S_{0.5}^3$. Thus $t_j^\tau = 0.5$ is not possible.
In Case II where $|S_{0.5}^1| < |S_{0.5}^3|$, all agents in $S_{0.5}^1$ are matched to agents in $S_{0.5}^3$. Each agent in $S_{0.5}^1$ has a contribution strictly smaller than 0.5 (by Claim 2 (b)) in both σ and τ. Since $p_i < 0.5$, we know $i \in S_{0.5}^1$. Thus it is not possible that $t_i^\sigma = 0.5$. ■

Observation **5.** Consider a preference profile $\succsim$ and allocations σ and τ. If τ Pareto dominates σ at $\succsim$, then $I_{i,t_i^\tau} \subseteq I_{i,t_i^\sigma}$ for all $i \in N$. This implies $D^\tau \subseteq D^\sigma$ and $U^\tau \subseteq D^\sigma$.

Lemma **6.** *Consider a preference profile $\succsim$ and allocations σ and τ. If σ satisfies Condition S and τ Pareto dominates σ, then τ satisfies Condition S.*

Proof: We assume for contradiction that τ does not satisfy Condition S. There are two cases. The first case is when τ violates Part 1 of Condition S, i.e. there exists $l \in L, h \in H$ such that $I_{h,t_h^\tau} \cap I_{l,t_l^\tau} \neq \emptyset$. Since $I_{l,t_l^\tau} \subseteq I_{l,t_l^\sigma}$ and $I_{h,t_h^\tau} \subseteq I_{h,t_h^\sigma}$ (by Observation 5), we have $I_{l,t_l^\sigma} \cap I_{h,t_h^\sigma} \neq \emptyset$. Thus σ violates Condition S. The second case is when τ violates Part 2 of Condition S, i.e. there exists $i \in N$ such that $0.5 \in I_{i,t_i^\tau}$. By Observation 5, $0.5 \in I_{i,t_i^\sigma}$. Thus σ violates Condition S. ■

Observation **6.** Consider an allocation σ which satisfies Condition S. For any $x \in [0, 0.5]$, if $x \notin D^\sigma \cup U^\sigma$, then $[t_i^\sigma \succsim_i x$ and $t_i^\sigma \succsim_i 1 - x]$ for all $i \in N$.

Lemma **7.** *Consider a preference profile $\succsim$. If σ satisfies Condition S and τ Pareto dominates σ at $\succsim$, then for all $x \in [0, 1]$,*

$$|\{i \in N | t_i^\tau = x\}| = |\{i \in N | t_i^\sigma = x\}|.$$

[26] All the claims proved are true for the allocation τ as well. The only change required in the arguments is to replace Condition (1) of the lemma by Condition (2).

Proof: Consider a preference profile $\succsim$ and allocations σ and τ. Assume σ satisfies Condition S and τ Pareto dominates σ at $\succsim$. Consider an $x \in [0,1]$. There are two possibilities.

(A) Let $x \in [0, 0.5]$. We consider three subcases.

1. $x \in [0, 0.5] \setminus [U^\sigma \cup D^\sigma]$.

   Since τ Pareto dominates σ, we have $U^\tau \subseteq U^\sigma$ and $D^\tau \subseteq D^\sigma$ (Observation 5). Thus $x \in [0, 0.5] \setminus [U^\tau \cup D^\tau]$. We know $x \notin D^\sigma \cup U^\sigma$. Observation 6 implies $t_i^\sigma \succsim_i x$ and $t_i^\sigma \succsim_i 1-x$ for all $i \in N$. Similarly for the allocation τ, we have $t_i^\tau \succsim_i x$ and $t_i^\tau \succsim_i 1-x$ for all $i \in N$.

   Applying Lemma 5, we have

$$|\{i \in N : t_i^\sigma = x\}| = |\{i \in N : t_i^\tau = x\}|.$$

2. $x \in U^\sigma$. Then there exists $h \in H$ such that $x \in I_{h,t_h^\sigma}$ and $1 - x \succ_h t_h^\sigma$. Since σ satisfies Condition S, we have $0.5 \notin U^\sigma$. Thus $x < 0.5$.

   For any $i \in N$ with $p_i = x$, it must be the case that $t_i^\sigma = x$. If not, then (i,h) will block σ with $(x, 1-x)$.

   Also for any $i \in N$ with $t_i^\sigma = x$, it must be the case that $p_i = x$. If not, the pair (i,h) will block σ.[27] This implies

$$|\{i \in N : t_i^\sigma = x\}| = |\{i \in N : p_i = x\}|.$$

   Note that all these agents must receive their peaks in τ as τ Pareto dominates σ. Thus,

$$|\{i \in N : t_i^\tau = x\}| \geq |\{i \in N : p_i = x\}| = |\{i \in N : t_i^\sigma = x\}|.$$

3. $x \in D^\sigma$. Then there exists $l \in L$ such that $x \in I_{l,t_l^\sigma}$.

   We will look at the agents who receive $1-x$ in σ. For all $i \in N$ such that $p_i^\sigma = 1-x$, we have $t_i^\sigma = 1-x$. Also for all $i \in N$ with $t_i^\sigma = 1-x$, it must be the case that $p_i = 1-x$. Thus,

$$|\{i \in N : t_i^\sigma = 1-x\}| = |\{i \in N : p_i = 1-x\}|.$$

   Since τ Pareto dominates σ, we have

$$|\{i \in N : t_i^\tau = 1-x\}| \geq |\{i \in N : p_i = 1-x\}| = |\{i \in N : t_i^\sigma = 1-x\}|.$$

[27]Suppose $p_i < x$. Then there exists $\epsilon > 0$ such that $p_i < x - \epsilon < x$ and $x - \epsilon \in I_{h,t_h^\sigma}$. The pair (i,h) blocks σ with $(x-\epsilon, 1-x+\epsilon)$. We can give a similar argument for the case $p_i > x$.

Note that $|\{i \in N : t_i^\tau = 1 - x\}| = |\{i \in N : t_i^\tau = x\}|$ and $|\{i \in N : t_i^\sigma = 1 - x\}| = |\{i \in N : t_i^\sigma = x\}|$ (Observation 4).

Therefore,

$$|\{i \in N : t_i^\tau = x\}| \geq |\{i \in N | p_i = 1 - x\}| = |\{i \in N | t_i^\sigma = 1 - x\}| = |\{i \in N | t_i^\sigma = x\}|.$$

We have shown that for every $x \in [0, 0.5]$,

$$|\{i \in N : t_i^\tau = x\}| \geq |\{i \in N : t_i^\sigma = x\}|.$$

(B) $x \in (0.5, 1]$.

Note that $|\{i \in N : t_i^\tau = x\}| = |\{i \in N : t_i^\tau = 1 - x\}|$ and $|\{i \in N : t_i^\sigma = x\}| = |\{i \in N : t_i^\sigma = x\}|$ (by Observation 4). Since $1 - x \in [0, 0.5]$, Case (A) is applicable. So $|\{i \in N : t_i^\tau = 1 - x\}| \geq |\{i \in N : t_i^\sigma = 1 - x\}|$. Combining the facts above, we have $|\{i \in N : t_i^\tau = x\}| \geq |\{i \in N | t_i^\sigma = x\}|$.

From (A) and (B), we know

$$|\{i \in N : t_i^\tau = x\}| \geq |\{i \in N | t_i^\sigma = x\}| \text{ for all } x \in [0, 1] \tag{1.1}$$

The sum of both the LHS and RHS of Equation 1.1 over all $x \in [0, 1]$ is $|N|$. Suppose there exists some x for which Equation 1.1 holds with strict inequality. Then there must exist another x for which the LHS of Equation 1.1 will be strictly less than the RHS. This contradicts 1.1. Thus Equation 1.1 holds with equality.
This completes the proof of the lemma. ■

Observation **7.** Consider an allocation σ that satisfies Condition S. For any agent $i \in N$, we have (a) if $t_i^\sigma < 0.5$ then $p_i < 0.5$ and (b) if $t_i^\sigma > 0.5$ then $p_i > 0.5$. To prove (a), assume $t_i^\sigma < 0.5$ and $p_i \geq 0.5$ for some agent i. Then 0.5 belongs to I_{i,t_i^σ} and σ violates Condition S. Similarly we can prove (b). Also if $p_i = 0.5$ then $t_i^\sigma = 0.5$.

Definition **8.** *Consider a preference profile $\succsim$ and two allocations $\sigma, \tau \in \Sigma$. Assume τ Pareto dominates σ at $\succsim$. We say τ is minimal if for all $\gamma \in \Sigma$ such that γ Pareto dominates σ at $\succsim$, we have*

$$|\{i \in N : t_i^\tau \neq t_i^\sigma\}| \leq |\{i \in N : t_i^\gamma \neq t_i^\sigma\}|.$$

Observation **8.** Consider a preference profile $\succsim$ and allocations $\sigma, \gamma \in \Sigma$ such that γ Pareto dominates σ at $\succsim$. Since $|N|$ is finite, there exists an allocation $\tau \in \Sigma$ such that τ Pareto dominates σ at $\succsim$ and is minimal. This allocation may not be unique.

We now complete the proof of Pareto efficiency.

Proof: Let σ be the allocation generated by the algorithm. We will prove the theorem by contradiction. Suppose there exists an allocation $\gamma \in \Sigma$ such that γ Pareto dominates σ. By Observation 8, we know there exists $\tau \in \Sigma$ such that τ Pareto dominates σ and is minimal. We will now work with the allocation τ and use it to show a contradiction.

Note that σ satisfies Condition S as it is generated by the algorithm. Since τ Pareto dominates σ, we know τ also satisfies Condition S (Lemma 6).

There are two cases to consider: (I) there exists an agent $l \in L$ such that $t_l^\tau \neq t_l^\sigma$ and (II) $t_l^\tau = t_l^\sigma$ for all $l \in L$.

Case I: There exists an agent $l \in L$ such that $t_l^\tau \neq t_l^\sigma$.

The SAM algorithm generates σ in K steps.[28] There are two possibilities. The first possibility is that there exists a low type agent l who is matched in Step 0 and $t_l^\sigma \neq t_l^\tau$.[29] We denote agent l as $l_{\bar{i}}$ where $\bar{i} = 0$. The second possibility is that all low type agents matched in Step 0 (if any) have the same contribution values in both σ and τ. Then there exists a low type agent who is matched in some Step $q \geq 1$ and has different contribution values in σ and τ. Let $\bar{i} \in \{1, \ldots, K\}$ be the smallest integer such that (i) $t_{l_{\bar{i}}}^\sigma \neq t_{l_{\bar{i}}}^\tau$ and (ii) $t_{l_i}^\sigma = t_{l_i}^\tau$ for all $i < \bar{i}$. Note that $l_{\bar{i}}$ is the agent matched in Step $\bar{i}$ of the algorithm.

There are three cases based on the contribution value of agent $l_{\bar{i}}$ in σ.

1. $t_{l_{\bar{i}}}^\sigma = p_{l_{\bar{i}}}$.

   Since τ Pareto dominates σ, we have $t_{l_{\bar{i}}}^\tau = p_{l_{\bar{i}}}$. This is a contradiction as by assumption the contribution values of agent $l_{\bar{i}}$ are different in σ and τ.

2. $t_{l_{\bar{i}}}^\sigma < p_{l_{\bar{i}}}$.

   Here $t_{l_{\bar{i}}}^\sigma$ is the infimum of the improvement set of $l_{\bar{i}}$ in σ. Since τ Pareto dominates σ and $t_{l_{\bar{i}}}^\sigma \neq t_{l_{\bar{i}}}^\tau$, we have $t_{l_{\bar{i}}}^\tau \in \overline{I_{l_{\bar{i}}, t_{l_{\bar{i}}}^\sigma}} \setminus \{t_{l_{\bar{i}}}^\sigma\}$. Note that $t_{l_{\bar{i}}}^\sigma < t_{l_{\bar{i}}}^\tau \leq 0.5$.[30]

   Let $t_{l_{\bar{i}}}^\tau = m^*$. Since σ satisfies Condition S and τ Pareto dominates it, applying Lemma 7 at $x = m^*$, we have

   $$|\{i \in N : t_i^\sigma = m^*\}| = |\{i \in N | t_i^\tau = m^*\}|.$$

   By assumption, $l_{\bar{i}}$ does not belong to the former set but belongs to the latter. Thus there exists an agent j who belongs to the former set and not to the latter. Note that $t_j^\sigma = m^* \neq t_j^\tau$.

[28] Recall in each step q (where $1 \leq q \leq K$), a low type agent l_q is matched. In Step 0, low type agents are matched if $|L| > |H|$.

[29] If there are several such agents, we choose an agent arbitrarily from this set.

[30] We know τ satisfies Condition S. By Observation 7 and $p_{l_{\bar{i}}} < 0.5$, we have $t_{l_{\bar{i}}}^\tau \leq 0.5$.

We claim that $j \in L$. To show this, we consider two cases. The first case is when $m^* < 0.5$. Suppose $j \notin L$. Then $p_j \geq 0.5$ and $0.5 \in I_{j,t_j^\sigma}$. This results in a contradiction as σ satisfies Condition S. Thus when $m^* < 0.5$, it must be the case that $p_j < 0.5$ and $j \in L$. The remaining case is when $m^* = 0.5$. We know $t_{l_{\bar{i}}}^\tau = m^* = 0.5$, $p_{l_{\bar{i}}} < 0.5$ and $t_j^\sigma = m^* = 0.5$. Applying Lemma 5 (Part (b)), we have $p_j \leq 0.5$. We will show that $p_j \neq 0.5$ if τ satisfies Condition S. Suppose $p_j = 0.5$. We know $t_j^\tau \neq t_j^\sigma = m^* = 0.5$. Then $p_j = 0.5 \in I_{j,t_j^\tau}$ and τ violates Condition S. Thus $p_j < 0.5$ and $j \in L$.

We know $l_{\bar{i}}, j \in L$. Agent j has different contribution values in σ and τ. By assumption, $l_{\bar{i}}$ is the first low type agent who has different contribution values in σ and τ. We first argue that $\bar{i} \neq 0$. If $\bar{i} = 0$, we have $t_{l_{\bar{i}}}^\sigma = 0.5$. Since $p_{l_{\bar{i}}} < 0.5$, we have a contradiction to the assumption $t_{l_{\bar{i}}}^\sigma < p_{l_{\bar{i}}}$. Thus $\bar{i} \geq 1$ and agent j is matched in a step strictly greater than Step $\bar{i}$ in the algorithm. By Lemma 4, we have $t_j^\sigma = m^* \leq t_{l_{\bar{i}}}^\sigma$. We have a contradiction as $t_{l_{\bar{i}}}^\sigma < t_j^\tau = m^* = t_j^\sigma$.

3. $t_{l_{\bar{i}}}^\sigma > p_{l_{\bar{i}}}$.

Let $t_{l_{\bar{i}}}^\sigma = s^*$. So $p_{l_{\bar{i}}} < t_{l_{\bar{i}}}^\sigma = s^* \leq 0.5$ (by Observation 7 and the fact that σ satisfies Condition S).

Since σ satisfies Condition S and τ Pareto dominates it, applying Lemma 7 at $x = s^*$, we have

$$|\{i \in N : t_i^\sigma = s^*\}| = |\{i \in N : t_i^\tau = s^*\}|.$$

Agent $l_{\bar{i}}$ belongs to the first set and not to the latter. Thus there exists an agent j who belongs to the latter set and not to the first. Note that $t_j^\tau = s^* = t_{l_{\bar{i}}}^\sigma \neq t_j^\sigma$.

We claim that $j \in L$. To show this, we consider two cases. The first case is when $s^* < 0.5$. Suppose $j \notin L$ and $p_j \geq 0.5$. Then $0.5 \in I_{j,t_j^\tau}$ and τ violates Condition S (Observation 7). Hence $p_j < 0.5$ and $j \in L$. The second case is when $s^* = 0.5$. We know $t_{l_{\bar{i}}}^\sigma = s^* = 0.5$, $p_{l_{\bar{i}}} < 0.5$ and $t_j^\tau = s^* = 0.5$. Applying Lemma 5 (Part (b)), we have $p_j \leq 0.5$. We will show that $p_j \neq 0.5$ if σ satisfies Condition S. Suppose $p_j = 0.5$. We know $t_j^\sigma \neq s^* = 0.5$. Then $p_j = 0.5 \in I_{j,t_j^\sigma}$ and σ violates Condition S. Thus $p_j < 0.5$ and $j \in L$. This establishes the claim that $j \in L$.

We now show that $e_j(s^*) \leq e_{l_{\bar{i}}}(s^*)$. To prove this claim, we consider two cases.

The first case is when $\bar{i} \geq 1$. Recall that both agents $l_{\bar{i}}$ and j have different contribution values in σ and τ. Since $\bar{i} \geq 1$ and agent $l_{\bar{i}}$ is the first low type agent to have different contribution values in σ and τ, agent j is matched in a step strictly greater than Step $\bar{i}$. Thus $j \in L_{\bar{i}}$.

Since $p_{l_{\bar{i}}} < t_{l_{\bar{i}}}^\sigma$, we know this is only possible in Substep $\bar{i}.1$ or Substep $\bar{i}.3.1$ of the algorithm (by Lemma 2). In both cases, $l_{\bar{i}}$ is the secondary agent. Thus $e_l(s^*) \leq e_{l_{\bar{i}}}(s^*)$ for all $l \in L_{\bar{i}} \setminus \{l_{\bar{i}}\}$. Since $j \in L_{\bar{i}}$, we have $e_j(s^*) \leq e_{l_{\bar{i}}}(s^*)$.

The second case is when $\bar{i} = 0$. Note that $t^{\sigma}_{l_{\bar{i}}} = s^* = 0.5 \neq t^{\sigma}_j$. This means that j is not matched in Step 0, when $l_{\bar{i}}$ is matched. Thus $e_j(0.5) \leq e_{l_{\bar{i}}}(0.5)$. This completes the proof of the claim.

Thus,

$$\overline{I_{l_{\bar{i}},t^{\sigma}_{l_{\bar{i}}}}} = [e_{l_{\bar{i}}}(s^*), s^*] \cap [0, 0.5] \subseteq [e_j(s^*), s^*] \cap [0, 0.5] = \overline{I_{j,t^{\tau}_j}}, \tag{1.2}$$

$$I_{l_{\bar{i}},t^{\sigma}_{l_{\bar{i}}}} = (e_{l_{\bar{i}}}(s^*), s^*) \cap [0, 0.5] \subseteq (e_j(s^*), s^*) \cap [0, 0.5] = I_{j,t^{\tau}_j}. \tag{1.3}$$

Since τ Pareto dominates σ, we have $t^{\tau}_{l_{\bar{i}}} \in \overline{I_{l_{\bar{i}},t^{\sigma}_{l_{\bar{i}}}}}$. This together with Fact 1.2 implies $t^{\tau}_{l_{\bar{i}}} \in \overline{I_{j,t^{\tau}_j}}$. So $t^{\tau}_{l_{\bar{i}}} \succsim_j t^{\tau}_j$. As τ Pareto dominates σ, we know $t^{\tau}_j \succsim_j t^{\sigma}_j$. Hence for agent j,

$$t^{\tau}_{l_{\bar{i}}} \succsim_j t^{\tau}_j \succsim_j t^{\sigma}_j. \tag{1.4}$$

Note that if $t^{\tau}_{l_{\bar{i}}} \in I_{l_{\bar{i}},t^{\sigma}_{l_{\bar{i}}}}$, then $t^{\tau}_{l_{\bar{i}}} \in I_{j,t^{\tau}_j}$ (by Fact 1.3). Thus $t^{\tau}_{l_{\bar{i}}} \succ_j t^{\tau}_j \succsim_j t^{\sigma}_j$ (as τ Pareto dominates σ). This implies $t^{\tau}_{l_{\bar{i}}} \succ_j t^{\sigma}_j$. Thus

$$\text{If } t^{\tau}_{l_{\bar{i}}} \succ_{l_{\bar{i}}} t^{\sigma}_{l_{\bar{i}}} \text{ then } t^{\tau}_{l_{\bar{i}}} \succ_j t^{\sigma}_j. \tag{1.5}$$

We will now construct an allocation $\delta \in \Sigma$ such that

(a) δ Pareto dominates σ, and

(b) $|\{i \in N : t^{\delta}_i \neq t^{\sigma}_i\}| < |\{i \in N : t^{\tau}_i \neq t^{\sigma}_i\}|$.

This will contradict the assumption that τ Pareto dominates σ and is minimal.

We construct δ as follows. The pairs in δ are defined as follows,

- the partner of $l_{\bar{i}}$ in δ is the partner of j in τ;
- the partner of j in δ is the partner of $l_{\bar{i}}$ in τ;
- for any agent s (different from $l_{\bar{i}}, j$ and their partners' in τ), the partner of s in δ is the same as her partner in τ.

To obtain the pairs in δ, we interchange the partners of agents $l_{\bar{i}}$ and j in τ and the partners of all other agents remain unchanged.

The contributions of the agents in δ are defined as,

- $t^{\delta}_s = t^{\tau}_s$ for all $s \in N \setminus \{l_{\bar{i}}, j\}$. The contribution of all such agents in δ is equal to their contribution in τ.

- $t^{\delta}_{l_{\bar{i}}} = t^{\sigma}_{l_{\bar{i}}} = s^*$. The contribution of $l_{\bar{i}}$ in δ is equal to her contribution in σ.
- $t^{\delta}_{j} = t^{\tau}_{l_{\bar{i}}}$. The contribution of agent j in δ is equal to the contribution of agent $l_{\bar{i}}$ in τ.

Note that for all $s \in N \setminus \{l_{\bar{i}}, j\}$, we have $t^{\delta}_{s} = t^{\tau}_{s} \succsim_s t^{\sigma}_{s}$ as τ Pareto dominates σ. The contribution of agent $l_{\bar{i}}$ is the same in δ and σ. For agent j, we have $t^{\delta}_{j} = t^{\tau}_{l_{\bar{i}}} \succsim_j t^{\sigma}_{j}$ (Fact 1.4). Thus all agents in N weakly prefer their contributions in δ to their contributions in σ.

In order to show that δ Pareto dominates σ, we need to show that there exists an agent who strictly improves in δ with respect to σ.[31] We consider three cases based on the agent who strictly prefers her contribution in τ to that in σ. Let $k \in N$ be the agent who strictly improves from σ to τ.

(a) $k \in N \setminus \{l_{\bar{i}}, j\}$. Here $t^{\delta}_{k} = t^{\tau}_{k} \succ_k t^{\sigma}_{k}$. Thus k also strictly improves in δ in comparison to σ.

(b) $k = l_{\bar{i}}$. Here $t^{\tau}_{l_{\bar{i}}} \succ_{l_{\bar{i}}} t^{\sigma}_{l_{\bar{i}}}$. Then Fact 1.5 implies $t^{\delta}_{j} = t^{\tau}_{l_{\bar{i}}} \succ_j t^{\sigma}_{j}$. Thus j strictly improves in δ in comparison to σ.

(c) $k = j$. Here $t^{\tau}_{j} \succ_j t^{\sigma}_{j}$. By Fact 1.4, $t^{\delta}_{j} = t^{\tau}_{l_{\bar{i}}} \succsim_j t^{\tau}_{j}$. Thus $t^{\delta}_{j} \succ_j t^{\sigma}_{j}$ and j strictly improves in δ in comparison to σ.

We have established that there exists an agent who strictly improves in δ with respect to σ.

We now show that τ is not minimal. Let K be the number of agents in $N \setminus \{l_{\bar{i}}, j\}$ who have different contribution values in σ and τ. By construction, K is also the number of agents in $N \setminus \{l_{\bar{i}}, j\}$ who have different contribution values in σ and δ.

Note that $|\{i \in N : t^{\tau}_{i} \neq t^{\sigma}_{i}\}| = K + 2$ and $|\{i \in N : t^{\delta}_{i} \neq t^{\sigma}_{i}\}| \leq K + 1$. So τ is not minimal and we have a contradiction.

Case II: For all $l \in L$, $t^{\tau}_{l} = t^{\sigma}_{l}$ Then there exists $h \in H$ such that $t^{\tau}_{h} \neq t^{\sigma}_{h}$.

The proof of Case II is virtually identical to that of Case I. We omit the details.

This completes the proof of the theorem. ■

[31] Note that this will also establish that σ and δ are distinct.

2 Welfare Maximisation in Stable Sharing

Based on work by P. Salmaso.

In this chapter we want to elaborate further on the model proposed in Chapter 1. Here we focus on the study of welfare implications of the SAM algorithm.

In Chapter 1 we introduced the notion of allocation, that is a specification of matching agents in pairs (or, simply, *matching*) and the individual contributions. To have a meaningful discussion about the welfare of an allocation we will restrict the preference domain to Euclidean single-peaked utility functions[1] with a cardinal interpretation. This allows us to define the utilitarian welfare of an allocation as the sum of agents' utilities at that allocation.

As a first preliminary result we show that the welfare of a stable allocation depends only on how the agents are matched and not on the individual contributions. More precisely, we prove that any two internally stable allocations in which agents are matched with the same partners provide the same utilitarian welfare.
In the setting of Euclidean single-peaked preferences, we will find that the SAM algorithm introduced in Chapter 1 can be recast in a more concise way. Our main result in this chapter is that the outcome of the SAM algorithm, in addition to being stable, also maximises the utilitarian welfare when agents have Euclidean single-peaked utility functions.

We will then explore the relation between strategy-proofness and welfare maximisation, showing a negative result about the possibility to have a social choice function that fulfills both.

Finally we discuss about two more egalitarian welfare functions, the Max-min and the weighted utilitarian welfare functions. We will show that an allocation based on the same matching as the outcome of the SAM algorithm and with agents splitting equally the utility among each pair maximises these welfare functions. However, we also show that the maximisation of those welfare functions is not necessarily compatible with stability.

[1] A special case of Euclidean preferences, which were introduced in Chapter 1.

In Section 1 and Section 2 we will present the model, together with a simplified description of the improvement sets defined in Chapter 1. In Section 3 we will discuss about matchings and prove the invariance of the Utilitarian Welfare among internally stable allocations that share the same matching. In Section 4 we will present the simplified version of the SAM algorithm, called Simple-SAM, and prove its equivalence with the SAM algorithm together with the main result of the chapter. Finally in Section 5 we will argue about strategy proofness and in Section 6 we will discuss about different welfare functions.

2.1 The model

The set of agents is $N = \{1, \ldots, n\}$ where n is even. Agents have to be assigned in pairs and each pair has to complete a task of unit value. No agent can remain unmatched and each agent has only one partner.

An allocation σ is a collection of triples, (i, j, t_i) where $i, j \in N$ and $t_i \in [0, 1]$. We interpret t_i as the contribution of agent i. The contribution of agent i's partner j is $t_j = 1 - t_i$. We refer to (t_i, t_j) as the contribution vector associated with the matched pair (i, j). We say $(i, j, t_i) \in \sigma$ if the pair (i, j) has the contribution vector (t_i, t_j) in σ. Let Σ denote the set of all feasible allocations.

Each agent i has a preference ordering $\succsim_i$ over her contribution.[2] We assume $\succsim_i$ is *Euclidean single-peaked*. The ordering $\succsim_i$ is Euclidean single-peaked if there exists a unique contribution $p_i \in [0, 1]$ such that $\succsim_i$ can be represented by a utility function of the following form,

$$u_i(x) = 1 - |p_i - x|. \tag{2.1}$$

p_i will be referred to as the peak of agent i in $\succsim_i$.
For the rest of the chapter we consider the order in $N = \{1, ..., n\}$ to be increasing in the peaks, i.e. $p_i < p_j \Rightarrow i < j$.[3] A preference profile $\succsim$ is an n-tuple of preferences $(\succsim_1, \ldots, \succsim_n)$. We call t_i^σ the contribution of agent i in the allocation σ.
Notice that, contrary to the model with continuous single-peaked preferences presented in Chapter 1, in the current model the peak of an agent determines univocally that agent's preferences. Moreover, if we interpret the utility functions of the agents in a cardinal way, we can define an *utilitarian welfare function* as

$$W(\sigma) = \sum_{i \in N} u_i(t_i^\sigma).$$

We say that an allocation σ is *welfare maximising* if it maximises the utilitarian welfare. I.e.

[2] The asymmetric and symmetric components of $\succsim_i$ are denoted by $\succ_i$ and $\sim_i$ respectively.

[3] We will loose this convention in Section 6, where we will discuss about incentive compatibility and deviations.

$\sigma \in \arg\max_{\tau\in\Sigma}(W(\tau))$.
We say, moreover, that two allocations τ and σ are *welfare equivalent* if $W(\tau) = W(\sigma)$.

2.1.1 The Improvement Sets

We now recall the notion of improvement set as first introduced in Chapter 1. This notion plays a key role in SAM algorithm.

As in Chapter 1, we partition agents into "high" type (H) and "low" type (L) agents depending upon whether their peaks are greater than or less than 0.5. Formally, $H = \{i \in N : p_i \geq 0.5\}$ and $L = \{i \in N : p_i < 0.5\}$. Since the agents in N are ordered increasingly according to their peaks, every low type agent will appear before every high type agent in N. Thus agents $\{1, ..., |L|\} \subseteq N$ are low type agents and agents $\{|L|+1, ..., n\} \subseteq N$ are high type agents.
We represent the peaks and the contributions of agents in the interval $[0, 0.5]$. The peak of a low type agent will be measured from left to right starting at 0, while the peak of a high type agent will be measured from right to left starting at 0.5.[4]

Consider an agent $i \in L$ with preference $\succsim_i$ (with peak p_i) and contribution t_i. We define the *improvement set* for i at t_i as follows,

$$I_{i,t_i} = \begin{cases} (2p_i - t_i, t_i) \cap [0, 0.5] & \text{if } t_i > p_i, \\ (t_i, 2p_i - t_i) \cap [0, 0.5] & \text{if } t_i < p_i, \\ \emptyset & \text{if } t_i = p_i. \end{cases}$$

Consider agent $i \in H$ with preference $\succsim_i$ (with peak p_i) and contribution t_i. We define the *improvement set* for i at t_i as follows,[5]

$$I_{i,t_i} = \begin{cases} (1 - 2p_i + t_i, 1 - t_i) \cap [0, 0.5] & \text{if } t_i < p_i, \\ (1 - t_i, 1 - 2p_i + t_i) \cap [0, 0.5] & \text{if } t_i > p_i. \\ \emptyset & \text{if } t_i = p_i \end{cases}$$

The assumption of Euclidean single-peakedness on $\succsim_i$ imposes structure on the improvement sets which we record below as an observation.

Observation **9.** The improvement set of an agent is a connected open subset of $[0, 0.5]$ or equivalently an open interval in $[0, 0.5]$. Moreover, if neither 0 or 0.5 belong to the

[4]The interval $[0, 0.5]$ can be thought of as a truncated one-dimensional Edgeworth box. For a low type agent, we are not interested in representing contributions greater than 0.5. Similarly we do not need to represent contributions smaller than 0.5 for a high type agent.

[5]This definition is equivalent to Definition 5 given in Chapter 1.

improvement set, and the improvement set is not empty, then the peak p_i is the midpoint of the improvement set if $i \in L$, and $1 - p_i$ is the midpoint of the improvement set if $i \in H$ instead. In fact the improvement set is an open ball in the topology of $[0, 0.5]$ with radius $|p_i - t_i|$ and center either p_i if $i \in L$, or $1 - p_i$, if $i \in H$.

2.1.2 Matchings and Welfare Maximisation

We call a *matching* a set of unordered pairs of agents such that every agent in N belongs to exactly one pair, i.e. it is a partition of N in sets of two elements.
Consider an allocation σ, i.e. a collection of triples $\{(i_1, j_1, t_{i_1}), (i_2, j_2, t_{i_2}), (i_3, j_3, t_{i_3}), ...\}$. Every triple consists of two agents, i and j, and a contribution, t_i. If we remove the contribution t_i from every triple in σ, what remains is a collection μ of disjoint pairs that is a partition of N too. Thus μ is a matching and it is univocally determined by σ. We say that the allocation σ is *based* on the matching μ.
We say that the agents i and j are *matched* in σ if they belong to the same triple in σ, or equivalently they belong to the same pair in μ, if σ is based on μ.
We are interested in welfare maximising allocations. In order to study them we start by recalling the definition of internal stability of an allocation.

Definition **9.** *A triple (i, j, t_i) is internally stable if there is no contribution t'_i such that $t'_i \succ_i t_i$ and $1 - t'_i \succ_j 1 - t_i$.*
An allocation σ is internally stable if it is composed by internally stable triples.

As remarked in Chapter 1, in any internally stable allocation agents who are matched must receive contributions falling on the same side of the respective peak. That is, for any internally stable allocation σ,

- if $p_i + p_j > 1$, then $t_i^\sigma \geq p_i$ and $t_j^\sigma \geq p_j$;
- if $p_i + p_j < 1$, then $t_i^\sigma \leq p_i$ and $t_j^\sigma \leq p_j$;
- if $p_i + p_j = 1$, then $t_i^\sigma = p_i$ and $t_j^\sigma = p_j$.

Internally stable allocations enjoy the following lemma.

Lemma **8.** *Let i and j be two agents allocated together in a triple (i, j, t_i) that is internally stable. Then $u_i(t_i) + u_j(t_j)$ follows the following rule:*

$$u_i(t_i) + u_j(t_j) = \begin{cases} 3 - p_i - p_j & \text{if } p_i + p_j > 1, \\ 1 + p_i + p_j & \text{if } p_i + p_j < 1, \\ 2 & \text{if } p_i + p_j = 1. \end{cases}$$

Proof: Let $p_i + p_j > 1$. In this case, at least one agent receives a contribution lower than her optimum. By internal stability, none of the agents receives a contribution higher than her peak. Thus $t_i \leq p_i$ and $t_j \leq p_j$; so $u_i(t_i) = 1 - |p_i - t_i| = 1 - p_i + t_i$ and $u_j(t_j) = 1 - |p_j - t_j| = 1 - p_j + t_j$. Thus

$$u_i(t_i) + u_j(t_j) = 1 - p_i + t_i + 1 - p_j + t_j = 3 - p_i - p_j.$$[6]

Let now $p_i + p_j < 1$. In this case, at least one agent receives a contribution higher than her optimum. By internal stability, none of the agents receives a contribution lower than her peak. Thus $t_i \geq p_i$ and $t_j \geq p_j$, so $u_i(t_i) = 1 - |p_i - t_i| = 1 + p_i - t_i$ and $u_j(t_j) = 1 - |p_j - t_j| = 1 + p_j - t_j$. Thus

$$u_i(t_i) + u_j(t_j) = 1 + p_i - t_i + 1 + p_j - t_j = 1 + p_i + p_j.$$[6]

If $p_i + p_j = 1$, then the only internally stable outcome is $t_i = p_i$ and $t_j = p_j$. $u_i(p_i) = 1$ and $u_j(p_j) = 1$, thus

$$u_i(t_i) + u_j(t_j) = 1 + 1 = 2.$$

■

This lemma has one interesting implication: for any given triple (i, j, t_i) the sum of the utilities of agents i and j does not depend on their contributions, as long as (i, j, t_i) is internally stable.

We can thus define the welfare of the pair (i, j) as $w(i, j) = u_i(t_i) + u_j(1 - t_i)$ for any internal stable triple (i, j, t_i).

It follows immediately from Lemma 8 that $1 \leq w(i, j) \leq 2$. Therefore our model is equivalent to a transferable utility matching model with value functions given by $w(i, j)$ as long ad agents' utility does not exceed 1.

Moreover the lemma has the following important corollary.

Corollary **2.** *Let σ and τ be two internally stable allocations based on the same matching μ. Then,*

$$W(\sigma) = W(\tau). \tag{2.2}$$

Proof: The welfare of σ is $W(\sigma) = \sum_{i \in N} u_i(t_i^\sigma)$. Since μ is a partition of N, this sum can be decomposed as $\sum_{(i,j) \in \mu} u_i(t_i^\sigma) + u_j(t_j^\sigma)$.

In the same way, the welfare of τ is $W(\tau) = \sum_{i \in N} u_i(t_i^\tau)$, and this sum can be decomposed as $\sum_{(i,j) \in \mu} u_i(t_i^\tau) + u_j(t_j^\tau)$.

By Lemma 8 however we have that $u_i(t_i^\tau) + u_j(t_j^\tau) = u_i(t_i^\sigma) + u_j(t_j^\sigma)$ for every pair in μ, thus $\sum_{(i,j) \in \mu} u_i(t_i^\sigma) + u_j(t_j^\sigma) = \sum_{(i,j) \in \mu} u_i(t_i^\tau) + u_j(t_j^\tau)$ and $W(\sigma) = W(\tau)$.

[6] Recall that i and j are matched, thus $t_i + t_j = 1$.

■

From now on we can then restrict to consider only internally stable allocations. In this way we can get rid of the specification of the allocation and focus on the matching. This is not a restricting hypothesis in looking for welfare maximising allocations since every welfare maximising allocation has to be internally stable. We can define *the welfare of a matching* $W(\mu)$ as the welfare of any internally stable allocation based on that matching. As a consequence we may refer to a matching to be *welfare maximising* when it has the maximal welfare among all matchings. Two matchings are *welfare equivalent* if they have the same welfare value. Notice that an allocation is welfare maximising if and only if

- it is internally stable, and
- it is based on a welfare maximising matching.

Notice also that two internally stable allocations are welfare equivalent if and only if they are based on welfare equivalent matchings.

Before proceeding we introduce the notion of *swap*.
Given an allocation μ and two agents i and j not matched in μ, let i' and j' be their respective partners. To swap i and j in μ means to build a matching μ' in which i and j are exchanged, i.e. the pairs $(i,i'),(j,j')$ are replaced by the pairs $(j,i'),(i,j')$.

We now want to introduce the concept of top-bottom matchings.[7]

DEFINITION **10.** *Given a preference profile* $\succsim$*, a top-bottom matching is a matching* μ *where, for every two pairs of agents* $(i,j),(h,k)\in\mu$*,*

- *either* $p_i \leq p_h$ *and* $p_j \geq p_k$*,*
- *or* $p_i \geq p_h$ *and* $p_j \leq p_k$*.*

Notice that the top-bottom matching is unique if all the peaks of the agents in N are distinct. The matching $\{(1,n),(2,n-1),...,(\frac{n}{2}),\frac{n}{2}+1)\}$ that associates the first agent with the last, the second agent with the second last, etc. is a top-bottom matching.
Moreover, every top-bottom matching can be obtained by swapping agents with the same peak, and thus the same preference profile.

THEOREM **2.** *Every top-bottom matching is welfare maximising.*

[7] We refer to this matching as a "top peak-bottom peak" matching in Chapter 1. We show, moreover, in Example 2 that there might be no stable allocation based on this matching in case of non-symmetric single-peaked preferences.

Proof: We will prove this result by induction on the cardinality of N. Recall that the cardinality of N is even. Moreover when $|N| = 2$ there is only one possible matching and that matching is top-bottom, thus the theorem is trivially true. Thus we treat $|N| = 4$ as the base case, and the inductive step is to prove that, if the statement holds for every set N such that $|N| = n$, then it also holds for every set N' such that $|N'| = n + 2$.

- Base Case. Let $N = \{1, 2, 3, 4\}$, recall that $p_1 \leq p_2 \leq p_3 \leq p_4$. We want to prove that $\{(1, 4), (2, 3)\}$ is a welfare maximising allocation.
  To do this we need to consider different cases:
  - If $p_1 + p_2 \geq 1$ then $p_i + p_j \geq 1$ for every pair i, j, thus $w(i, j) = 3 - p_i - p_j$. So whichever matching we choose $W(\mu) = 6 - \sum_{i=1}^{4} p_i$. All the matchings are welfare equivalent, so every matching is welfare maximising.
  - Conversely if $p_3+p_4 \leq 1$ then $p_i+p_j \leq 1$ for every pair i, j, thus $w(i, j) = 1+p_i+p_j$. So whichever matching we choose $W(\mu) = 2 + \sum_{i=1}^{4} p_i$. All the matchings are welfare equivalent, so every matching is welfare maximising.
  - If $p_1+p_2 < 1$ and $p_1+p_3 \geq 1$ then for every pair $(i, j) \neq (1, 2)$ $w(i, j) = 3-p_i-p_j$ while $w(1, 2) = 1 + p_1 + p_2$.
    In this case $W((1, 3)(2, 4)) = W((1, 4)(2, 3)) = 6-\sum_{i=1}^{4} p_i$, whereas $W((1, 2)(3, 4)) = 4 + p_1 + p_2 - p_3 - p_4$. So $W((1, 4)(2, 3)) - W((1, 2)(3, 4)) = 2 - 2(p_3 + p_4) \leq 0$ since $p_3 + p_4 \geq 1$. Thus $W((1, 4)(2, 3)) \geq W((1, 2)(3, 4))$.
  - If $p_1+p_3 < 1$ and both $p_1+p_4 \geq 1$ and $p_2+p_3 \geq 1$ we have $W((1, 2)(3, 4)) = 4+p_1+p_2-p_3-p_4$, $W((1, 3)(2, 4)) = 4+p_1+p_3-p_2-p_4$ and $W((1, 4)(2, 3)) = 6-\sum_{i=1}^{4} p_i$. We can conclude as in the previous point that $W((1, 4)(2, 3))-W((1, 2)(3, 4)) \leq 0$. Moreover $W((1, 4)(2, 3)) - W((1, 3)(2, 4)) = 2 - 2(p_2 + p_4) \leq 0$ since $p_2 + p_4 \geq 1$
  - If $p_1 + p_4 < 1$ and $p_2 + p_3 > 1$ we have that $W((1, 2)(3, 4)) = 4 + p_1 + p_2 - p_3 - p_4$, $W((1, 3)(2, 4)) = 4 + p_1 + p_3 - p_2 - p_4$, and $W((1, 4)(2, 3)) = 4 + p_1 + p_4 - p_3 - p_2$. Since $p_1 + p_4 \geq p_1 + p_3 \geq p_1 + p_2$ and $p_2 + p_3 \leq p_2 + p_4 \leq p_3 + p_4$ we have that $W((1, 4)(2, 3)) \geq W((1, 3)(2, 4)) \geq W((1, 2)(3, 4))$.

  The remaining cases are similar to the ones already considered.

- Inductive Step. Let μ be a matching that involves $n + 2$ agents. We want to prove that there exists a top-bottom matching $\bar{\mu}$ such that $W(\bar{\mu}) \geq W(\mu)$.
  First, consider the agents 1 and $n + 2$. If they are not matched in μ, we call i and j their partners, respectively. Thus $(1, i)$ and $(j, n+2)$ belong to μ. Define the allocation μ' where i and $n + 2$ are swapped, so that $(1, n + 2)$ and (j, i) belong to μ'. Let $S = \{(1, i), (j, n + 2)\}$. Then

$$W(\mu') = w(1, n + 2) + w(j, i) + \sum_{p \in \mu \setminus S} w(p),$$

$$W(\mu) = w(1,i) + w(j,n+2) + \sum_{p\in\mu\setminus S} w(p).$$

In the base case we proved that $w(1,n+2) + w(j,i) \geq w(1,i) + w(j,n+2)$, thus $W(\mu') \geq W(\mu)$.
If the agents 1 and $n+2$ are matched in μ, then we simply let $\mu' = \mu$.
Let us now consider $\mu' \setminus \{(1,n+2)\}$, that is a matching within n agents, $\{2,...,n+1\}$. By the inductive hypothesis there exists a top-bottom matching ν on the set $\{2,....,n+1\}$ such that $W(\nu) \geq W(\mu' \setminus \{(1,n+2)\})$. Thus, let $\bar{\mu} = \nu \cup \{(1,n+2)\}$. Therefore, $W(\bar{\mu}) \geq W(\mu') \geq W(\mu)$. Moreover, $\bar{\mu}$ is by construction a top-bottom matching.

Since for every matching there exists a top-bottom matching with greater or equal welfare, and all top-bottom matchings are welfare equivalent, it follows that top-bottom matchings are welfare maximising.

■

2.2 The Simple-SAM Algorithm

In this section, we provide a formal description of an algorithm that allocates agents in a stable and Pareto efficient way. We will show that this algorithm, which we call Simple-SAM algorithm, is equivalent to the SAM algorithm presented in Chapter 1 in the case of agents with Euclidean single-peaked preferences. In particular, this section can be seen as an example of how the more complicated SAM algorithm works.

In the rest of the section, we adopt the following convention: whenever we write a triple (i, j, t_i) in the description of an allocation, we assume $i < j$.[8]

The Simple-SAM algorithm has $K + 1$ steps, including Step 0.
In Step 0, we match the agents around the median of N in such a way that we remain with the same number of high type and low type agents. We define also the sets D_1 and U_1, which are the union of the improvement sets of low type agents and of high type agents, respectively, matched in this step.
Let $K = \min\{|H|, |L|\}$. If $K \neq \frac{n}{2}$ we call M the set of agents $\{K + 1, ..., n - K\}$.
We have to distinguish three cases:

1. $|H| = |L|$, i.e. $M = \emptyset$. In this case $K = \frac{n}{2}$ and we do not match any agent. Therefore $U_1 = \emptyset$ and $D_1 = \emptyset$.

[8] Suppose agents 1 and 2 are paired in an allocation. Let $p_1 = 0.4$, $p_2 = 0.7$ and their contributions in the allocation be $t_1 = 0.1$, $t_2 = 0.9$. We shall write the triple as $(1, 2, 0.1)$.

2. $|H| > |L|$, i.e. $M \subseteq H$. We match agents in M using a top-bottom matching. Then we assign to each pair a contribution of 0.5. Here $U_1 = \cup_{i \in M} I_{i,0.5}$ and $D_1 = \emptyset$.

3. $|L| > |H|$, i.e. $M \subseteq L$. We match agents in M using a top-bottom matching. Then we assign to each pair a contribution of 0.5. Here $D_1 = \cup_{i \in M} I_{i,0.5}$ and $U_1 = \emptyset$. [9]

At each Step q, $1 \leq q \leq K$, we form a pair consisting of a low type agent and a high type agent, agent $K + 1 - q$ and agent $n - K + q$, respectively. We denote these agents by l_q and h_q, respectively. The algorithm is provided three inputs:

- peaks of agents h_q and l_q,
- the union $U_q \subseteq [0, 0.5]$ of the improvement sets of H-type agents matched until Step q,
- the union $D_q \subseteq [0, 0.5]$ of the improvement sets of L-type agents matched until Step q.

Then, we have to consider five cases:

1. if $p_{l_q} < 1 - p_{h_q}$ and $p_{l_q} \in U_q$, then we define $t_{l_q} := p_{l_q}$;
2. if $p_{l_q} < 1 - p_{h_q}$ and $p_{l_q} \notin U_q$, then we define $t_{l_q} := \min\{1 - p_{h_q}, \inf U_q\}$;
3. if $p_{l_q} > 1 - p_{h_q}$ and $1 - p_{h_q} \in D_q$, then we define $t_{l_q} := 1 - p_{h_q}$;
4. if $p_{l_q} > 1 - p_{h_q}$ and $1 - p_{h_q} \notin D_q$, then we define $t_{l_q} := \min\{p_{l_q}, \inf D_q\}$;
5. if $p_{l_q} = 1 - p_{h_q}$, then we define $t_{l_q} := p_{l_q}$.

At this point, we add to the allocation the triple (l_q, h_q, t_{l_q}). Note that $t_{h_q} = 1 - t_{l_q}$. Finally we define the new union sets of improvement sets for the following step,

- the new D_{q+1} set is $D_q \cup I_{l_q, t_{l_q}}$,
- the new U_{q+1} set is $U_q \cup I_{h_q, t_{h_q}}$.

THEOREM **3.** *The Simple-SAM Algorithm produces the same outcome as the SAM algorithm in this setup. That is, when the SAM algorithm is applied to an allocation of agents with Euclidean single-peaked preferences, together with the appropriate specifications on the tie-breaking rule $\succ^N$ and on the matching in Step* 0 *(see below), it reduces to the Simple-SAM algorithm.*

We will use the following specifications to prove Theorem 3.

[9] Recall that the definition of $I_{i,t}$ is not the same if $i \in H$ or $i \in L$.

- $\succ^N$ is the order induced by the distance from the median $\frac{n+1}{2}$, from lower distance to higher. In case of a tie we can specify $\succ^N$ both to prefer the lowest agent or the highest agent.[10]
  For example, if $n = 10$, agent 7 will be preferred to agent 9 but not to agent 5, since the latter is nearer to $\frac{n+1}{2} = 5.5$.

- At step 0 we allocate the required agents starting from the ones with the higher priority according to $\succ^N$, thus creating a top-bottom matching.
  For example, if the agents in $\bar{L}$ are $\{\frac{n}{2}-1, \frac{n}{2}, \frac{n}{2}+1, \frac{n}{2}+2\}$ we form the triples $(\frac{n}{2}, \frac{n}{2}+1, 0.5)$ and $(\frac{n}{2}-1, \frac{n}{2}+2, 0.5)$.[11]

Before proceeding to the proof of the theorem, we need the following lemma, related to the notion of equivalent contribution $e_i(t)$ stated in Chapter 1.

Lemma **9.** *a) Let $i, j \in L$ and suppose $i < j$. Then for every $t \in [0,1]$ $e_i(t) \leq e_j(t)$.*
b) Let $i, j \in H$ and suppose $i > j$. Then for every $t \in [0,1]$ $e_i(t) \leq e_j(t)$.

Proof: a) First notice that for every $k \in N, t \in [0,1]$, $|2p_k - t - p_k| = |t - p_k|$ and $2p_k - t = t \Leftrightarrow p_k = t$. Thus, if $2p_k - t \in [0, 0.5]$, $2p_k - t = e_k(t)$. If $2p_k - t \notin [0, 0.5]$ then we have two possible cases,

- if $2p_k - t < 0$, then $e_k(t) = -\epsilon$;

- if $2p_k - t > 0.5$, then $e_i(t) = 0.5 + \epsilon$;

where ϵ is any small positive number, as in the definition of $e_i(t)$.
Since $i < j$, $p_i \leq p_j$, thus $2p_i - t \leq 2p_j - t$. Let us consider the following cases.

- If $2p_i - t < 0$, then $e_i(t) = -\epsilon$.[12] But $e_j(t)$ is greater or equal to $-\epsilon$ for every t, hence we conclude.

- If $2p_j - t > 0.5$ then $e_j(t) = +\epsilon$.[13] But $e_i(t)$ is lower or equal to $0.5 + \epsilon$ for every t, hence we conclude.

- If $0 \leq 2p_i - t \leq 2p_j - t \leq 0.5$ then trivially $2p_i - t = e_i(t)$ and $2p_j - t = e_j(t)$.

[10]It is not important to specify the preference between two agents with the same distance from $\frac{n+1}{2}$ because this case can never happen along the SAM algorithm.

[11]Later in this chapter we will show that the agents to be allocated in step 0 of the SAM algorithm, if any, are always a set of agents that is "central" in N, i.e. the set of all the agents ranging from $\frac{n}{2} - a$ to $\frac{n}{2} + a + 1$ for some $0 < a < \frac{n}{2}$.

[12]This is always the case when $2p_j - t < 0$ since $2p_i - t \leq 2p_j - t$.

[13]This is always the case when $2p_i - t > 0.5$ since $2p_i - t \leq 2p_j - t$.

Proof of Point (b) is identical.

■

Proof: We prove Theorem 3 by induction on the step q.
We want to prove that at the end of each step q the outcomes of the two algorithms are the same, i.e. the agents matched at step q are the same and are matched in the same way by the two algorithms, each agent matched at step q receives the same contribution value in both algorithm, and the sets U_{q+1} and D_{q+1} are also the same in both algorithms. Additionally, the sets L_{q+1} and H_{q+1} defined at each step of the SAM algorithm will be shown to be the sets $\{1, ..., q-k+1\}$ and $\{n-k+q, ..., n\}$, respectively.
Along the proof we will sometimes recall the SAM algorithm, using italics.

- Base Case. We will prove first that the SAM algorithm and the Simple-SAM algorithm produce the same allocation at Step 0.

  There are three possibilities to consider.

  1. $|H| = |L|$. In this case both the SAM algorithm and the Simple-SAM algorithm do not match any agent.
  2. $|H| > |L|$. *Compute* $e_i(0.5)$ *for all* $i \in H$. *Pick the* $|H| - |L|$ *agents whose equivalents* $e_i(0.5)$ *are closest to* 0.5. *Ties are broken using the ordering* $\succ^N$.
     In this case $K = |L|$, so agents from 1 to K are low type agents, while agents from $K+1$ to n are high type agents. By Lemma 9 the high type agents with the highest $e_i(0.5)$ are the agents with the lowest peak. Since the order on N is increasing in the peaks, $\bar{H}$ consists of agents from $K+1$ to $n-K$. Ties are solved using $\succ^N$. Agents within $K+1$ and $n-K$ are the agents in N nearest to the median.
  3. $|H| < |L|$. *Compute* $e_i(0.5)$ *for all* $i \in L$. *Pick the* $|L| - |H|$ *agents whose equivalents* $e_i(0.5)$ *are closest to* 0.5. *Ties are broken using the ordering* $\succ^N$.
     In this case $K = |H|$, so agents from 1 to $n-K$ are low type agents, while agents from $n-K+1$ to n are high type agents. By Lemma 9 the low type agents with the highest $e_i(0.5)$ are the agents with the highest peak. Since the order on N is increasing in the peaks, $\bar{L}$ consists of agents from $K+1$ to $n-K$.[14]

  Recall that we imposed that the matching at this step in the SAM algorithm should follow a top-bottom matching, just as in the Simple-SAM algorithm.
  Finally notice that in both algorithms D_1 and U_1 are defined as the union of the improvement sets of the low type agents and high type agents, respectively, matched at step 0.

[14] Agents within $K+1$ and $n-K$ are the agents in N nearest to the median.

- Inductive Step.
  - First we prove that the primary agent of SAM algorithm is always the agent of his type with peak nearest to 0.5 among the unassigned agents, i.e. either the low type agent with the highest peak or the high type agent with the lowest peak. [15]
    If the primary agent is chosen at step $q.1$, then she is the agent with lowest peak among the high type agents with peak lower than $1 - \inf D_q$, i.e. the high type agent with the lowest peak.
    If the primary agent is chosen at step $q.2$, then she is the agent with highest peak among the low type agents with peak higher than $\inf U_q$, i.e. the low type agent with the lowest peak.
    If the primary agent is chosen at step $q.3$, then she is the agent with peak nearest to 0.5 by construction.
  - Secondly, we show that the secondary agent of the SAM algorithm is the agent of her type with the peak nearest to 0.5. Notice that at every substep the secondary agent is chosen as the still available agent with the highest equivalent $e_i(t)$, for some t that depends on the substep where she is chosen.
    In Lemma 9 we proved that this agent is the agent with the highest peak if she is an low type agent or the agent with the lowest peak if she is an hihg type agent. In both cases she is the available agent of her kind with the peak nearest to 0.5. Since agents in N are ordered according to their peak and ties are resolved in favour of the agent nearest to the median, l_q is the higher agent in L_q (that is, $K - q + 1$), and h_q is the lower agent in H_q (that is, $n - K + q$). Thus the agents matched at any step are the same in both algorithms.
  - We remove the agents $K - q + 1$ and $n - K + q$ from L_q and H_q, respectively, remaining with $L_{q+1} = \{1, ..., K - q\}$ and $H_{q+1} = \{n - K + q + 1, ..., n\}$.
  - We now need to prove that the agents receive the same contribution values according to the two algorithms.
    Notice that if $p_{l_q} \geq 1 - p_{h_q}$ and l_q is the primary agent in the SAM algorithm, or if $p_{l_q} \leq 1 - p_{h_q}$ and h_q is the primary agent in the SAM algorithm, then the contributions of the two algorithm coincide.
    Let us now consider the case where $p_{l_q} < 1 - p_{h_q}$ and l_q is the primary agent. This is possible only if the matching is added to the allocation at step $q.2$. In this case $t_{l_q} = \min\{p_{l_q}, \inf D_q\}$. We want then to prove that $p_{l_q} \leq \inf D_q$, with $p_{l_q} \in U_q$, which would imply $t_{l_q} = p_{l_q}$, as is stated in case 1 of the Simple-SAM algorithm. Lemma 1 shows that $1 - p_{h_q} < \sup U_q^1$ and $p_{l_q} < \sup D_q^1$,[16]; thus one

[15] Thus the one nearest to the median since ties are broken this way.

[16] Recall that U_1^q is the lowest connected component of U_q and D_1^q is the lowest connected component of

has $\inf U_q^1 = \inf U_q < p_{l_q} < 1 - p_{h_q} < \sup U_1^q$, and p_{l_q} belongs to $U_q^1 \subseteq U_q$. Therefore, either $p_{l_q} < \inf D_q$. or $p_{l_q} \in D_q \cap U_q$, but in Chapter 1 we proved that the two sets are disjoint, hence $p_{l_q} < \inf D_q$.
The case where $p_{l_q} > 1 - p_{h_q}$ and $1 - p_{h_q} \in D_q$ is analogous.

– Finally, notice that in both algorithms D_{q+1} and U_{q+1} are defined as the union of the improvement sets of the low type agents and high type agents, respectively, matched until step q.

■

2.3 Discussion

2.3.1 Incentive Compatibility

In the definition of the algorithm we assumed that the preferences of the agents are known to the Social Planner or, equivalently, that the agents report truthfully their preferences. However one can wonder what happens if we drop this hypothesis and consider agents that strategically report their preferences.
Unfortunately in this case there is a trade-off between strategy-proofness and welfare maximisation, as we will show in this section.

Since we need to compare different preference profiles for the same set of agents, we will drop the convention that the agents in N are ordered increasingly in their peaks.

Let $\mathcal{P}$ be the set of all possible preference profiles. A *social choice function* is a function $f : \mathcal{P} \to \Sigma$ that associates an allocation $f(\succsim)$ to any preference profile $\succsim$.
We say that a social choice function is *strategy-proof* if no agent can improve by misreporting her preferences. That is, a social choice function is not strategy-proof if there exists two preference profiles $\succsim = (\succsim_1, ..., \succsim_i, ..., \succsim_n)$ and $\succsim' = (\succsim_1, ..., \succsim_i', ..., \succsim_n)$ that differ only for agent i such that $f(\succsim') \succ_i f(\succsim)$. Otherwise, f is strategy-proof.

Theorem **4.** *If $|N| \geq 4$, then any utilitarian welfare maximising social choice function is not strategy-proof.*

Proof: Let $N = \{1, 2, 3, 4\}$ and let f be a welfare maximising social choice function.
Let us now consider the preference profiles $\succsim$ and $\succsim'$ induced by the following peaks shown in table 2.1,

We can compute the values shown in Table 2.2 for each pair of agents.

D_q.

	p_1	p_2	p_3	p_4
$\succsim$	0.43	0.42	0.43	0.6
$\succsim'$	0.41	0.42	0.43	0.6

Table 2.1: Peaks of agents.

	$w(1,2)$	$w(1,3)$	$w(2,3)$	$w(1,4)$	$w(2,4)$	$w(3,4)$
$\succsim$	1.85	1.86	1.85	1.97	1.98	1.97
$\succsim'$	1.83	1.84	1.85	1.99	1.98	1.97

Table 2.2: Value functions for each pair of agents.

Notice that for preference profile $\succsim'$, the only welfare maximising matching is $\{(1,4)(2,3)\}$. In fact $W(\{(1,4)(2,3)\}) = 3.84$ which is greater than $W(\{(1,3)(2,4)\}) = 3.82$ and $W(\{(1,2)(3,4)\}) = 3.80$. Thus, $f(\succsim')$ is based on $\{(1,4)(2,3)\}$ and is internally stable. Instead, according to $\succsim$, the only welfare maximising matching is $\{(1,3),(2,4)\}$. In fact, $W(\{(1,3)(2,4)\}) = 3.84$ which is greater than $W(\{(1,4)(2,3)\}) = 3.82$ and $W(\{(1,2)(3,4)\}) = 3.82$. Thus $f(\succsim)$ is based on $\{(1,3)(2,4)\}$ and is internally stable. In this case, since $t_1 + t_3$ is equal to 1, at least one agent between 1 and 3 receives a contribution greater than 0.5 in $f(\succsim)$. We can consider this agent, without loss of generality, to be agent 1. Then $u_1(t_1) \leq u_1(0.5) = 0.93$. If agent 1 were matched with agent 4, though, in an internally stable way, as it happens in $f(\succsim')$, $u_1(t_1)$ would be at least $u_1(1-p_4) = 1.97$. Thus, there would be a profitable deviation for agent 1 to declare that her preferences are $\succsim'_1$.
If $|N| > 4$, we can suppose that the other agents have perfectly complementary peaks, thus they have to be matched within themselves in any welfare maximising allocation.

■

2.3.2 Different Social Welfare Functions

In section 1 we have introduced the utilitarian welfare function. This welfare function was first theorized by Bentham (1789) and is still widely used. However, utilitarian welfare is only one possible choice among many different ways to compute the welfare of an allocation. In this section we discuss what changes in the analysis of our Euclidean single-peaked preferences model for some other common choices of welfare function.

A different class of welfare functions is the class of *weighted utilitarian welfare functions.*

The weighted utilitarian welfare functions can be expressed in the following form,

$$W(\sigma) = \sum_{i \in N} f(u_i(t_i^{\sigma})),$$

where $f : [0, 1] \to \mathbb{R}$ is an increasing and concave function.
These welfare functions can be seen as more egalitarian than the utilitarian welfare function, since an increase in utility for an agent with low starting utility would increase the welfare function more than the same increase in utility for an agent with higher starting utility.

The most extremely egalitarian welfare function is the *max-min welfare utility function.* Given an allocation σ, the Max-min welfare function is defined as

$$W(u_1, ..., u_n) = \min(u_1, ..., u_n).$$

Thus the max-min welfare function increases only if the utility of the lowest-utility agents increases.

Theorem **5.** *Let $\succsim$ be a preference profile. Let the welfare function be the Max-min welfare function. Then, there always exists a welfare maximising allocation based on a top-bottom matching.*

The intuition is as follows. Let us fix a matching μ. Let σ_μ be an allocation based on μ that splits equally the utility among the matched agents. Then σ_μ is trivially welfare-maximal within the set of allocations based on μ. We can define the welfare of any matching μ as the welfare of σ_μ, that is we define the Max-min welfare of μ as $\min\{\frac{w_p}{2}\}_{p \in \mu}$. Thus a welfare maximising matching is a matching that maximises $\min\{w_p\}_{p \in \mu}$.
Proof: We will prove this theorem by induction, in a similar way as we proved Theorem 2. The first non-trivial case is $|N| = 4$ and the inductive step takes as hypothesis the statement for $|N| = n$ and proves it for $|N| = n + 2$.

- Base Case. Let $N = \{1, 2, 3, 4\}$ such that $p_1 \leq p_2 \leq p_3 \leq p_4$. We want to prove that $(1, 4), (2, 3)$ is a welfare maximising matching.
  Let us start by noting that

$$p_1 + p_2 \leq p_1 + p_3 \leq p_1 + p_4 \leq p_2 + p_4 \leq p_3 + p_4, \tag{2.3}$$

$$p_1 + p_2 \leq p_1 + p_3 \leq p_2 + p_3 \leq p_2 + p_4 \leq p_3 + p_4. \tag{2.4}$$

  Without loss of generality we can assume that $\min\{w(1, 4), w(2, 3)\} = w(1, 4)$. Let us distinguish three cases.

– If $p_1 + p_4 < 1$, then

$$w(1,4) = 1 + p_1 + p_4 \geq 1 + p_1 + p_3 = w(1,3) \geq \min\{w(1,3), w(2,4)\},$$

$$w(1,4) = 1 + p_1 + p_4 \geq 1 + p_1 + p_2 = w(1,2) \geq \min\{w(1,2), w(3,4)\}.$$

Thus $\{(1,4),(2,3)\}$ is welfare maximal.

– If $p_1 + p_4 > 1$, then

$$w(1,4) = 3 - p_1 - p_4 \geq 3 - p_2 - p_4 = w(2,4) \geq \min\{w(1,3), w(2,4)\}.$$

$$w(1,4) = 3 - p_3 - p_4 \geq 3 - p_3 - p_4 = w(3,4) \geq \min\{w(1,2), w(3,4)\}.$$

Thus $\{(1,4),(2,3)\}$ is welfare maximal.

– If $p_1 + p_4 = 1$, then $w(1,4)$ is maximal and $\{(1,4),(2,3)\}$ is welfare maximal.

- Inductive Step. Let μ be a matching that involves $n+2$ agents. We want to prove that there exists a top-bottom matching $\bar{\mu}$ such that $W(\bar{\mu}) \geq W(\mu)$.
First, consider the agents 1 and $n+2$. If they are not matched we call i and j their partners, respectively. So $(1,i)$ and $(j,n+2)$ belong to μ.
Let us consider the allocation μ' where we swap i and $n+2$, so that $(1,n+2)$ and (j,i)[17] belong to μ'. Let $S = \{(1,i),(j,n+2)\}$.

  In the base case we proved that

$$\min\{w(1,n+2)w(j,i)\} \geq \min\{w(1,i), w(j,n+2)\},$$

$$\min\{p\}_{p \in \{(1,n+2),(i,j)\} \cup \mu \setminus S} \geq \min\{p\}_{p \in \mu}.$$

  Thus $W(\mu') \geq W(\mu)$.
If the agents 1 and $n+2$ are matched in μ, then we simply let $\mu' = \mu$.
Let us now consider $\mu' \setminus \{(1,n+2)\}$, that is a matching within the n agents $\{2,...,n+1\}$.
By the inductive hypothesis there exists a top-bottom matching ν on the set $\{2,....,n+1\}$ such that $W(\nu) \geq W(\mu' \setminus \{(1,n+2)\})$. Thus, let $\bar{\mu} = \nu \cup \{(1,n+2)\}$. Therefore, $W(\bar{\mu}) \geq W(\mu') \geq W(\mu)$. Moreover, $\bar{\mu}$ is by construction a top-bottom matching.

■

We can prove a similar result for symmetric concave separable welfare functions.

Theorem **6.** *Let $\succsim$ be a preference profile. Let $W(\sigma)$ be a weighted utilitarian welfare function. There always exists a welfare maximising allocation based on a top-bottom matching.*

[17] Since these are unordered pairs, this is the same pair as (i,j) in the case $p_i < p_j$.

Before proceeding to the proof of Theorem 6 we need to prove the following lemma regarding concave functions.

Lemma **10.** *Let I be an in interval in $\mathbb{R}$ and let $f : I \to \mathbb{R}$ be an increasing concave function.*
Let $x_1, x_2, x_3, x_4 \in I$ such that $x_1 \leq x_2 \leq x_4$ and such that $x_1 + x_4 \leq x_2 + x_3$, then $f(x_1) + f(x_4) \leq f(x_2) + f(x_3)$.

Proof: Let us start with the case $x_1 + x_4 = x_2 + x_3$. Thus $x_3 = x_4 + x_1 - x_2$.
If $x_1 = x_4$, then $x_1 = x_2 = x_3 = x_4$ and the thesis is trivially true.
If $x_1 \neq x_4$, we define $\alpha = \frac{x_2 - x_1}{x_4 - x_1}$. Thus $1 - \alpha = \frac{x_4 - x_1 - x_2 + x_1}{x_4 - x_1} = \frac{x_4 - x_2}{x_4 - x_1}$. By definition of concave function,

$$f(\alpha x_1 + (1-\alpha)x_4) \geq \alpha f(x_1) + (1-\alpha)(x_4),$$
$$f((1-\alpha)x_1 + \alpha x_4) \geq (1-\alpha)f(x_1) + \alpha f(x_4).$$

Thus

$$f(\alpha x_1 + (1-\alpha)x_4) + f((1-\alpha)x_1 + \alpha x_4) \geq \alpha f(x_1) + (1-\alpha)f(x_4) + (1-\alpha)f(x_1) + \alpha f(x_4). \quad (2.5)$$

Obviously,

$$\alpha f(x_1) + (1-\alpha)f(x_4) + (1-\alpha)f(x_1) + \alpha f(x_4) = f(x_1) + f(x_4).$$

Let us now concentrate on the first term of Equation (2.5),

$$(1-\alpha)x_1 + \alpha x_4 = \frac{(x_4 - x_2)x_1 + (x_2 - x_1)x_4}{x_4 - x_1} = \frac{x_2 x_4 - x_2 x_1}{x_4 - x_1} = x_2,$$

$$(1-\alpha)x_4 + \alpha x_1 = \frac{(x_4 - x_2)x_4 + (x_2 - x_1)x_1 + x_4 x_1 - x_4 x_1}{x_4 - x_1} = x_4 + x_1 - x_2 = x_3.$$

Thus

$$f(\alpha x_1 + (1-\alpha)x_4) + f((1-\alpha)x_1 + \alpha x_4) = f(x_3) + f(x_2).$$

If $x_3 > x_4 + x_1 - x_2$ then, since f is increasing,

$$f(x_2) + f(x_3) \geq f(x_2) + f(x_4 + x_1 - x_2) \geq f(x_1) + f(x_4).$$

■

This lemma has as corollary the well known results on concave functions that

$$\frac{f(x_1) + f(x_2)}{2} \leq f\left(\frac{x_1 + x_2}{2}\right). \quad (2.6)$$

for every concave function f and every x_1, x_2 in the domain of f.

We can now prove Theorem 6.

Proof: First notice that, by Equation 2.6, given a matching μ, one allocation that maximises the welfare based on μ is again the allocation σ_μ that split the values of the pairs equally.

The welfare of μ is $W(\sigma_\mu) = \sum_{p\in\sigma} 2f(\frac{w(p)}{2})$. Thus for the remainder of the proof we will assume that this allocation is chosen.

We will prove this theorem by induction, in a similar way as we proved Theorem 2. As in that case the first non-trivial case is $|N| = 4$ and the inductive step takes as hypothesis the statement for $|N| = n$ and proves it for $|N| = n+2$.

- Base Case. Let $N = \{1,2,3,4\}$ such that $p_1 \leq p_2 \leq p_3 \leq p_4$, we want to prove that $\mu = (1,4),(2,3)$ is a welfare maximising matching.
  Let $\mu' = \{p_1, p_2\}$ be a different matching on N. By Theorem 2 we know that $w(1,4)+w(2,3) \geq w(p_1)+w(p_2)$, by Theorem 5 we know that $\min\{w(1,4), w(2,3)\} \geq \min\{w(p_1), w(p_2)\}$. Thus by Lemma 10 $W(\mu) \geq W(\mu')$.

- Inductive Step. Let μ be a matching that involves $n+2$ agents. We want to prove that there exists a top-bottom matching $\bar{\mu}$ such that $W(\bar{\mu}) \geq W(\mu)$.
  First, consider the agents 1 and $n+2$. If they are not matched in μ, we call i and j their partners, respectively. Thus $(1,i)$ and $(j,n+2)$ belong to μ. Define the allocation μ' where i and $n+2$ are swapped, so that $(1,n+2)$ and (j,i) belong to μ'. Let $S = \{(1,i),(j,n+2)\}$. Then
  $$W(\mu') = 2f\left(\frac{w(1,n+2)}{2}\right) + 2f\left(\frac{w(j,i)}{2}\right) + \sum_{p\in\mu\setminus S} 2f\left(\frac{w(p)}{2}\right),$$
  $$W(\mu) = 2f\left(\frac{w(1,i)}{2}\right) + 2f\left(\frac{w(j,n+2)}{2}\right) + \sum_{p\in\mu\setminus S} 2f\left(\frac{w(p)}{2}\right).$$
  In the base case we proved that $2f(\frac{w(1,n+2)}{2}) + 2f(\frac{w(j,i)}{2}) \geq 2f(\frac{w(1,i)}{2}) + 2f(\frac{w(j,n+2)}{2})$, thus $W(\mu') \geq W(\mu)$.
  If the agents 1 and $n+2$ are matched in μ, then we simply let $\mu' = \mu$.
  Let us now consider $\mu' \setminus \{(1,n+2)\}$, that is a matching within the n agents $\{2,...,n+1\}$. By the inductive hypothesis there exists a top-bottom matching ν on the set $\{2,....,n+1\}$ such that $W(\nu) \geq W(\mu' \setminus \{(1,n+2)\})$. Thus, let $\bar{\mu} = \nu \cup \{(1,n+2)\}$. Therefore, $W(\bar{\mu}) \geq W(\mu') \geq W(\mu)$. Moreover, $\bar{\mu}$ is by construction a top-bottom matching.

Since for every matching there exists a top-bottom matching with greater or equal welfare, and all top-bottom matchings are welfare equivalent, it follows that top-bottom matchings are welfare maximising.

■

Unfortunately the maximisation of these other welfare functions is not compatible with stability as the following example will show.
Let us first recall the definition of stability from Chapter 1.

Definition **11.** *Let σ be an allocation and $i, j \in N$ be agents with contributions t_i and t_j respectively in σ. Then the pair (i, j) weakly blocks σ if there exists a contribution vector (t'_i, t'_j) with $t'_i + t'_j = 1$, $t'_i \succsim_i t_i$ and $t'_j \succsim_j t_j$ with either $t'_i \succ_i t_i$ or $t'_j \succ_j t_j$. An allocation is strongly stable if it cannot be weakly blocked by any pair of agents.*

Example 1. Let $N = \{1, 2, 3, 4\}$. Agents' peaks are summarized in Table 2.3

p_1	p_2	p_3	p_4
0.1	0.4	0.4	0.4

Table 2.3: Peaks of agents in Example 1.

We conjecture that, according to this preference profile, in every stable allocations σ the contribution t_i for every agent $i \in N$ should be 0.5.
Every allocation is formed of two pairs, thus the contribution values of two non matched agents: t_i^σ and t_j^σ are greater or equal to 0.5.
Suppose by contradiction that $t_j^\sigma > 0.5$, then there exists $0 < \epsilon \leq 0.1$ such that $0.5 < 0.5 + \epsilon < t_j^\sigma$. In this case agents i and j can block the allocation with $(i, j, 0.5 - \epsilon)$. Thus, $t_i^\sigma = t_j^\sigma = 0.5$ and the contributions of agents' i and j partners must be 0.5 too. In particular, the utilities of the agents in any stable allocation are the same, summarized in table 2.4.

Let us now consider the allocation $\tau = \{(1, 2, 0.35), (3, 4, 0.5)\}$. The utilities of agents in τ are summarized in Table 2.4 too.

	p_1	p_2	p_3	p_4
σ	0.6	0.9	0.9	0.9
τ	0.75	0.75	0.9	0.9

Table 2.4: Utilities of agents in τ and σ.

The max-min welfare of τ is 0.75, while the max-min welfare of σ is 0.6. Thus σ does not maximise the max-min welfare function.
Let us consider now a concave separable symmetric welfare function like

$$W(\gamma) = \sum_{i \in N} f(u_i(t_i^\gamma)),$$

where $f : [0, 1] \to \mathbb{R}$ is a increasing strictly concave function.
Recall that Equation 2.6 holds as an equality only if $x_1 = x_2$. Thus $f(0.6) + f(0.9) < f(0.75) + f(0.75)$ and $W(\sigma) < W(\tau)$.
Notice that τ and σ have, however, the same utilitarian welfare: 3.3. □

Example 1 shows an important limitation of more egalitarian welfare functions in our model. To maximise these kind of functions in many cases we are required to split equally the utility within the pairs, but this may harm stability: in fact, in our model some agents, in particular those whose peaks are nearest to the median 0.5, have more opportunities to block an allocation. Thus, to ensure the stability it is necessary to grant them a more favourable treatment.

3 Stable and Efficient Task Assignment to Pairs

Based on work by A. Nicolò, P. Salmaso, A. Sen, S. Yadav.

In many situations, agents are matched in teams in order to perform a task. Agents have preferences over the task that they are asked to complete, as well as over the partners that they are assigned to work with. Forming stable teams is often important - it ensures that agents do not decide to abandon their assignments and do better for themselves.

We study a model in which a centralised authority matches agents in pairs and assigns them a task. We are interested in mechanisms that satisfy stability, efficiency, and provide incentives to agents to truthfully reveal their preferences.
This problem shares some features with two-sided matching models, like the roommate problem, since agents have preferences over their potential partners. It also has common features with one-sided matching models like the house allocation model and the object assignment model, because a task has to be assigned to each pair of agents. In this sense, this problem is a hybrid of the two classical models. In this general setting it is very likely to have instances in which no stable allocations exist due to the presence of cycles in agents' preferences. For this reason we turn our attention to a setting which is still general and describes interesting real problems, and such that stable allocations always exist.

Individual preferences can be described by a graph in which each agent is a node. If two nodes (agents) have a link they are "friends". In this case they may have a set of tasks, possibly empty, that these agents like to perform together. Preferences over tasks are dichotomous but not separable, because the tasks that agent i likes to complete with agent j can be different from the tasks that agent i would like to complete with another partner $k \neq j$. The preferences over tasks are assumed to be *pairwise symmetric* among agents, i.e. the set of tasks that agent i would like to complete with j coincides with the set of tasks that j would like to do with i. Thus, for any pair of agents, there exists a set of common good tasks (possibly empty). If agents are not linked, the set of common good tasks is empty. Agent i's preferences are defined over all partner-object tuples and the tuple in which agent i remains alone. Partner-object tuples belong to three equivalence classes. The first equivalence class consists of tuples where the agent is paired with a friend and this pair is assigned an object from their common set. The second class contains the tuples in which the agent is matched with a friend, but the task they have to perform does not belong to the common set. The third equivalence class consists of tuples where the agent is matched with someone she is not friends with. We assume that each agent prefers to remain alone than being matched with someone with whom she is not linked.

This setting describes many interesting situations. Consider for instance a police department that assigns officers to different geographical areas (precincts) and neighborhood-based enforcement personnel in each precinct assume responsibility for public safety management within their geographic area. Officers have preferences over the precincts they are assigned and patrol an area either alone or with a partner. Officers prefer to have a partner as long as they trust on her and we assume that this feeling of confidence is reciprocal. A different example is the case of partnerships among institutions, firms, or research centers that are promoted by some national or international agency. In case of agencies in the EU, for instance, only partnerships between institutions in two different countries can apply for funding (i.e. can be linked, in our terminology); institutions like research centres have preferences over both the projects and the potential partners, and apply to a call only with partners with whom they may successfully carry out a project, even if they may decide to propose projects that are less compelling to them but have higher probability to be funded. Our setting applies to those cases in which only bilateral partnerships can be created and each institution can only belong to at most one funded partnership.
To solve this centralised matching problem we propose an algorithm, the Object Constrained Maximal Matching Algorithm (OCMMA), that generates a Pareto efficient and weak core assignment. We also characterise the social choice function that associates the OCMMA assignment at every preference profile by means of four axioms: OCMMA is the unique assignment rule that satisfies Pareto efficiency, a restricted version of Maskin Monotonicity, an invariance property with respect to deleted links and such that the outcome belongs to the weak core.

3.1 Relationship to Existing Literature

In this chapter we consider a variant of the model proposed in Nicolò et al. (2019). Both works analyse a model where agents have to be matched in pairs, and each pair must be assigned an object. The agents have preferences over (partner, project) tuples. Both models have dichotomous preferences over partners and objects. The set of possible partners is partitioned into good and bad partners, and the set of projects is partitioned into good and bad projects. However, while the preference domain in Nicolò et al. (2019) is separable over partners and projects, i.e. the marginal component preferences over partners and projects are independent, in the present work we assume that agents care more about partners than objects, and that the partition in good and bad objects can depend on the partner.

Finally, in the two works different set of properties are considered. In particular, efficiency may not be reached in the algorithm presented in Nicolò et al. (2019), contrary to the algorithm presented here.

These models relate to the classical roommate problem first introduced by Gale and Shapley (1962) (see Roth and Sotomayor (1992) for a discussion) but they allow agents to have preferences over both the roommate and the room. In this they reflect classical assignment problems (Shapley and Scarf (1974)), with the crucial difference that agents are replaced by couples of agents endogenously formed in our model.

A model more closely related to ours is Sethuraman and Smilgins (2015). In this unpublished paper the authors extend the classical two sided marriage problem by including a set of objects. Their model divides the agents in two sets, men and women, and each agent has a preference over pairs formed by an object and an agent of the opposite gender. Men, women and objects are considered to be in equal number and a matching is a collection of triples (man, woman, object). The authors show that if there are no well specified property rights on the object assigned to a pair and each agent can join a new pair with an object previously assigned to her, then stability may be absent. The authors then assume that either only men or women have ownership on the assigned object and can block an allocation by proposing to share the owned object with a different partner of the opposite gender. In this case the authors show that there always exists at least one stable matching.

Our model is different from the above one in four ways: first we consider a roommate problem, second we provide the possibility for agents to remain unmatched, third there are more objects than the maximal number of pairs, and fourth we impose stronger restrictions on admissible preference profiles while relaxing the conditions on admissible blockings.

An approach similar to Sethuraman and Smilgins (2015) has been followed by Combe (2017). In this work, the author shows that a stable matching might fail to exist in a two-sided model unless ownership on the objects is introduced. Moreover, efficiency is not always reached in his model, as opposed to what we find in our model.

A more similar approach to the one presented in this chapter is followed by Burkett et al. (2018). The authors propose a model of room assignment to pairs and investigate different two-stage mechanisms that first match agents in pairs and then assign to each pair a room according to a priority order. The preferences of the agents are similar to the one presented in our model: the authors assume that the agents involved in a pair give identical value to the pair, similarly to what we call pairwise alignment. Moreover, they assume that all the agents assign equal value to the same room, whereas in our model preferences on objects may depend on the pairing. The notion of stability is defined at the first stage, since agents can block a matching only before receiving a room. The authors show that, if priority order is determined at the first stage, stability may fail to exist, while a symmetric random priority order determined after the first stage is ex-post stable. Notice, however, that their notion of stability is different from ours. Moreover, it should be noted that we investigate many different properties such as efficiency, restricted Maskin monotonicity, etc.

Another related model is a variation of the stable roommate model called "stable activities", proposed by Cechlárová and Fleiner (2005). In this model agents, once matched, can choose between different activities to perform as a pair, and their preferences are defined over partner-activity pairs. In the same paper, equivalence to the original roommate problem by Gale and Shapley (1962) has been proved. While in their model each activity can be allocated an arbitrary number of times, in our model they can be allocated at most once. For our model we do not expect equivalence with the original roommate problem to hold, since our model takes many features and restrictions from assignment models, which would be trivial without the assumption that an object can be allocated a maximum number of times.

Pycia (2012) showed a very general result on the existence of stable coalitions. From his work we take the concept of pairwise alignment, that is fundamental in our setting as well. However, the presence of objects in our model make it not completely embeddable in his model. In fact, the notion of stability in Pycia (2012) considers a blocking coalition only a coalition where every agent involved strictly improves. If we consider objects as agents in our setting, however, they clearly cannot strictly improve. Moreover, we do not restrict ourselves to prove only results about coalition stability and we show a full characterization that involves many more axioms.

Another stream of papers that is loosely linked to our model concerns threesome matchings. In these models agents are divided in three groups and a matching is a collection of triples of agents belonging to different groups. Alkan (1988) considered a model where agents have preferences over pairs of agents of the other two groups and showed that a stable matching may not exists even when agents' preferences are separable. Biró and McDermid (2010) consider a similar model where preferences are cyclic, thus agents in the first group only care about the agent in the second group they are matched with, etc. However, these models are different from ours, as the absence of objects make them more similar to Pycia (2012) model than to our model.

Finally Raghavan (2018) considers an allocation problem where objects are assigned to pairs of agents but only have preferences over objects and not on their partner.

3.2 The Model

Let there be a finite set of agents $N = \{1, 2, \dots, i, j, \dots, n\}$ and a finite set of objects denoted by A. We assume that there is a "sufficiently large number" of objects; in particular $|A| \geq \frac{|N|}{2}$.

In our model, each agent is either assigned a partner and object or remains alone. Remaining alone is interpreted as being assigned the special object a^*, which is the outside option for all agents. A *triple* consists either of two distinct agents and an object or an agent, the null set and the object a^*. Formally, a triple t is either (i, j, a) where $i, j \in N$,

$i \neq j$ and $a \in A$ or $(i, \emptyset, a^*)$. The former signifies that i and j have been paired together with the alternative a; the latter indicates that i is unmatched. We shall write $i \in t$ if t is either of the form (i, j, a) or $(i, \emptyset, a^*)$. Abusing notation slightly, we shall also write $a \in t$ if $t = (i, j, a)$ and $a^* \in t$ if $t = (i, \emptyset, a^*)$. Let T denote the set of all triples. For any i, T_i denotes the triples to which i belongs, i.e. $T_i = \{t \in T | i \in t\}$.

Let $S \subseteq N$ with $S \neq \emptyset$. The collection of triples $\alpha \subset T$ is an *assignment for S* (or a *partial assignment*) if the following conditions hold: (i) for all $i \in S$, there exists a unique $t \in \alpha$ such that $i \in t$ (ii) for every $i \notin S$, there does not exist $t \in \alpha$ such that $i \in t$ and (iii) for all $a \neq a^*$, $|\{t \in \alpha | a \in t\}| \leq 1$. According to the definition, no agent outside S is part of any triple in α. Every agent in S either remains alone or is matched to another unique agent in S. Finally, every object that is not the outside option is assigned to at most one triple in α, i.e. such an object cannot be assigned to two pairs in S. If α is an assignment for S, we shall write $S = N(\alpha)$. Suppose $S = N(\alpha)$. Then $A(\alpha) = \{a \in t | t \in \alpha\}$, i.e. $A(\alpha)$ is the set of objects assigned to the members of $N(\alpha)$. Note that $a^* \in A(\alpha)$ in case some members of $S(\alpha)$ are unmatched in α. Let $u^\alpha = A \setminus A(\alpha)$; it is the set of objects not assigned in α. The partial assignments σ, τ are said to be mutually consistent if $N(\sigma) \cap N(\tau) = \emptyset$ and $A(\sigma) \cap A(\tau) = \emptyset$.

Let Σ denote the set of all partial assignments. A particular class of partial assignments is of special interest. A partial assignment is *complete* if $N(\alpha) = N$.[1] A complete assignment will be referred to simply as an assignment. Let Σ_c denote the set of (complete) assignments. Clearly $\Sigma_c \subset \Sigma$.

3.2.1 Preferences

Each agent has a preference ordering over all triples that she belongs to. In the standard model, it is customary to start with agent preferences and define a *preference profile* as a collection of agent preferences. In our model, we shall find it convenient to do the reverse - to start with the notion of a preference profile and deduce preferences from it.

A *preference profile* $\succsim$ is a weighted undirected graph consisting of the following elements:

1. A set of nodes which is the set of agents N.

2. A set of edges or links denoted by $L_{\succsim}$. If agents $i, j \in N$ are linked, we say $(i, j) \in L_{\succsim}$. These links are undirected in the sense that no distinction is made between (i, j) and (j, i).

3. A "weight" associated for each link $(i, j) \in L_{\succsim}$, which is a set of objects denoted by $A_{\succsim}(i, j)$. We allow for the possibility that $A_{\succsim}(i, j) = \emptyset$. In this case, we refer to the link (i, j) as an empty link. We denote the set of empty links by $E_{\succsim}$.

[1]Note that in this case, requirement (ii) is satisfied vacuously.

A pair of agents (i, j) are "friends" (or are linked) if $(i, j) \in L_{\succsim}$. By assumption, the notion of friendship/linkedness is symmetric - if i is a friend of j, then j is a friend of i. The set $A_{\succsim}(i, j)$ is the set of tasks/objects that the linked pair (i, j) would like to perform/own together. Our assumption regarding the symmetry of friendship makes it natural to assume that the tasks that i likes to perform with j are "exactly" the same as the ones that j likes to perform with i. An important feature of our model is that an agent may like to perform different tasks with different agents she is linked to. For instance, University i likes to collaborate on projects in Physics and Chemistry with University j, but with University k on projects in Economics and Management.[2] We also allow for the possibility that i and j are linked but have no tasks that they would like to perform together.

A preference profile $\succsim$ induces a preference ordering $\succsim_i$ for all $i \in N$. The ordering $\succsim_i$ is defined over T_i. The asymmetric component of $\succsim_i$ is denoted by $\succ_i$. For this purpose, we define the sets $H_1(\succsim, i)$, $H_2(\succsim, i)$ and $H_3(\succsim, i)$ below.

1. $H_1(\succsim, i) = \{(i, j, a) \in T_i : (i, j) \in L_{\succsim} \text{ and } a \in A_{\succsim}(i, j)\}$.

2. $H_2(\succsim, i) = \{(i, j, a) \in T_i : (i, j) \in L_{\succsim} \text{ and } a \notin A_{\succsim}(i, j)\}$.

3. $H_3(\succsim, i) = \{(i, j, a) \in T_i : (i, j) \notin L_{\succsim}\}$.

Here $H_1(\succsim, i)$ is the set of triples where i is matched to a friend and is given an object that both like. The set $H_2(\succsim, i)$ is the set of triples where i is matched to a friend but is not given an object that they both like. A triple where i is matched to a friend with whom she has an empty link is also included in this set. Finally, $H_3(\succsim, i)$ consists of triples where i is matched to an agent with whom she has no link. Observe that the only triple in T_i that does not belong to any of the sets $H_k(\succsim, i)$, $k \in \{1, 2, 3\}$ is the triple $(i, \emptyset, a^*)$ where i is unmatched.

The preference ordering $\succsim_i$ is constructed as follows: the sets $H_k(\succsim, i)$, $k \in \{1, 2, 3\}$ form equivalence classes with $H_1(\succsim, i)$ ranked strictly above $H_2(\succsim, i)$, which in turn is ranked strictly above $H_3(\succsim, i)$. In addition, the triple $(i, \emptyset, a^*)$ is ranked strictly above triples in $H_3(\succsim, i)$ and strictly below triples in $H_2(\succsim, i)$. More formally,

$$H_1(\succsim, i) \succ_i H_2(\succsim, i) \succ_i (i, \emptyset, a^*) \succ_i H_3(\succsim, i).$$

Observation **10.** The preferences of agents satisfy a property of *pairwise alignment*, similar to the one defined in Pycia (2012).[3] In particular, for any agent pair (i, j), $x \in A$ and $k \in \{1, 2, 3\}$, we have $[(i, j, x) \in H_k(\succsim, i)] \Leftrightarrow [(i, j, x) \in H_k(\succsim, j)]$.

[2] This means $A_{\succsim}(i, j)$ may be different from $A_{\succsim}(i, k)$ when $(i, j), (i, k) \in L_{\succsim}$.

[3] Note that original Pycia property is defined in a context where coalition of more than two agents have to be admissible, and is silent in matching models.

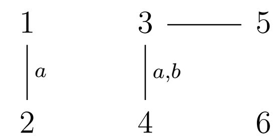

Figure 3.1: Preference Profile $\succsim$ in Example 2.

For any preference profile $\succsim$ and $k \in \{1, 2, 3\}$, we let $H_k(\succsim)$ denote the set of all triples that are ranked in equivalence class k by some agent, i.e. $H_k(\succsim) \coloneqq \cup_{i \in N} H_k(\succsim, i)$. We also denote the set of all preference profiles $\succsim$ by $\mathcal{R}$. We illustrate these notions by means of an example.

Example 2. Suppose that $N = \{1, 2, 3, 4, 5, 6\}$ and $A = \{a, b, c\}$ The graph in Figure 3.1 represents the preference profile $\succsim$.

Note that $L_{\succsim} = \{(1, 2), (3, 4), (3, 5)\}$, $A_{\succsim}(1, 2) = \{a\}$, $A_{\succsim}(3, 4) = \{a, b\}$, while $A_{\succsim}(3, 5) = \emptyset$ (i.e. $(3, 5)$ is an empty link). The graph specifies the preferences of all agents. For example, the first indifference class for agents 1 and 3 are $H_1(\succsim, 1) = \{(1, 2, a)\}$ and $H_1(\succsim, 3) = \{(3, 4, a), (3, 4, b)\}$, respectively. The second indifference class for agents 1 and 3 are $H_2(\succsim, 1) = \{(1, 2, b), (1, 2, c)\}$ and $H_2(\succsim, 3) = \{(3, 4, c), (3, 5, a), (3, 5, b), (3, 5, c)\}$, respectively. The third indifference class for agent 1 consists of triples where she is matched with agents 3, 4, 5 or 6 with any object. Similarly, the third indifference class for agent 3 consists of triples where he is matched with agents 1, 2 or 6 with any object. Both agents prefer being alone to any triple in their third indifference class. Thus $(1, \emptyset, a^*) \succ_1 (1, 6, a)$, for instance. Note that $H_1(\succsim) = \{(1, 2, a), (3, 4, a), (3, 4, b)\}$ and $H_2(\succsim) = \{(1, 2, b), (1, 2, c), (3, 4, c), (3, 5, a), (3, 5, b), (3, 5, c)\}$. □

3.3 Axioms

In this section, we introduce various axioms pertaining to a problem. A social choice function (SCF) associates an assignment with every preference profile. Formally, it is a function $f : \mathcal{R} \to \Sigma_c$. For any preference profile $\succsim \in \mathcal{R}$ and $i \in N$, let $f_i(\succsim)$ denote the triple to which agent i belongs to in the assignment $f(\succsim)$.

3.3.1 Blocking, Stability and Efficiency

The notion of stability is a familiar one in the matching literature.[4] An assignment is in the (weak) core if it cannot be (strongly) blocked by any coalition.[5] A coalition can block an assignment if there exists a "feasible" partial assignment for the coalition which makes every member of the coalition strictly better off. The notion of feasibility depends critically on the objects that are assumed to be available to the coalition. For our purpose, we will assume that the objects available to the coalition for blocking an assignment are of two kinds - all unassigned objects and objects that are assigned to pairs where the agents in the pair are not in the first indifference class. For instance, consider the assignment $(1, 2, a), (3, 5, c), (4, 6, b)$ in Example 2. The set of objects available to the coalition $S = \{3, 5\}$ for blocking this assignment is $\{b, c\}$. Although b is assigned to the pair $(4, 6)$, we allow it to be stolen by S since neither 4 nor 6 "like" b. We therefore have a permissive notion of blocking which is appropriate since we will prove a possibility result. We describe this formally below.

Let σ and $\succsim$ be an assignment and a preference profile respectively. Define the set $A_{\sigma,\succsim} = A \setminus \{a \in A : \exists i, j \in N$ such that $(i, j, a) \in \sigma$ and $(i, j, a) \in H_1(\succsim)\}$. We shall assume that $A_{\sigma,\succsim}$ is the set of objects available to any coalition S for blocking the assignment σ.

Definition **12.** *A coalition S strongly blocks the assignment σ at preference profile $\succsim$ if there exists a partial assignment σ such that (i) $N(\sigma') = S$, (ii) $A(\sigma') \subseteq A_{\sigma,\succsim}$ and (iii) every agent in $i \in S$ is strictly better-off in σ than in σ according to $\succsim_i$. An assignment is in the weak core if it is not strongly blocked by any coalition.*

According to our definition of the weak core, any agent i who is placed in the indifference class $H_1(\succsim, i)$ will never be part of any blocking coalition. Also agents who are placed in $H_3(\succsim, i)$ will block by breaking the pair ane remaining on their own.
A related notion is Pareto efficiency.

Definition **13.** *An assignment σ Pareto dominates an allocation τ at $\succsim$ if for all $i \in N$, i is at least as well-off in τ than in σ and there exists at least an agent k who is strictly better-off in τ than in σ. An assignment is Pareto efficient if there does not exist an assignment that Pareto dominates it.*

Stability and Pareto efficiency are independent in our model. Consider the assignment σ with the triples $(1, 2, a), (3, 5, b), (4, \emptyset, a^*), (6, \emptyset, a^*)$ in Example 2. Observe that the coalition $\{3, 4\}$ can block with the partial assignment $(3, 4, b)$. However, this assignment is Pareto efficient. Suppose τ Pareto dominates σ. Then, there must exist an agent who is strictly better-off in τ than in σ. Since agents 1 and 2 are in the first class in σ, they should remain

[4] See Roth and Sotomayor (1992).

[5] All over the chapter the weak core is the only notion of core we consider, thus we will call it simply "core" and the strong blocking simply "blocking".

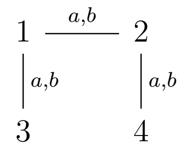

Figure 3.2: Preference Profile $\succsim'$ in Example 3.

in the first class in τ as well. Thus $(1,2,a)$ must belong to τ. Since agent 5 is in the second indifference class in σ, she must remain in the second class, i.e. she is with agent 3 in τ. The two remaining agents, 4 and 6, are unmatched in σ and must remain so in τ in order to be as well-off as before. Therefore, no agent strictly improves in τ and τ does not Pareto dominate σ.

The next example shows that an assignment can be in the weak core but not Pareto-efficient.

Example 3. Suppose that $N = \{1,2,3,4\}$ and $A = \{a,b\}$. The graph in Figure 3.2 represents the preference profile $\succsim'$.

Consider the assignment σ with the triples $(1,2,a), (3,\emptyset,a^*), (4,\emptyset,a^*)$. Observe σ is in the weak core because agents 1 and 2 are in their top indifference class and $3,4$ cannot improve. On the other hand, $\tau := \{(1,3,a),(2,4,b)\}$ Pareto dominates σ. □

We will say that a SCF is stable if its value at any preference profile belongs to the weak core. Similarly a SCF is Pareto efficient if its value at every preference profile is Pareto efficient.

3.3.2 Maskin Monotonicity

Maskin monotonicity (MM) is a well-known axiom in allocation theory. It was introduced in Maskin (1999) as a necessary condition for the Nash implementability of social choice correspondences. It can also be justified as an axiom in its own right. It has been used in in Kojima and Manea (2010) to characterise the Deferred Acceptance Algorithm.

Before defining the axiom formally, we introduce another piece of notation. The ordering $\succsim_i$ is defined over triples to which i belongs. This can be extended to assignments in an obvious way: for any pair of assignments $\sigma, \tau \in \Sigma_c$, we say $\sigma \succsim_i \tau$ if $t \succsim t'$, where t and t' are the (unique) triples to which i belongs in σ and τ, respectively.

Definition **14.** *The SCF f satisfies Maskin Monotonicity (MM) if, for all $\succsim, \succsim' \in \mathcal{R}$,*

$$[f(\succsim) \succsim_i \sigma \Rightarrow f(\succsim) \succsim'_i \sigma \text{ for all } \sigma \in \Sigma_c \text{ and } i \in N] \implies [f(\succsim) = f(\succsim')].$$

Let $\succsim$ be a preference profile and let $f(\succsim) = \tau$. Suppose $\succsim'$ is another preference profile with the property that all assignments that are less preferred to τ under $\succsim$ for all agents are also less preferred to τ under $\succsim'$ for all agents. If f satisfies Maskin Monotonicity, then f must also choose τ under $\succsim'$.

The MM condition has an important implication that distinguishes it from strategic properties such as strategy-proofness and group strategy-proofness. Suppose $\succsim, \succsim'$ are two preference profiles which differ only in the preferences of a strict subset of agents $S \subset N$. Let f be an SCF which satisfies Maskin Monotonicity. Assume further that the assignment $f(\succsim)$ "improves" in $\succsim'$ for all agents in S. Maskin Monotonicity implies that the assignments of all agents in $N \setminus S$ also do not change. This implication cannot be deduced if f satisfies only group strategy-proofness. Our characterization result depends heavily on this feature of Maskin Monotonicity.

The MM condition is a very strong condition in our model, as the following proposition shows. A similar result appears in Saijo (1987) for the unrestricted preference domain.

Proposition **6.** *Let f be a social choice function that satisfies Maskin monotonicity. Then f is constant.*

Proof: Let $\succsim, \succsim' \in \mathcal{R}$. Suppose $f(\succsim)(\succsim')$. Let $\succsim'' \in \mathcal{R}$ be the preference profile where all agents are linked and each pair likes all objects, i.e. for any agent $i \in N$, all triples except staying alone belong to $H_1(\succsim'', i)$. Observe that $f(\succsim) \succsim_i \sigma \implies f(\succsim) \succsim''_i \sigma$ for all $\sigma \in \Sigma_c$ and $i \in N$. Since f satisfies Maskin Monotonicity, we have $f(\succsim) = f(\succsim'')$. By an identical argument, we conclude $f(\succsim') = f(\succsim'')$. Thus $f(\succsim) = f(\succsim'') = f(\succsim')$ - a contradiction. ■

We define a weaker version of the MM condition below.

Definition **15.** *Let $\succsim, \succsim' \in \mathcal{R}$ be such that (i) for all $t \in f(\succsim)$, t is in the same equivalence class in $\succsim'_i$ as in $\succsim_i$ for all $i \in N$ and (ii) for all $t \notin f(\succsim)$, t is in the same or lower equivalence class in $\succsim'_i$ as in $\succsim_i$ for all $i \in N$. If the SCF f satisfies Resticted Maskin Monotonicity (RMM), then $f(\succsim') = f(\succsim)$.*

The RMM condition can be restated as follows. Suppose agents i, j are linked and get an object a that they like in $f(\succsim)$. In $\succsim'$, they must continue to be linked and to like a while they may no longer like other previously liked objects. Agents i, j who are linked but not assigned a commonly liked object, must continue to remain linked in $\succsim'$. Other links which are not used in the assignment $f(\succsim)$ may be broken. New links cannot be added nor can linked agents add new objects to their set of commonly liked objects. The RMM condition requires the same assignment to be chosen in $\succsim'$. It is clear that the RMM condition is weaker than MM since the antecedent of the former is weaker.

3.3.3 Invariance with respect to deleted links

This axiom imposes restrictions on the SCF when a pair of agents who have an empty link snap that link if they are assigned together by the SCF. The axiom requires that such a change should not affect the assignments of agents who are already in the highest indifference class. Changing the assignments of agents who were in their highest indifference class will not serve any purpose in improving the well-being of the agents who are no longer friends. In fact, the breaking of this link frees an extra object for possible assignment to other agents.

Definition **16.** *The preference profiles $\succsim, \succsim' \in \mathcal{R}$ are (i,j)-empty link variants if (i) $(i,j) \in E_{\succsim}$, (ii) $L_{\succsim'} = L_{\succsim} \setminus \{(i,j)\}$ and (iii) $A_{\succsim'}(k,l) = A_{\succsim}(k,l)$ for all $(k,l) \in L_{\succsim'}$. A SCF f satisfies Invariance with respect to Deleted Links (IDL) if for all $\succsim, \succsim' \in \mathcal{R}$ that are (i,j)-empty link variants and $(i,j,a) \in f(\succsim)$ for some $a \in A$, then $f_k(\succsim) = f_k(\succsim')$ for all $k \in N(f(\succsim) \cap H_1(\succsim))$.*

The IDL axiom is independent of the MM axiom. Suppose that $(i,j,a) \in f(\succsim)$, where the link between i and j is empty. If the link between i and j snaps, then the triple (i,j,a) declines in both $\succsim'_i$ and $\succsim'_j$. Under these circumstances, the antecedent of MM does not apply.

3.4 Object Constrained Maximal Matching Algorithm

In this section, we describe the Object Constrained Maximal Matching Algorithm (OCMMA). The algorithm is a simple sequential procedure which selects a weak core assignment. In the next section, we will show that the OCMMA assignment is the unique assignment satisfying the axioms elucidated in the previous section.

A partial assignment is (fully) contained in $H_1(\succsim)$ if all agents in the partial assignment are in the first indifference class according to $\succsim$. The set of all partial assignments with this property is denoted by $F_1(\succsim)$, i.e. $F_1(\succsim) = \{\sigma \in \Sigma : \sigma \subseteq H_1(\succsim)\}$. The set $F_2(\succsim)$ is defined similarly, i.e. $F_2(\succsim) = \{\sigma \in \Sigma : \sigma \subseteq H_2(\succsim)\}$. Every assignment in $F_1(\succsim)$ is composed entirely of triples in $H_1(\succsim)$ and every assignment in $F_2(\succsim)$ is composed entirely of triples in $H_2(\succsim)$.

Returning to Example 2, the set $F_1(\succsim)$ consists of the following sets - $\{(1,2,a)\}$, $\{(3,4,a)\}$, $\{(3,4,b)\}$ and $\{(1,2,a),(3,4,b)\}$. The set $F_2(\succsim)$ contains many elements such as $\{(3,5,a)\}$, $\{(3,5,b)\}$, $\{(1,2,c),(3,5,b)\}$, $\{(1,2,b),(3,4,c)\}$ etc.

Let $\succsim, \succsim'$ be such that $F_1(\succsim) \subseteq F_1(\succsim')$. Every allocation composed of only one triple $\{t\}$ that belongs to $F_1(\succsim)$ also belongs to $F_1(\succsim')$. Thus all triples in $H_1(\succsim)$ also belong to $H_1(\succsim')$ and $H_1(\succsim) \subseteq H_1(\succsim')$. If $H_1(\succsim) \subseteq H_1(\succsim')$, every partial assignment σ entirely composed of triples in $H_1(\succsim)$ is composed of triples in $H_1(\succsim')$. Thus $F_1(\succsim) \subseteq F_1(\succsim')$. Similarly we can show $F_2(\succsim) \subseteq F_2(\succsim') \Leftrightarrow H_2(\succsim) \subseteq H_2(\succsim')$. We record these facts below.

Observation **11.** We have $[F_1(\succsim) \subseteq F_1(\succsim') \Leftrightarrow H_1(\succsim) \subseteq H_1(\succsim')]$ and $[F_2(\succsim) \subseteq F_2(\succsim') \Leftrightarrow H_2(\succsim) \subseteq H_2(\succsim')]$.

We have $[F_1(\succsim) = F_1(\succsim') \Leftrightarrow H_1(\succsim) = H_1(\succsim')]$ and $[F_2(\succsim) = F_2(\succsim') \Leftrightarrow H_2(\succsim) = H_2(\succsim')]$.

We define the sets $\mathcal{F}_k$, $k \in \{1, 2\}$, as follows,

$$\mathcal{F}_k = \{F \subseteq \Sigma : \exists \succsim \in \mathcal{R} \text{ such that } F = F_k(\succsim)\}.$$

An important observation is that $\mathcal{F}_1 = \mathcal{F}_2$. In order to see this, pick an arbitrary preference profile $\succsim$. Let $\succsim^c$ be the preference profile such that (i) $L_{\succsim^c} = L_{\succsim}$ and (ii) $A_{\succsim^c}(i,j) = A \setminus A_{\succsim}(i,j)$ for all $(i,j) \in L_{\succsim}$. As a consequence, $H_1(\succsim^c) = H_2(\succsim)$, $H_2(\succsim^c) = H_1(\succsim)$ and $H_3(\succsim^c) = H_3(\succsim)$. These imply $F_1(\succsim^c) = F_2(\succsim)$ and $F_2(\succsim^c) = F_1(\succsim)$, thanks to Observation 11. Let $\mathcal{F} := \mathcal{F}_1 = \mathcal{F}_2$.

The OCMMA algorithm is defined by an admissible collection $\langle C_1, \{C_2^\sigma\}_{\sigma\in\Sigma}\rangle$ where

1. $C_1 : \mathcal{F} \to \Sigma$ is a Stage 1 choice function and
2. $C_2^\sigma : \mathcal{F} \to \Sigma$ is a Stage 2 choice function parameterised by the partial assignment $\sigma \in \Sigma$.

Definition **17.** *The collection $\langle C_1, \{C_2^\sigma\}_{\sigma\in\Sigma}\rangle$ is admissible if it satisfies the following conditions.*

1. *Feasibility: For all $F \in \mathcal{F}$ and $\sigma \in \Sigma$, the partial assignments σ and $C_2^\sigma(F)$ are mutually consistent.*
2. *Stage 1 maximality: For all $F \in \mathcal{F}$, there does not exist $\sigma \in F$ such that $N(C_1(F)) \subset N(\sigma)$.*
3. *Stage 2 maximality: For all $F \in \mathcal{F}$, $\sigma \in \Sigma$, $\tau \in F$, If $N(\tau) \supset N(C_2^\sigma(F))$ then τ and σ are not mutually consistent.*
4. *Contraction consistency [6]: For all $\sigma \in \Sigma$ and for all $F, F' \in \mathcal{F}$ such that $F' \subseteq F$, we have*
   (a) $[C_1(F) \in F'] \Rightarrow [C_1(F) = C_1(F')]$ and
   (b) $[C_2^\sigma(F) \in F'] \Rightarrow [C_2^\sigma(F) = C_2^\sigma(F')]$.

[6] This property is referred to as Sen's Property α in the literature on the rationalizability of choice functions (see Rubinstein (2012)).

We now describe the OCMMA algorithm. Let $\langle C_1, \{C_2^\sigma\}_{\sigma\in\Sigma}\rangle$ be an admissible collection. Fix an arbitrary preference profile $\succsim$. The assignment induced by the OCMMA algorithm is described below.

Step I: Apply C_1 to $F_1(\succsim)$ to generate the partial assignment $\sigma_1 = C_1(F_1(\succsim))$.

Step II: Apply apply $C_2^{\sigma_1}$ to $F_2(\succsim)$ to generate $\sigma_2 = C_2^{\sigma_1}(F_2(\succsim))$.

Step III: Match the remaining agents to the outside option, i.e. $\sigma_3 = \{(i, \emptyset, a^*)\}_{i\notin N(\sigma_1\cup\sigma_2)}$.

The outcome of the algorithm is $\sigma_1 \cup \sigma_2 \cup \sigma_3$.

The feasibility condition in Definition 17 ensures that the partial assignments generated in Steps I and II are mutually consistent and the algorithm is well-defined. The Stage 1 maximality condition in Definition 17 requires that a "largest possible" partial assignment (in terms of set inclusion) is chosen in Step I. The Stage 2 maximality condition is very similar to its Stage 1 counterpart, except that it is the largest possible but amongst those that are feasible in Stage 2. The contraction consistency condition is well known in choice theory. If the choice from a set belongs to a particular subset, then the choice from the subset must coincide with that from the larger set. In our setting the contraction consistency property, moreover, implies an even stronger property, as the following lemma shows.

Lemma **11.** *Let $\langle C_1, \{C_2^\sigma\}_{\sigma\in\Sigma}\rangle$ be an admissible collection. Then for all $\sigma \in \Sigma$ and for all $F, F' \in \mathcal{F}$,*

1. $[C_1(F) \in F'] \wedge [C_1(F') \in F] \Rightarrow C_1(F) = C_1(F')$ *and*
2. $[C_2^\sigma(F) \in F'] \wedge [C_2^\sigma(F') \in F] \Rightarrow C_2^\sigma(F) = C_2^\sigma(F')$.

Proof: Let us first prove part 1 of the lemma. We claim that there exists $F'' \in \mathcal{F}$ such that $[C_1(F) \in F''] \wedge [C_1(F') \in F'']$ and $F'' \subseteq F \cap F'$. In fact, let $\succsim$ and $\succsim'$ be two preference profiles such that $F_1(\succsim) = F$ and $F_1(\succsim') = F'$, and let $\succsim''$ be a preference profile such that $H_1(\succsim'') = C_1(F) \cup C_1(F')$. Notice that

- $C_1(F)$ belongs to $F_1(\succsim)$ and to $F_1(\succsim')$, thus it is contained in both $H_1(\succsim)$ and $H_1(\succsim')$,
- $C_1(F')$ belongs to $F_1(\succsim)$ and to $F_1(\succsim')$, thus it is contained in both $H_1(\succsim)$ and $H_1(\succsim')$

Thus $H_1(\succsim'') = C_1(F) \cup C_1(F') \subseteq H_1(\succsim) \cap H_1(\succsim')$. By Observation 11 this implies that $F'' = F_1(\succsim'') \subseteq F_1(\succsim) \cap F_1(\succsim') = F \cap F'$ and this proves the claim. By Contraction consistency then $C_1(F) = C_1(F'') = C_1(F')$.
The proof of part 2 of the lemma is analogous. ■

We illustrate OCMMA with an example.

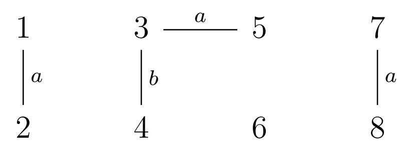

Figure 3.3: Preference Profile $\succsim$ in Example 4.

Example 4. Let $N = \{1, 2, \ldots, 8\}$ and $A = \{a, b, c, d\}$. Consider the preference profile whose graph is in figure 3.3.

We define the sets $H_1(\succsim)$, $H_2(\succsim)$ and $H_3(\succsim)$ using the preference profile $\succsim$. Observe that $H_1(\succsim) = \{(1, 2, a), (3, 4, b), (3, 5, a), (7, 8, a)\}$. The other sets can be defined similarly.

Consider a complete order $\succ^{\Sigma}$ defined on the set of partial assignment Σ with the following property: for all $\sigma, \tau \in \Sigma$ such that $N(\sigma) \supset N(\tau) \rightarrow \sigma \succ^{\Sigma} \tau$.

The OCMMA algorithm chooses assignments in three steps as follows. In Step I, it chooses a maximal partial assignment (σ_1) using the order from the set $F_1(\succsim)$. In Step II, it removes the agents and objects who have been assigned in Step I; then, it defines the set $F_2(\succsim)$ appropriately over the unassigned agents and objects and it chooses the maximal partial assignment (σ_2) from the set $F_2(\succsim)$. Finally, in Step III, all remaining agents remain in their own.

The set $F_1(\succsim)$ contains partial assignments, like $\{(1, 2, a), (3, 4, b)\}$, $\{(3, 5, a)\}$ etc. Note that both these assignments are maximal with respect to set inclusion. Assume $(3, 5, a) \succ^{\Sigma} \{(1, 2, a), (3, 4, b)\}$. Then $\sigma_1 = \{(3, 5, a)\}$.

In Step II, the partial assignment $\{(1, 2, b), (7, 8, c)\}$ may be chosen by $\succ^{\Sigma}$ from the set $F_2(\succsim)$. Finally in Step III, the OCMMA forms the triples $(4, \emptyset, a^*)$ and $(6, \emptyset, a^*)$. □

A feature of this example is that the Stage 1 and Stage 2 choice functions are induced by a complete order on the set of all partial assignments Σ. It is well known from standard decision theory that if all possible subsets of partial assignments could be generated as $F_1(\succsim)$ then contraction consistency would imply the existence of such an ordering.[7] However this is not the case in our model as shown in the example below.

Example 5. Let $N = \{1, 2, 3, 4, 5, 6\}$ and $A = \{a, b, c\}$. Consider the preference profiles $\succsim^1$, $\succsim^2$ and $\succsim^3$ in Figure 3.4.

Suppose $C_1(F_1(\succsim^1)) = \{(1, 2, a)(3, 5, b)\} = \alpha$, $C_1(F_1(\succsim^2)) = \{(3, 4, b)(2, 6, c)\} = \beta$ and $C_1(F_1(\succsim^3)) = \{(5, 6, c)(2, 3, a)\} = \gamma$. Observe that $\beta \in F_1(\succsim^1)$, $\gamma \in F_1(\succsim^2)$ and $\alpha \in F_1(\succsim^3)$. However there does exist a preference profile $\succsim \in R$ such that $F_1(\succsim) = \{\alpha, \beta, \gamma\}$. Any preference profile $\succsim$ with the property that $\alpha, \beta, \gamma \in F_1(\succsim)$ would look like the profile $\succsim^4$ in Figure 3.5. But then δ belongs to $F_1(\succsim^4)$ where $\delta = \{(1, 2, a), (3, 4, b), (5, 6, c)\}$. We can have $C_1(F_1(\succsim^4)) = \delta$ without contradicting contraction consistency. □

[7]See Rubinstein (2012).

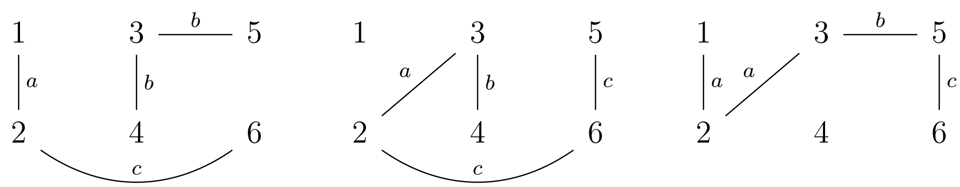

Figure 3.4: Preference profiles $\succsim^1$, $\succsim^2$, $\succsim^3$ in Example 5.

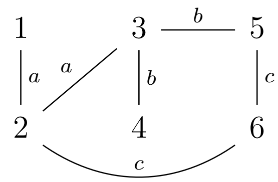

Figure 3.5: Preference profile $\succsim^4$ in Example 5.

3.5 The Result

We can now state our main result.

Theorem **7.** *A SCF f is Pareto efficient, satisfies RMM, IDL and belongs to the weak core if and only if there exists an admissible collection $\langle C_1, \{C_2^\sigma\}_{\sigma\in\Sigma}\rangle$ such that the outcome of the OCMMA algorithm induced by this collection at the preference profile $\succsim$ is $f(\succsim)$.*

Proof: We begin by showing that the assignment generated by OCMMA from the admissible collection satisfies the axioms mentioned above. Let $\succsim$ be an arbitrary preference profile and $\sigma = \sigma_1 \cup \sigma_2 \cup \sigma_3$ be the assignment generated by the algorithm at $\succsim$ as described earlier.

Pareto efficiency: We assume for contradiction that σ is not Pareto efficient, i.e. there exists $\tau \in \Sigma_c$ such that τ Pareto dominates σ. We will show that $N(\tau \cap H_1(\succsim)) = N(\sigma_1)$ and $N(\tau \cap H_2(\succsim)) = N(\sigma_2)$.

Since τ Pareto dominates σ, we have $N(\sigma_1) \subseteq N(\tau \cap H_1(\succsim))$. It is clear that $\tau \cap H_1(\succsim) \in F(\succsim)$. By Stage 1 maximality, we know $N(\sigma_1) \not\subset N(\tau \cap H_1(\succsim))$. Therefore $N(\sigma_1) = N(\tau \cap H_1(\succsim))$.

We will now show $N(\tau \cap H_2(\succsim)) = N(\sigma_2)$. Pareto efficiency of τ implies that $N(\sigma_2) \subseteq N(\tau \cap H_2(\succsim))$. Suppose $\tau \cap H_2(\succsim) \in F_2(\succsim)$. Then, by Stage 2 maximality, $N(\sigma_2) \not\subset N(\tau \cap H_2(\succsim))$. Thus $N(\sigma_2) = N(\tau \cap H_2(\succsim))$. Suppose $\tau \cap H_2(\succsim) \notin F_2(\succsim)$. Construct a partial assignment α as follows: $N(\alpha) = N(\tau \cap H_2(\succsim))$ and the agents in α are paired exactly as in τ; however, they are assigned objects in $A \setminus A(\sigma_1)$. By construction, $\alpha \in F_2(\succsim)$. By Stage 2 maximality, we have $N(\sigma_2) \not\subset N(\alpha)$. Since $N(\alpha) = N(\tau \cap H_2(\succsim))$, it follows that $N(\sigma_2) \not\subset N(\tau \cap H_2(\succsim))$. Therefore, $N(\sigma_2) = N(\tau \cap H_2(\succsim))$.

Since $N(\sigma_1) \subseteq N(\tau \cap H_1(\succsim))$ and $N(\sigma_2) = N(\tau \cap H_2(\succsim))$, the remaining agents in $N(\sigma_3)$ cannot strictly improve in τ compared to σ. Therefore, τ does Pareto dominate σ contradicting our hypothesis.

Weak core: Consider a coalition S which blocks σ. Since agents must strictly improve by blocking, none of the agents $k \in S$ belong to any triple in σ_1. Pick an arbitrary agent $i \in S$. Let (i, j, a) be the triple in the partial assignment used to block σ. Suppose i improves to $H_1(\succsim, i)$. By pairwise alignment, her partner j must also improve to $H_1(\succsim, j)$. Furthermore $a \notin A(\sigma_1)$. But then (i, j, a) should have been assigned in Step 1 by virtue of Stage 1 maximality.

Suppose i improves to $H_2(\succsim, i)$. By pairwise alignment, her partner j also improves to $H_2(\succsim, j)$. This implies that agents i and j belong to triples in σ_3. Since both agents strictly improve by blocking, $(i, j) \in L_{\succsim}$. This in turn implies that i and j should have been paired together with some object in $A \setminus A(\sigma_1)$ by OCMMA in Step 2 in order to satisfy Stage 2 maximality.

IDL: Consider $\succsim, \succsim' \in \mathcal{R}$ such that $\succsim, \succsim'$ are (i, j)-empty link variants and $(i, j, a) \in f(\succsim)$ for some $a \in A$. By the definition of (i, j)-variants, we know $H_1(\succsim) = H_1(\succsim')$. This implies $F_1(\succsim) = F_1(\succsim')$. The OCMMA procedure implies $\sigma_1 = C_1(F_1(\succsim)) = F_1(\succsim')$. Therefore $f_k(\succsim) = f_k(\succsim')$ for all $k \in N(f(\succsim) \cap H_1(\succsim))$.

RMM: Let $\succsim, \succsim' \in \mathcal{R}$ be such that (i) for all $t \in f(\succsim)$, t is in the same equivalence class in $\succsim'_i$ as in $\succsim_i$ for all $i \in N$ and (ii) for all $t \notin f(\succsim)$, t is in the same or lower equivalence class in $\succsim'_i$ as in $\succsim_i$ for all $i \in N$. Recall $f(\succsim) = \sigma = \sigma_1 \cup \sigma_2 \cup \sigma_3$. Let $f(\succsim') = \sigma' = \sigma'_1 \cup \sigma'_2 \cup \sigma'_3$.

By the construction of $\succsim'$, $H_1(\succsim', i) \subseteq H_1(\succsim, i)$ for all $i \in N$. This implies $H_1(\succsim') \subseteq H_1(\succsim)$. By Observation 11, we have $F_1(\succsim') \subseteq F_1(\succsim)$. Also $\sigma_1 \in F_1(\succsim')$ by the construction of $\succsim'$. Since $\sigma_1 = C_1(F_1(\succsim))$ and $C_1(\succsim) \in F_1(\succsim') \subseteq F_1(\succsim)$, contraction consistency implies $\sigma_1 = C_1(F_1(\succsim')) = \sigma'_1$.

Note that some triples in $H_1(\succsim)$ can decline to $H_2(\succsim')$ and some in $H_2(\succsim)$ can decline further in $\succsim'$. The latter can occur only if some triples not in σ_2 break their links in $\succsim$. Construct an intermediate preference profile $\hat{\succsim}$ such that (i) $L_{\succsim} = L_{\hat{\succsim}}$ and (ii) $A_{\succsim'}(i, j) = A_{\hat{\succsim}}(i, j)$ for all $(i, j) \in L_{\succsim'} \cap L_{\hat{\succsim}}$. This implies $H_1(\hat{\succsim}) = H_1(\succsim')$ and $H_2(\succsim) \subseteq H_2(\hat{\succsim})$. Let $\hat{\sigma} = \hat{\sigma}_1 \cup \hat{\sigma}_2 \cup \hat{\sigma}_3$ be the assignment chosen by OCMMA at $\hat{\succsim}$. The arguments in the previous paragraph imply that $\sigma_1 = \hat{\sigma}_1$. Once again the choice function used in Stage 2 is the same in both $\succsim$ and $\hat{\succsim}$. By definition, $\hat{\sigma}_2$ and $\hat{\sigma}_1 = \sigma_1$ are mutually consistent. Since $L_{\succsim} = L_{\hat{\succsim}}$, we have $\hat{\sigma}_2 \in H_2(\succsim)$. These facts imply $\hat{\sigma}_2 \in F_2(\succsim)$. Also $F_2(\succsim) \subseteq F_2(\hat{\succsim})$ by Observation 11. By contraction consistency, it must be the case that $\sigma_2 = \hat{\sigma}_2$.

Returning to $\hat{\succsim}, \succsim'$, we note that $\sigma'_1 = \hat{\sigma}_1$. In addition, $H_2(\succsim') \subseteq H_2(\hat{\succsim})$. However a link that is present in $\hat{\succsim}$ and not in $\succsim$ does not belong to $\hat{\sigma}_2$. Thus $\hat{\sigma}_2 \in H_2(\succsim')$. We already know that $\hat{\sigma}_2$ is mutually consistent with $\hat{\sigma}_1 = \sigma'_1$. These facts imply that $\hat{\sigma}_2 \in F_2(\succsim')$.

By Observation 11, $F_2(\succsim') \subseteq F_2(\hat{\succsim})$. Contraction consistency implies that $\hat{\sigma}_2 = \sigma_2'$; hence, $\sigma_2' = \sigma_2$. We have already established $\sigma_1' = \sigma_1$. Thus $\sigma_3' = \sigma_3$, as all agents in Step III of OCMMA remain on their own.

Let f be a stable and efficient social choice function that satisfies RMM and IDL. We will show that there exists an admissible collection $\langle C_1, \{C_2^\sigma\}_{\sigma\in\Sigma}\rangle$ such that the outcome of the OCMMA algorithm induced by this collection at the preference profile $\succsim$ is $f(\succsim)$.
For the remainder of the proof we will call a link l such that $A_{\succsim}(l) = A$ a *full link* in $\succsim$. The set of full links in $\succsim$ are denoted as $F_{\succsim}(l)$.

Observation **12.** *Let $\succsim$ and $\succsim'$ be two preference profiles.*

- *If $H_1(\succsim) \subseteq H_1(\succsim')$, then*
  - *every link l in $L_{\succsim}$ that is not an empty link belongs to $L_{\succsim'}$ too;*
  - *for every link $l \in L_{\succsim} \cap L_{\succsim'}$ one has $A_{\succsim}(l) \subseteq A_{\succsim'}(l)$.*
- *If $H_1(\succsim) = H_1(\succsim')$, then*
  - *every link l in $L_{\succsim}$ that is not an empty link belongs to $L_{\succsim'}$ too; conversely, every link l in $L_{\succsim'}$ that is not an empty link belongs to $L_{\succsim}$ too;*
  - *for every link $l \in L_{\succsim} \cap L_{\succsim'}$ one has $A_{\succsim}(l) = A_{\succsim'}(l)$.*
- *If $H_2(\succsim) \subseteq H_2(\succsim')$, then*
  - *every link l in $L_{\succsim}$ that is not a full link belongs to $L_{\succsim'}$ too;*
  - *for every link $l \in L_{\succsim} \cap L_{\succsim'}$ one has $A_{\succsim}(l) \supseteq A_{\succsim'}(l)$.*
- *If $H_2(\succsim) = H_2(\succsim')$, then*
  - *every link l in $L_{\succsim}$ that is not a full link belongs to $L_{\succsim'}$ too; conversely, every link l in $L_{\succsim'}$ that is not a full link belongs to $L_{\succsim}$ too;*
  - *for every link $l \in L_{\succsim} \cap L_{\succsim'}$ one has $A_{\succsim}(l) = A_{\succsim'}(l)$.*

Along this proof we will need to compare different preference profiles several times. For later convenience, we introduce the following definitions.

- Given an allocation σ and a link (i,j), we say that (i,j) is *active in σ* if agents i and j are matched in σ. Otherwise, we say that the link (i,j) is *inactive in σ*.
- Given a preference profile $\succsim$ and a set of links $S \subseteq L_{\succsim}$, we say that we obtain the preference profile $\succsim'$ by *removing the set L from $\succsim$* when $L_{\succsim'} = L_{\succsim} \setminus L$ and $A_{\succsim'}(l) = A_{\succsim}(l)$ for every link $l \in L_{\succsim'}$.

- Given a preference profile $\succsim$ and a set of triples $T \subseteq H_1(\succsim)$ we say that we obtain the preference profile $\succsim'$ by *downgrading the set* T *in* $\succsim$ when $H_1(\succsim') = H_1(\succsim) \setminus T$ and $H_2(\succsim') = H_2(\succsim) \cup T$.

Observation **13.** *Note that RMM can be reformulated as follows:*
"If we obtain $\succsim'$ from $\succsim$ by downgrading a set T of triples in $H_1(\succsim) \setminus f(\succsim)$ and removing a set $L \subset L_{\succsim}$ of inactive links from $\succsim$, then $f(\succsim') = f(\succsim)$."

First, we prove the following lemma.

Lemma **12.** *Let $\succsim$ and $\succsim'$ be two preference profiles such that $F_1(\succsim) = F_1(\succsim')$, then* $f(\succsim) \cap H_1(\succsim) = f(\succsim') \cap H_1(\succsim')$.

Proof: If $\succsim = \succsim'$ there is nothing to prove, thus let us suppose that the two preference profiles are different.
Notice that, by Observation 11, $F_1(\succsim) = F_1(\succsim') \Rightarrow H_1(\succsim) = H_1(\succsim')$. From now on we will call the latter set $H_1(\succsim)$. By Observation 12, for every non-empty link l in $L_{\succsim}$, l belongs to $L_{\succsim'}$ too, and $A_{\succsim}(l) = A_{\succsim'}(l)$, the opposite holds too. Therefore, $\succsim$ and $\succsim'$ differ at most by their set of empty links, $E_{\succsim}$ and $E_{\succsim'}$, respectively.
Let us consider first the case where we obtain $\succsim'$ by removing the empty link $(i, j) \in E_{\succsim}$ from $\succsim$. We now have two options.

- If i and j are not matched in $f(\succsim)$, then, by Restricted Maskin Monotonicity, $f(\succsim) = f(\succsim')$. In particular $f(\succsim) \cap H_1(\succsim) = f(\succsim') \cap H_1(\succsim')$.

- Otherwise, there exists $a \in A$ such that $(i, j, a) \in f(\succsim)$. In this case, by IDL, $f|_{H_1(\succsim)}(\succsim) = f|_{H_1(\succsim)}(\succsim')$. This means that $f(\succsim) \cap H_1(\succsim) \subseteq f(\succsim') \cap H_1(\succsim)$.
  Actually, one has $f(\succsim) \cap H_1(\succsim) = f(\succsim') \cap H_1(\succsim)$. In fact, let us suppose, by contradiction, that $f(\succsim) \cap H_1(\succsim) \subset f(\succsim') \cap H_1(\succsim)$. Then, there exists a triple $t = (k, h, b) \in (f(\succsim') \cap H_1(\succsim)) \setminus (f(\succsim) \cap H_1(\succsim))$. Since $f(\succsim')$ is an allocation, the agents h and k do not belong to $N(f(\succsim) \cap H_1(\succsim))$, thus they are not in the first class in $f(\succsim)$; moreover, one has $a \notin A(f(\succsim) \cap H_1(\succsim))$, thus the object a is available for any coalition to block the assignment $f(\succsim)$. In particular, the coalition $\{h, k\}$ can block the assignment $f(\succsim)$ using (h, k, a). Contradiction, because f is stable.

Let us consider now the general case where $H_1(\succsim) = H_1(\succsim')$, while $E_{\succsim} \neq E_{\succsim'}$.
In this case we can build a sequence of preference profiles $\succsim_0, \succsim_1, \succsim_2, ..., \succsim_k$ such that

- the preference profile $\succsim_0$ is equal to $\succsim$,

- the preference profile $\succsim_k$ is equal to $\succsim'$,

- for every $n \in \{0, ..., k-1\}$ either $\succsim_{n+1}$ is obtained from $\succsim_n$ by removing an empty link (i_n, j_n), or $\succsim_n$ is obtained from $\succsim_{n+1}$ by removing an empty link (i_n, j_n).

By the previous case, for every $n \in \{0, ..., k-1\}$ one has $f(\succsim_n) \cap H_1(\succsim_n) = f(\succ_{n+1}) \cap H_1(\succsim_{n+1})$. Thus, $f(\succsim) \cap H_1(\succsim) = f(\succsim') \cap H_1(\succsim')$.

■

As a consequence of this lemma we can define a choice function $C_1(F_1(\succsim))$ as $f(\succsim) \cap H_1(\succsim)$.

Lemma **13.** *$C_1(F_1(\succsim))$ satisfies contraction consistency.*

Proof: Let us suppose that $F_1(\succsim) \supset F_1(\succsim')$ and $f(\succsim) \cap H_1(\succsim) \in F_1(\succsim')$. We want to prove that $f(\succsim) \cap H_1(\succsim) = f(\succsim') \cap H_1(\succsim')$.
Notice that, by Remark 11, $F_1(\succsim) \supset F_1(\succsim') \Rightarrow H_1(\succsim) \supset H_1(\succsim')$.
Since $C_1(F_1(\succsim))$ and $C_1(F_1(\succsim'))$ do not depend on the particular choice of $\succsim$ and $\succsim'$, but only on the specific sets $F_1(\succsim)$ and $F_1(\succsim')$, we can restrict ourselves to consider the case where $\succsim'$ and $\succsim$ have the same set of links $L_{\succsim}$.[8] Thus, $\succsim'$ is obtained from $\succsim$ by removing a set $S \subseteq H_1(\succsim)$ of triples. Notice that every triple $t \in f(\succsim) \cap H_1(\succsim)$ belongs to $H_1(\succsim')$, hence t does not belong to S.
Thus, by Restricted Maskin Monotonicity, $f(\succsim) = f(\succsim')$. Since $f(\succsim) \cap H_1(\succsim) \in F_1(\succsim')$, $f(\succsim) \cap H_1(\succsim) = f(\succsim') \cap H_1(\succsim')$.

■

Lemma **14.** *$C_1(F_1(\succsim))$ respects stage 1 maximality.*

Proof: Let $A \in \mathcal{F}$ and $\sigma, \tau \in A$ be partial allocations such that $N(\sigma) \subset N(\tau)$. We want to prove that $C_1(A) \neq \sigma$. Let us consider a preference profile $\succsim$ such that $H_1(\succsim) = \tau \cup \sigma$ and such that $\succsim$ does not have empty links, so that the only links in $\succsim$ are within agents in $N(\tau)$. Notice that $F_1(\succsim) \subseteq A$.
Let us suppose by contradiction that $C_1(A) = \sigma$. Thus, by contraction consistency, $C_1(F_1(\succsim)) = \sigma$, i.e. $f(\succsim) \cap H_1(\succsim) = \sigma$. Let us consider $\bar{\tau}$ as an allocation that coincides with τ on $N(\tau)$ and that completes it giving the outside option to every agent not in $N(\tau)$, i.e. $\bar{\tau} = \tau \cup \{(i, \emptyset, a^*)\}_{i \in N \setminus N(\tau)}$. $\bar{\tau}$ Pareto dominates $f(\succsim)$ since:

- agents in $N(\sigma)$ are in the first indifference class according to both $f(\succsim)$ and $\bar{\tau}$,
- agents in $N(\tau) \setminus N(\sigma)$[9] are in the first indifference class according to $\bar{\tau}$ and not in the first indifference class according to $f(\succsim)$,
- agents in $N \setminus N(\tau)$ have no links in $\succsim$, so their best option is the outside option, which is provided by $\bar{\tau}$.

[8]Note that some of the links in $L_{\succsim}$ may be empty links in $\succsim$ or in $\succsim'$.
[9]This set is not empty, as by assumption $N(\sigma) \subset N(\tau)$.

We have thus a contradiction since the outcomes of f are Pareto efficient, thus $f(\succsim) \cap H_1(\succsim) \neq \sigma$ and $C_1(A) \neq \sigma$.

■

Lemma **15.** *Let* $\succsim$ *and* $\succsim'$ *be two preference profiles such that*

- $C_1(F_1(\succsim)) = C_1(F_1(\succsim'))$,
- $F_2(\succsim) = F_2(\succsim')$.

Then $f(\succsim) \cap H_2(\succsim) = f(\succsim') \cap H_2(\succsim')$.

Proof: If $\succsim = \succsim'$ there is nothing to prove, thus let us suppose that the two preference profiles are different.
Notice that, by Remark 11, $F_2(\succsim) = F_2(\succsim') \Rightarrow H_2(\succsim) = H_2(\succsim')$. From now on we will also call the latter set $H_2(\succsim)$. Thus for every non-full link l in $L_{\succsim}$, l belongs to $L_{\succsim'}$ too and $A_{\succsim}(l) = A_{\succsim'}(l)$; the converse holds, too. Therefore, $\succsim$ and $\succsim'$ differ at most by their set of full links, $F_{\succsim}$ and $F_{\succsim'}$, respectively.
Let $\sigma = C_1(F_1(\succsim)) = C_1(F_1(\succsim'))$. Notice that for every full link (i, j) that is present in $\succsim$ and not in $\succsim'$,

- i and j are not matched in σ, since $\sigma \in F_1(\succsim) \cap F_1(\succsim')$, and
- either i or j are in $N(\sigma)$, since otherwise they could block $f(\succsim)$ with any object.

Therefore i and j are not matched in $f(\succsim)$, and (i, j) is inactive in $f(\succsim)$. An analogous reasoning holds for every full link (i, j) that is present in $\succsim'$ and not in $\succsim$.

Thus, if we call $\succsim''$ the preference profile that has none of these links, i.e. such that

- $L_{\succsim''} = L_{\succsim} \cap L_{\succsim'}$, and
- for every link $l \in L_{\succsim''}$, $A_{\succsim''}(l) = A_{\succsim}(l) = A_{\succsim'}(l)$,

then, by RMM, $f(\succsim) = f(\succsim'') = f(\succsim')$.

■

We can then define $C_2^{\sigma}(A) = C_2^{C_1(F_1(\succsim))}(F_2(\succsim))$ for every $\sigma \in \Sigma$ and $A \in \mathcal{F}$ such that there exists a preference profile $\succsim$ such that $f(\succsim) \cap H_1(\succsim) = \sigma$ and $F_2(\succsim) = A$.
All the other cases are never reached by f, thus, once proved that $C_2^{\sigma}(A)$ respects stage 2 maximality and contraction consistency, we can complete C_2^{σ} in a way that respects these property even on sets where the OCMMA Algorithm never applies it.

The feasibility property follows directly from the fact that $f(\succsim)$ is an allocation.

LEMMA **16.** C_2^{σ} *satisfies contraction consistency.*

Proof: Let $\succsim$ and $\succsim'$ be two preference profiles such that

$$(C_1(F_1(\succsim)) = C_1(F_1(\succsim'))) \wedge (F_2(\succsim) \subseteq F_2(\succsim')) \wedge (C_2(F_2(\succsim') \in F_2(\succsim))$$

Let $\sigma = C_1(F_1(\succsim)) = C_1(F_1(\succsim'))$ and suppose that the only full links in both $\succsim$ and $\succsim'$ are the ones that involve agents matched in σ. By Observation 11, $F_2(\succsim) \subseteq F_2(\succsim') \Rightarrow H_2(\succsim) \subseteq H_2(\succsim')$. Thus,

- $L_{\succsim} \subseteq L_{\succsim'}$, and
- for every link $(i,j) \in L_{\succsim}$, $S_{\succsim}(i,j) \supseteq S_{\succsim'}(i,j)$.

Let us now consider the preference profile $\succsim''$ such that

- $L_{\succsim''} = L_{\succsim} \subseteq L_{\succsim'}$, and
- for every link $(i,j) \in L_{\succsim''}$, $S_{\succsim''}(i,j) = S_{\succsim'}(i,j) \subseteq S_{\succsim}(i,j)$.

Notice that $F_1(\succsim'') \subseteq F_1(\succsim')$ and that $\sigma \in F_1(\succsim'')$. Thus, by contraction consistency of C_1, $C_1(F_1(\succsim'')) = \sigma$.
We can obtain $\succsim''$ by removing the set of links $L := L_{\succsim'} \setminus L_{\succsim}$ from $\succsim''$. Now, one has that

- $\sigma \in F_1(\succsim)$, thus no link in L is active in σ,
- $C_2^{\sigma}(F_2(\succsim')) \in F_2(\succsim)$, thus no link in L is active in $C_2^{\sigma}(F_2(\succsim')) = f(\succsim') \cap H_2(\succsim')$,
- $f(\succsim')$ is Pareto efficient, so there are no active links in $f(\succsim')$ that are not in the first or second indifference class, i.e. that are not active either in σ or in $C_2^{\sigma}(F_2(\succsim'))$.

Thus no link in L is active in $f(\succsim')$ and, by RMM, $f(\succsim'') = f(\succsim')$.
We can obtain $\succsim''$ by downgrading the triples in the set $T := H_1(\succsim) \setminus H_1(\succsim')$. Since $f(\succsim) \cap H_1(\succsim) = \sigma \in F_1(\succsim')$, none of the triple in T belongs to $f(\succsim)$; thus, by RMM, $f(\succsim) = f(\succsim')$. In particular, $f(\succsim) \cap H_2(\succsim) = f(\succsim') \cap H_2(\succsim')$. ■

LEMMA **17.** $C_2^{\sigma}(F_2(\succsim))$ *satisfies the stage 2 maximality.*

Proof: Suppose, by contradiction, that there exists an allocation $\tau \subseteq H_2(\succsim)$, feasible with $\sigma := C_1(F_1(\succsim))$, such that $N(\tau) \supset N(C_2^{\sigma}(F_2(\succsim)))$.
We claim that, in this case, $\tilde{\tau} := \sigma \cup \tau \cup \{(i, \emptyset, a^*)\}_{i \in N \setminus N(\tau \cup \sigma)}$ Pareto dominates $f(\succsim)$. In fact,

- agents in $N(C_1(F_1(\succsim)))$ are in the first class both in $f(\succsim)$ and in $\tilde{\tau}$,

- agents in $N(C_2(F_2(\succsim)))$ are in the second indifference class both in $f(\succsim)$ and in $\tilde{\tau}$,
- agents in $N(\tau) \setminus N(C_2(F_2(\succsim)))$ are in the second indifference class in $\tilde{\tau}$ and either get the outside option or are in the third class in $f(\succsim)$, and
- all the remaining agents get the outside option in τ and either get the outside option or are in the third class in $f(\succsim)$.

Contradiction, since f is Pareto efficient. ■

■

3.6 Strategic Properties of the OCMMA

In this section we investigate the strategic properties of the OCMMA, in particular we discuss group strategy proofness.

We assume that the preferences of an agent are private information, and can be misreported by an agent if she believes this could be advantageous. However, the assumption of pairwise alignment of preferences introduces some complications in the standard model as it imposes a restriction on preference *profiles*: individual announcements of preferences may lead to profile announcements that are inconsistent with pairwise alignment. In particular, an agent's preferences are completely determined by preferences of other agents, thus she cannot misreport without creating a preference profile that is not admissible. However, a pair or even a group can coalize and misreport. Thus in this context it is more natural to think about the strategic properties of the algorithm in term of group strategy proofness.

In the following, we show that the OCMMA satisfies group strategy proofness. Let $\succsim$ denote a preference profile where $\succsim = (\succsim_1, \ldots, \succsim_n)$. Consider a non-empty coalition of agents $C \subseteq N$.

Definition **18.** *An SCF f is manipulable at $\succsim$ by a non-empty coalition $C \subseteq N$ if there exists $\tilde{\succsim}$ such that $\tilde{\succsim}_i = \succsim_i$ for all $i \notin C$ and $f(\succsim') \succ_i f(\succsim)$ for all $i \in C$. A SCF f is group strategy-proof if it is not manipulable by any non-empty coalition $C \subseteq N$ at any preference profile $\succsim$.*

Theorem **8.** *The SCF induced by the OCMMA algorithm is group strategy-proof.*

Proof: Let f be the SCF induced by the OCMMA algorithm. Consider an arbitrary non-empty coalition $C \subseteq N$. Let $\succsim$ be a preference profile. We assume by contradiction that there exists $\tilde{\succsim}$ such that $\tilde{\succsim}_i = \succsim_i$ for all $i \notin C$ and such that $f(\tilde{\succsim}) \succ_i f(\succ)$ for all $i \in C$.

Let $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ be the assignment generated by the OCMMA at the preference profile $\succsim$. Let N_1 be the set of agents who are allocated in Step 1 of the mechanism at $\succsim$, i.e. N_1 is the set of agents in $N(\sigma_1)$. Let A_1 denote the set of objects allocated in Step 1 of the mechanism at $\succsim$, so A_1 is $A(\sigma_1)$.
Let $\tilde{\sigma} = (\tilde{\sigma}_1, \tilde{\sigma}_2, \tilde{\sigma}_3)$ be the assignment generated by the OCMMA at the preference profile $\tilde{\succsim}$. Let $\tilde{N}_1$ be the set of agents in $\tilde{\sigma}_1$. Let $\tilde{N}_1$ and $\tilde{A}_1$ denote the set of agents and the set of objects allocated in Step 1 of the mechanism at $\tilde{\succsim}$.
First notice that no agent is in the third indifference class in σ, thus no agent can improve by getting the outside option in $\tilde{\sigma}$, and every agent in C has to be matched in $\tilde{\sigma}$.

Claim 1: $C \cap N_1 = \emptyset$.

Proof: This is a trivial observation: since all agents in C must strictly improve when they misreport and the agents in N_1 are in the first indifference class, they cannot improve. ■

Claim 2: For any pair of agents $i, j \in C$ such that $(i, j, a) \in \tilde{\sigma}$, we have $(i, j, a) \in H_1(\succsim)$.

Proof: Since agents i, j strictly improve by misreporting, $(i, j, a) \notin H_3(\succsim)$, thus $(i, j) \in L_{\succsim}$. Since the outcome of the OCMMA is stable, we know that either agent i or agent j is in the second indifference class in $f(\succsim)$,[10] otherwise they could block with any object[11]. Suppose w.l.o.g. that agent $i \in H_2(\succsim)$ in $f(\succsim)$; in $\tilde{\sigma}$ agent i should be better off, thus $(i, j, a) \in H_1(\succ_i)$. By pairwise alignment, we have $(i, j, a) \in H_1(\succ_j)$. Thus $(i, j, a) \in H_1(\succ)$. ■

Claim 3: $\sigma_1 = \tilde{\sigma}_1$. Thus $N_1 = \tilde{N}_1$

Proof: We know by definition that $\tilde{\sigma}_1 \in F_1(\tilde{\succsim})$. Let us suppose by contradiction that $\tilde{\sigma}_1 \in F_1(\tilde{\succsim}) \setminus F_1(\succsim)$, this implies that $\tilde{\sigma}_1$ is contained in $H_1(\tilde{\succsim})$ but it is not contained in $H_1(\succsim)$. In particular there exists a triple (i, j, a) in $\tilde{\sigma}_1$ that is in $H_1(\tilde{\succsim}) \setminus H_1(\succsim)$. Agents i and j are thus misreporting since their preferences on (i, j, a) are different from $\tilde{\succsim}$ to $\succsim$, so $i, j \in C$ and by claim 2 $(i, j, a) \in H_1(\succsim)$. Contradiction. Thus both $\tilde{\sigma}_1 \in F_1(\tilde{\succsim}) \cap F_1(\succsim)$ and $\sigma_1 \in F_1(\tilde{\succsim}) \cap F_1(\succsim)$. By Proposition 11 $\tilde{\sigma}_1 = \sigma_1$. ■

Since $f(\succsim)$ is stable, however, it is not possible that $(i, j, a) \in H_1(\succsim)$ and $i, j \notin N_1$, $a \notin A_1$. Thus, $C \subseteq H_1(\succsim) \cap f(\tilde{\succsim}) \subseteq N_1$, but by claim 1, $C \cap N_1 = \emptyset$. ■

[10]Recall that they cannot be in the first indifference class in $f(\succsim)$ by claim 1.

[11]Recall that $|A| \geq \frac{|N|}{2}$, thus there always exists an object $x \notin A_1$ if $N_1 \neq N$.

Bibliography

Abdulkadiroğlu, A. and T. Sönmez (1999): "House allocation with existing tenants," *Journal of Economic Theory*, 88, 233–260.

Abdulkadiroğlu, A. and T. Sönmez (2003): "School choice: A mechanism design approach," *American Economic Review*, 93, 729–747.

Alkan, A. (1988): "Nonexistence of stable threesome matchings," *Mathematical Social Sciences*, 16, 207–209.

Bentham, J. (1789): *An Introduction to the Principles of Morals and Legislation*, T.Payne and Son.

Bergantiños, G., J. Massó, I. M. De Barreda, and A. Neme (2015): "Stable partitions in many division problems: the proportional and the sequential dictator solutions," *Theory and Decision*, 79, 227–250.

Biró, P. and E. McDermid (2010): "Three-sided stable matchings with cyclic preferences," *Algorithmica*, 58, 5–18.

Burkett, J., F. X. Flanagan, and A. L. Griffith (2018): "Allocating group housing," *Social Choice and Welfare*, 50, 581–596.

Cechlárová, K. and T. Fleiner (2005): "On a generalization of the stable roommates problem," *ACM Transactions on Algorithms (TALG)*, 1, 143–156.

Ching, S. (1994): "An alternative characterization of the uniform rule," *Social Choice and Welfare*, 11, 131–136.

Combe, J. (2017): "Matching with Ownership," *Available at SSRN 3071879.*

Egerváry, J. (1931): "Matrixok kombinatorius tulajdonságairól," *Matematikai és Fizikai Lapok*, 38, 16–28.

Eriksson, K. and J. Karlander (2001): "Stable outcomes of the roommate game with transferable utility," *International Journal of Game Theory*, 29, 555–569.

Gale, D. and L. S. Shapley (1962): "College admissions and the stability of marriage," *The American Mathematical Monthly*, 69, 9–15.

Gensemer, S., L. Hong, and J. S. Kelly (1996): "Division rules and migration equilibria," *Journal of Economic Theory*, 69, 104–116.

Kojima, F. and M. Manea (2010): "Axioms for deferred acceptance," *Econometrica*, 78, 633–653.

König, D. (1916): "Über graphen und ihre anwendung auf determinantentheorie und mengenlehre," *Mathematische Annalen*, 77, 453–465.

Lovász, L. and M. D. Plummer (2009): *Matching theory*, vol. 367, American Mathematical Soc.

Mas-Colell, A., M. D. Whinston, J. R. Green, et al. (1995): *Microeconomic Theory*, vol. 1, Oxford University Press New York.

Maskin, E. (1999): "Nash equilibrium and welfare optimality," *The Review of Economic Studies*, 66, 23–38.

Nicolò, A., A. Sen, and S. Yadav (2019): "Matching with partners and projects," *Journal of Economic Theory*, 184, 104942.

Petersen, J. (1891): "Die Theorie der regulären graphs," *Acta Mathematica*, 15, 193.

Pycia, M. (2012): "Stability and preference alignment in matching and coalition formation," *Econometrica*, 80, 323–362.

Raghavan, M. (2018): "Pairwise Allocation Via Partner Trading," .

Roth, A. E. (1984): "The evolution of the labor market for medical interns and residents: a case study in game theory," *Journal of Political Economy*, 92, 991–1016.

Roth, A. E. and E. Peranson (1999): "The redesign of the matching market for American physicians: Some engineering aspects of economic design," *American Economic Review*, 89, 748–780.

Roth, A. E., T. Sönmez, and M. U. Ünver (2004): "Kidney exchange," *The Quarterly Journal of Economics*, 119, 457–488.

Roth, A. E. and M. Sotomayor (1992): "Two-sided matching," *Handbook of game theory with economic applications*, 1, 485–541.

Rubinstein, A. (2012): *Lecture notes in microeconomic theory: the economic agent*, Princeton University Press.

Saijo, T. (1987): "On constant Maskin monotonic social choice functions," *Journal of Economic Theory*, 42, 382–386.

Scarf, H. E. (1967): "The core of an N person game," *Econometrica*, 35, 50–69.

Schummer, J. and W. Thomson (1997): "Two derivations of the uniform rule and an application to bankruptcy," *Economics Letters*, 55, 333–337.

Sethuraman, J. and A. Smilgins (2015): "Two-sided matchings with Objects," *Mimeo.*

Shapley, L. and H. Scarf (1974): "On cores and indivisibility," *Journal of Mathematical Economics*, 1, 23–37.

Shapley, L. and R. Vohra (1991): "On Kakutani's fixed point theorem, the KKMS theorem and the core of a balanced game," *Economic Theory*, 1, 108–116.

Shapley, L. S. and M. Shubik (1971): "The assignment game I: The core," *International Journal of Game Theory*, 1, 111–130.

Sönmez, T. (1994): "Consistency, monotonicity, and the uniform rule," *Economics Letters*, 46, 229–235.

Sprumont, Y. (1991): "The division problem with single-peaked preferences: a characterization of the uniform allocation rule," *Econometrica*, 59, 509–519.

Teo, C.-P. and J. Sethuraman (1998): "The geometry of fractional stable matchings and its applications," *Mathematics of Operations Research*, 23, 874–891.