SPECTRAL STABILITY OF THE STEKLOV PROBLEM

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ABSTRACT. This paper investigates the stability properties of the spectrum of the classical Steklov problem under domain perturbation. We find conditions which guarantee the spectral stability and we show their optimality. We emphasize the fact that our spectral stability results also involve convergence of eigenfunctions in a suitable sense according with the definition of connecting system by [21]. The convergence of eigenfunctions can be expressed in terms of the H^1 strong convergence. The arguments used in our proofs are based on an appropriate definition of compact convergence of the resolvent operators associated with the Steklov problems on varying domains.

In order to show the optimality of our conditions we present alternative assumptions which give rise to a degeneration of the spectrum or to a discontinuity of the spectrum in the sense that the eigenvalues converge to the eigenvalues of a limit problem which does not coincide with the Steklov problem on the limiting domain.

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1. INTRODUCTION

In this paper we consider the spectral stability of the classical Steklov problem [18], namely

$$\begin{cases} \Delta u = 0, & \text{in } \Omega\,, \\ u_\nu = \lambda u, & \text{on } \partial\Omega\,, \end{cases} \tag{1}$$

where u_ν denotes the normal derivative of u on $\partial\Omega$ and λ is a real parameter, see [15] for a survey on this subject. Here and in the sequel, the domain for the Steklov problem will always be a bounded domain in $\mathbb{R}^N$ with $N \geq 2$ with appropriate conditions on the regularity of its boundary.

By a solution of (1) we mean a function $u \in H^1(\Omega)$ such that

$$\int_\Omega \nabla u \nabla v \, dx = \lambda \int_{\partial\Omega} uv \, dS\,, \qquad \text{for any } v \in H^1(\Omega)\,, \tag{2}$$

where $H^1(\Omega)$ denotes the standard Sobolev space of functions in $L^2(\Omega)$ with first order weak derivatives in $L^2(\Omega)$. It is well-known that (1) can be interpreted as an eigenvalue problem with respect to the parameter λ: we recall that λ is an eigenvalue for (1) if (1) admits a nontrivial solution in the sense of (2). That nontrivial solution is called eigenfunction for λ.

We recall that the set of the eigenvalues of (1) is a countable set of isolated nonnegative real numbers which may be ordered in an increasing sequence diverging to $+\infty$. As customary, we agree to repeat each eigenvalue as many times as its multiplicity:

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq \dots \tag{3}$$

We also recall that the first positive eigenvalue λ_1 admits the following variational characterization

$$\lambda_1 = \inf_{v \in \mathbb{H}(\Omega)} \frac{\int_\Omega |\nabla v|^2 \, dx}{\int_{\partial\Omega} v^2 \, dS} \tag{4}$$

where $\mathbb{H}(\Omega) := \left\{ v \in H^1(\Omega) \setminus H^1_0(\Omega) : \int_\Omega v \, dx = 0 \right\}$. By a classical argument it can be shown that all the other successive eigenvalues admit a minimax characterization as shown in more details in Section 2.5.

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In this article we study the stability of the eigenvalues and eigenfunctions of (1) with respect to domain perturbation. More precisely if $\{\Omega_\varepsilon\}_{0<\varepsilon\le\varepsilon_0}$ is a family of domains which converges to a domain Ω in a suitable sense, denoting by $\{\lambda_n^\varepsilon\}_{n=1}^\infty$, $\{\lambda_n\}_{n=1}^\infty$ the eigenvalues of the Steklov problem in Ω_ε and Ω respectively, we say that we have stability for the eigenvalues, if for any $n\ge 1$, $\lambda_n^\varepsilon\to\lambda_n$ as $\varepsilon\to 0$. We are also interested in what we will call spectral convergence of S_ε to S where S_ε, S denote the resolvent operators associated with (1) in Ω_ε and Ω respectively, see Section 2.3 for the precise definitions of these operators. Exploiting the results contained in [21], one can deduce that spectral convergence of those operators implies not only convergence of eigenvalues but also convergence of the corresponding eigenfunctions in a suitable sense, see Theorem 2.5 for more details.

We note that the behaviour of the eigenvalues of (1) subject to domain perturbation has been discussed in [8] which provides sharp conditions for their stability. Sufficient conditions ensuring the stability of the resolvent operators (associated with the corresponding Dirichlet-to-Neumann map) in a class of star-shaped domains are given in [20] where the question whether one could obtain the same results in a general setting is considered "out of reach". We also cite the very recent paper [9] concerning the asymptotic behaviour of the Steklov problem on dumbbell domains.

The aim of the present paper is to study not only the stability of the eigenvalues as done in [8] but also the stability of the eigenfunctions, and to give an answer to the question raised in [20] by proving stability results for the resolvent operators mentioned above in a more general class of domains. Moreover, we prove that our conditions are sharp by analysing the behaviour of the resolvent operators in a limiting case and we study the degeneration phenomena appearing when the strength of the boundary perturbations exceeds the threshold corresponding to that case. We also apply our results to the homogenization of problem (1) subject to periodic boundary perturbations.

Our results are in the spirit of the results of [1, 2, 3, 12] concerning the solutions of second order linear and nonlinear boundary problems with Robin type boundary conditions. Namely, in [12] the authors consider the problem

$$\begin{cases} -\Delta u = f(u), & \text{in } \Omega,\\ u_\nu + \beta u = 0, & \text{in } \partial\Omega, \end{cases}$$

where β is a fixed positive constant and provide stability and instability results for the solutions u upon perturbation of Ω. In particular, they identify the conditions on the perturbations of Ω which cause the degeneration of Robin boundary conditions to Dirichlet boundary conditions. They also discuss the case where a deformation of the coefficient β appears in the limiting problem. Similar results have been obtained in [1, 2, 3] for nonlinear boundary conditions of the form $u_\nu + g(x,u) = 0$.

We observe that the resolvent operators S_ε act on functional spaces which depend on the parameter ε so that, in order to provide a reasonable notion of compact convergence for operators acting on different functional spaces, we make use of suitable "connecting systems" which allow to pass from the varying Hilbert spaces defined on Ω_ε to the limiting fixed Hilbert space defined on Ω. This approach is made possible thanks to a number of notions and results which go back to the works of F. Stummel [19] and G. Vainniko [21] and which have been further implemented in [4, 11]. See also the recent paper [7]. All these notions are discussed in detail in Section 2.1. These abstract results are based on the notion of E-compact convergence of operators, possibly acting on different Hilbert spaces.

In order to prove stability results and the corresponding E-compact convergence of the operators S_ε, we consider domains Ω_ε, Ω belonging to uniform classes of domains with $C^{0,1}$ boundaries - see Definition 2.6 - and we require that the boundaries of Ω_ε converge to the boundary of Ω in the sense of (16)-(17). We observe that if the boundaries of Ω_ε converge to the boundary of Ω in C^1 then conditions (16)-(17) are satisfied. Conditions (16)-(17) make possible the construction in Section 2.4 of a family of linear continuous operators $E_\varepsilon : H^1(\Omega)\to H^1(\Omega_\varepsilon)$, i.e. the connecting system, which allows us to treat operators defined on $H^1(\Omega_\varepsilon)$ and $H^1(\Omega)$ simultaneously. In fact, the family of operators $\{E_\varepsilon\}$ is a connecting system in the sense of [21] provided that $H^1(\Omega_\varepsilon)$ is endowed with the equivalent norm

$$\|u\|_\varepsilon := \left(\int_{\Omega_\varepsilon} |\nabla u|^2 \, dx + \int_{\partial\Omega_\varepsilon} u^2 \, dS\right)^{1/2} \qquad \text{for any } u \in H^1(\Omega_\varepsilon) .$$

In particular, the family of operators $\{E_\varepsilon\}$ makes possible the definition of the notion of E-convergence for a sequence $u_\varepsilon \in H^1(\Omega_\varepsilon)$ to a function $u \in H^1(\Omega)$, i.e.

$$\|u_\varepsilon - E_\varepsilon u\|_\varepsilon \to 0, \ \text{ as } \varepsilon \to 0, \tag{5}$$

which allows to overcome the problem of having different functional spaces to deal with. We note that, similarly to [14], the operators E_ε are constructed by pasting together suitable pull-back operators defined by means of appropriate local diffeomorphisms.

In is important to observe that the operators E_ε satisfy the following property: for any $\varepsilon > 0$ there exists an open set $K_\varepsilon \subset \Omega \cap \Omega_\varepsilon$ such that $(E_\varepsilon u)(x) = u(x)$ for all $x \in K_\varepsilon$ and such that $|(\Omega_\varepsilon \cup \Omega) \setminus K_\varepsilon| \to 0$ as $\varepsilon \to 0$. (Here and in the sequel $|A|$ denotes the Lebesgue measure of any measurable set $A \subset \mathbb{R}^N$.) This, combined with (5), implies the more familiar strong convergence

$$\|u_\varepsilon - u\|_{H^1(\Omega_\varepsilon\cap\Omega)} \to 0, \ \text{ as } \varepsilon \to 0 \tag{6}$$

as we state in Proposition 4.4.

The main results of the paper are contained in Section 3; we proceed by describing in details the meaning of their contents.

The first main result is Theorem 3.1 in which we prove the spectral stability of (1) under the validity of conditions (16)-(17). More precisely we prove the E-compact convergence of S_ε to S as $\varepsilon \to 0$. On the base of the general Theorem 2.5, this implies the spectral convergence of S_ε to S, hence the convergence of the eigenvalues and the E-convergence of the eigenfunctions in the sense of Theorem 2.5. In particular the eigenfunctions converge in the sense of (6).

The subsequent results aim to show the optimality of conditions assumed in Theorem 3.1.

Theorem 3.2 can be considered an extension of Theorem 3.1 since we assume again the validity of (16) but we replace (17) with the weaker condition (46). Indeed, if we assume the validity of (16) and (17) simultaneously then (46) holds true with $\gamma_j = \sqrt{1 + |\nabla_{x'} g_j|^2}$ so that the function γ defined in (47) satisfies $\gamma \equiv 1$ on $\partial\Omega$. In such a case the eigenvalues λ_n^ε converge to λ_n as $\varepsilon \to 0$ and spectral stability is proved. But whenever the function $\gamma \not\equiv 1$ on $\partial\Omega$, convergence of the eigenvalues to the natural limit problem fails to be true thus giving rise to a discontinuity phenomenon. In Proposition 3.4 (ii), we exhibit an explicit example where the function $\gamma \not\equiv 1$. We note that, assumption (46) is, mutatis mutandis, the condition used in [12, Theorem 4.4] and in [2, 3].

Then in Theorem 3.3, we assume the alternative conditions (53)-(54) and we prove degeneration of eigenvalues by showing that $\lambda_n^\varepsilon \to 0$ as $\varepsilon \to 0$ for any $n \geq 1$. An explicit example in which (53)-(54) hold true can be found in Proposition 3.4 (iii). We also note that condition (54) was used in [1, 12] to prove the degeneration of the Robin problem to the Dirichlet problem.

Finally, we provide a more clear picture of the conditions contained in Theorems 3.1-3.3 by considering a particular case in which the domains Ω_ε and Ω are in the form

$$\Omega_\varepsilon = \{(x', x_N) \in \mathbb{R}^N : x' \in W , \ -1 < x_N < g_\varepsilon(x')\} , \qquad \Omega = W \times (-1, 0)$$

where W is a cuboid or a bounded domain in $\mathbb{R}^{N-1}$ of class $C^{0,1}$, $g_\varepsilon(x') = \varepsilon^\alpha b(x'/\varepsilon)$ for any $x' \in W$ and $b : \mathbb{R}^{N-1} \to [0, +\infty)$ is a nonconstant Y-periodic function with $Y = \left(-\frac{1}{2}, \frac{1}{2}\right)^{N-1}$ the unit cell in $\mathbb{R}^{N-1}$. Note that this type of periodic perturbations is classical in homogenization theory, in particular in the study of boundary homogenization problems, see e.g., [5, 13] and the references therein. In this particular situation, the conditions introduced in Theorems 3.1-3.3 find a clear representation depending on the value assumed by the exponent α. More precisely, it is proved in Proposition 3.4 that the assumptions of the three main theorems correspond to the cases $\alpha > 1$, $\alpha = 1$ and $0 < \alpha < 1$ respectively. Taking into account what was shown in Proposition 3.4, the statements of Theorems 3.1-3.3 can be unified in a single result contained in Theorem 3.5. This theorem shows spectral stability for $\alpha > 1$, degeneration in the case $0 < \alpha < 1$ and a discontinuity phenomenon in the limiting case $\alpha = 1$ in the sense that the eigenvalues of (59) converge to the eigenvalues of the modified problem (60) whose eigenvalues are given by the eigenvalues of (59) divided by the constant $C_b = \int_Y \sqrt{1+|\nabla_{x'} b(x')|^2}\, dx'$.

We note that in our previous paper [14], we considered the spectral stability of certain Steklov problems for the biharmonic operator. Although the proof of Theorem 3.1 is based on a method similar to the one used for the proof of the corresponding stability results in [14], the part of the present paper concerning the discontinuity and the degeneration phenomenon - namely, Theorems 3.2, 3.3 - is completely different and is specifically designed for problem (1).

We conclude this section by explaining how this paper is organized. In Section 2 we state some well-known basic results about spectral stability for operators defined on abstract Hilbert spaces, we introduce the main assumptions about the perturbed domains Ω_ε and the limit domain Ω, we contruct the resolvent operator S associated with (1), we construct a connecting system acting from $H^1(\Omega)$ into $H^1(\Omega_\varepsilon)$ and finally we show that the classical minimax characterization of eigenvalues applies to the Steklov spectrum. Section 3 is devoted to the statements of the main results already described above. Section 4 is devoted to the proof of Theorem 3.1 and to the statement and the proof of Proposition 4.4. Sections 5-6 are devoted to the proofs of Theorems 3.2-3.3 respectively, Section 7 to the proof of Proposition 3.4 and finally Section 8 to the proof of Theorem 3.5.

2. Preliminaries and notation

2.1. **A general approach to spectral stabilty.** The study of the spectral stability of (1) is reduced to the study of suitable families $\{B_\varepsilon\}_{0<\varepsilon\le\varepsilon_0}$ of non-negative compact self-adjoint operators defined in Hilbert spaces $\mathcal{H}_\varepsilon$ associated with the domains Ω_ε.

In order to follow this approach we recall here the notion of E-convergence. As already mentioned in the Introduction, we follow the approach of [21] and the successive development by [4], [11].

According to the notation used in [4] (see also [14]), we denote by $\mathcal{H}_\varepsilon$ a family of Hilbert spaces for $\varepsilon \in [0, \varepsilon_0]$ and we assume that there exists a family of linear operators $E_\varepsilon : \mathcal{H}_0 \to \mathcal{H}_\varepsilon$ such that

$$\|E_\varepsilon u\|_{\mathcal{H}_\varepsilon} \stackrel{\varepsilon\to 0}{\longrightarrow} \|u\|_{\mathcal{H}_0}\,, \qquad \text{for all } u \in \mathcal{H}_0\,. \tag{7}$$

Definition 2.1. *We say that a family $\{u_\varepsilon\}_{0<\varepsilon\le\varepsilon_0}$, with $u_\varepsilon \in \mathcal{H}_\varepsilon$, E-converges to $u \in \mathcal{H}_0$ if $\|u_\varepsilon - E_\varepsilon u\|_{\mathcal{H}_\varepsilon} \to 0$ as $\varepsilon \to 0$. We write this as $u_\varepsilon \stackrel{E}{\longrightarrow} u$.*

Definition 2.2. *Let $\{B_\varepsilon \in \mathcal{L}(\mathcal{H}_\varepsilon) : \varepsilon \in (0, \varepsilon_0]\}$ be a family of linear and continuous operators. We say that $\{B_\varepsilon\}_{0<\varepsilon\le\varepsilon_0}$ converges to $B_0 \in \mathcal{L}(\mathcal{H}_0)$ as $\varepsilon \to 0$ if $B_\varepsilon u_\varepsilon \stackrel{E}{\longrightarrow} B_0 u$ whenever $u_\varepsilon \stackrel{E}{\longrightarrow} u$. We write this as $B_\varepsilon \stackrel{EE}{\longrightarrow} B_0$.*

Definition 2.3. *Let $\{u_\varepsilon\}_{0<\varepsilon\le\varepsilon_0}$ be a family such that $u_\varepsilon \in \mathcal{H}_\varepsilon$. We say that $\{u_\varepsilon\}_{0<\varepsilon\le\varepsilon_0}$ is precompact if for any sequence $\varepsilon_n \to 0$ there exist a subsequence $\{\varepsilon_{n_k}\}$ and $u \in \mathcal{H}_0$ such that $u_{\varepsilon_{n_k}} \stackrel{E}{\longrightarrow} u$.*

Definition 2.4. *We say that $\{B_\varepsilon\}_{0<\varepsilon\le\varepsilon_0}$ with $B_\varepsilon \in \mathcal{L}(\mathcal{H}_\varepsilon)$ and B_ε compact, converges compactly to a compact operator $B_0 \in \mathcal{L}(\mathcal{H}_0)$ if $B_\varepsilon \stackrel{EE}{\longrightarrow} B_0$ and for any family $\{u_\varepsilon\}_{0<\varepsilon\le\varepsilon_0}$ such that $u_\varepsilon \in \mathcal{H}_\varepsilon$, $\|u_\varepsilon\|_{\mathcal{H}_\varepsilon} = 1$, we have that $\{B_\varepsilon u_\varepsilon\}_{0<\varepsilon\le\varepsilon_0}$ is precompact in the sense of Definition 2.3. We write this as $B_\varepsilon \stackrel{C}{\longrightarrow} B_0$.*

We now recall some notations already used in [14]. If B is a non-negative compact self-adjoint operator in a infinite dimensional Hilbert space $\mathcal{H}$, its spectrum is a finite or a countably infinite set and all non-zero elements of the spectrum are positive eigenvalues of finite multiplicity. When the spectrum is a countably infinite set, the eigenvalues can be represented by a non-increasing sequence μ_n, $n \in \mathbb{N}$, such that $\lim_{n\to\infty} \mu_n = 0$. As usual we agree to repeat each eigenvalue in the sequence μ_n, $n \in \mathbb{N}$ as many times as its multiplicity.

We also define the notion of generalized eigenfunction: given a finite set of m eigenvalues $\mu_n, \dots, \mu_{n+m-1}$ with $\mu_n \neq \mu_{n-1}$ and $\mu_{n+m-1} \neq \mu_{n+m}$, we call generalized eigenfunction (associated with $\mu_n, \dots, \mu_{n+m-1}$) any linear combination of eigenfunctions associated with the eigenvalues $\mu_n, \dots, \mu_{n+m-1}$.

We now state the following theorem which is a simplified rephrased version of [21, Theorem 6.3], see also [4, Theorem 4.10], [6, Theorem 5.1], [11, Theorem 3.3] and [14, Theorem 1].

Theorem 2.5. *Let $\{B_\varepsilon\}_{0<\varepsilon\le\varepsilon_0}$ and B_0 be non-negative compact self-adjoint operators in the Hilbert spaces $\mathcal{H}_\varepsilon$, $\mathcal{H}_0$ respectively. Assume that their eigenvalues are given by the sequences $\mu_n(\varepsilon)$ and $\mu_n(0)$, $n \in \mathbb{N}$, respectively. If $B_\varepsilon \xrightarrow{C} B_0$ then we have spectral convergence of B_ε to B_0 as $\varepsilon \to 0$ in the sense that the following statements hold:*

(i) *For every $n \in \mathbb{N}$ we have $\mu_n(\varepsilon) \to \mu_n(0)$ as $\varepsilon \to 0$.*
(ii) *If $u_n(\varepsilon)$, $n \in \mathbb{N}$, is an orthonormal sequence of eigenfunctions associated with the eigenvalues $\mu_n(\varepsilon)$ then there exists a sequence ε_k, $k \in \mathbb{N}$, converging to zero and orthonormal sequence of eigenfunctions $u_n(0)$, $n \in \mathbb{N}$ associated with $\mu_n(0)$, $n \in \mathbb{N}$ such that $u_n(\varepsilon_k) \xrightarrow{E} u_n(0)$.*
(iii) *Given m eigenvalues $\mu_n(0), \dots, \mu_{n+m-1}(0)$ with $\mu_n(0) \neq \mu_{n-1}(0)$ and $\mu_{n+m-1}(0) \neq \mu_{n+m}(0)$ and corresponding orthonormal eigenfunctions $u_n(0), \dots, u_{n+m-1}(0)$, there exist m orthonormal generalized eigenfunctions $v_n(\varepsilon), \dots, v_{n+m-1}(\varepsilon)$ associated with $\mu_n(\varepsilon), \dots, \mu_{n+m-1}(\varepsilon)$ such that $v_{n+i}(\varepsilon) \xrightarrow{E} u_{n+i}(0)$ for all $i = 0, 1, \dots, m-1$.*

2.2. **Classes of domains.** According to [14], in order study the spectral convergence for the Steklov eigenvalue problem, we shall consider uniform families of domains with some prescribed common properties. In this perspective we recall the notion of atlas from [10], see also [5, Section 5] and [14, Section 2.2]. According to [5, 10], given a set $V \subset \mathbb{R}^N$ and a number $\delta > 0$ we write

$$V_\delta := \{x \in V : d(x, \partial V) > \delta\}. \tag{8}$$

Definition 2.6. [10, Definition 2.4] *Let $\rho > 0$, $s, s' \in \mathbb{N}$ with $s' < s$. Let $\{V_j\}_{j=1}^s$ be a family of open cuboids (i.e. rotations of rectangle parallelepipeds in $\mathbb{R}^N$) and $\{r_j\}_{j=1}^s$ be a family of rotations in $\mathbb{R}^N$. We say that $\mathcal{A} = (\rho, s, s', \{V_j\}_{j=1}^s, \{r_j\}_{j=1}^s)$ is an atlas in $\mathbb{R}^N$ with parameters $\rho, s, s', \{V_j\}_{j=1}^s, \{r_j\}_{j=1}^s$, briefly an atlas in $\mathbb{R}^N$. Moreover, we say that a bounded domain Ω in $\mathbb{R}^N$ belongs to the class $C^{k,\gamma}(\mathcal{A})$ with $k \in \mathbb{N}$ and $\gamma \in [0,1]$ if the following conditions are satisfied:*

(i) *$\Omega \subset \cup_{j=1}^s (V_j)_\rho$ and $(V_j)_\rho \cap \Omega \neq \emptyset$ where $(V_j)_\rho$ is meant in the sense given in* (8) *;*
(ii) *$V_j \cap \partial\Omega \neq \emptyset$ for $j = 1, \dots, s'$ and $V_j \cap \partial\Omega = \emptyset$ for $s'+1 \le j \le s$;*
(iii) *for $j = 1, \dots, s$ we have*

$$r_j(V_j) = \{x \in \mathbb{R}^N : a_{ij} < x_i < b_{ij}, i = 1, \dots, N\}, \quad j = 1, \dots, s;$$
$$r_j(V_j \cap \Omega) = \{x = (x', x_N) \in \mathbb{R}^N : x' \in W_j, a_{Nj} < x_N < g_j(x')\}, \quad j = 1, \dots, s'$$

where $x' = (x_1, \dots, x_{N-1})$, $W_j = \{x' \in \mathbb{R}^{N-1} : a_{ij} < x_i < b_{ij}, i = 1, \dots, N-1\}$ and the functions $g_j \in C^{k,\gamma}(\overline{W_j})$ for any $j \in 1, \dots, s'$ with $k \in \mathbb{N} \cup \{0\}$ and $0 \le \gamma \le 1$. Moreover, for $j = 1, \dots, s'$ we assume that $a_{Nj} + \rho \le g_j(x') \le b_{Nj} - \rho$, for all $x' \in \overline{W_j}$.

Finally we say that Ω if of class $C^{k,\gamma}$ if it is of class $C^{k,\gamma}(\mathcal{A})$ for some atlas $\mathcal{A}$. In the sequel $C^{k,0}$ will be simply denoted by C^k.

Let Ω be a bounded domain in $\mathbb{R}^N$ of class $C^{0,1}$. The Hilbert space $H^1(\Omega)$ is naturally endowed with the scalar product

$$\int_\Omega \nabla u \nabla v \, dx + \int_\Omega uv \, dx \qquad \text{for any } u, v \in H^1(\Omega).$$

However, taking into account the structure of problem (1), it appears reasonable to replace the classical scalar product of $H^1(\Omega)$ with another one in which the scalar product in $L^2(\Omega)$ of the two functions $u, v \in H^1(\Omega)$ is replaced by the scalar product in $L^2(\partial\Omega)$ of their traces on $\partial\Omega$. In the next lemma we recall that these two scalar products are equivalent in $H^1(\Omega)$.

Lemma 2.7. *Let Ω be a bounded domain in $\mathbb{R}^N$ of class $C^{0,1}$. Then we have:*

(i) *there exists a constant $C(N, \Omega)$ depending only on N and Ω such that*

$$\int_{\partial\Omega} u^2 \, dS \le C(N, \Omega) \left(\int_\Omega |\nabla u|^2 \, dx + \int_\Omega u^2 \, dx \right) \qquad \textit{for any } u \in H^1(\Omega);$$

more precisely, if $\mathcal{A}$ is an atlas as in Definition 2.6 such that Ω is of class $C^{0,1}(\mathcal{A})$, the dependence of $C(N,\Omega)$ on Ω occurs through the atlas $\mathcal{A}$ and the $C^{0,1}$ norms of the functions g_j introduced in the same definition.

(ii) *there exists a constant $C(N, \mathrm{diam}(\Omega))$ depending only on N and $\mathrm{diam}(\Omega)$, where $\mathrm{diam}(\Omega)$ denotes the diameter of Ω, such that*

$$\int_\Omega u^2\,dx \le C(N,\mathrm{diam}(\Omega))\left(\int_\Omega |\nabla u|^2\,dx + \int_{\partial\Omega} u^2\,dS\right) \qquad \text{for any } u \in H^1(\Omega)\,;$$

(iii) *the following scalar product*

$$\int_\Omega \nabla u \nabla v\,dx + \int_{\partial\Omega} uv\,dS \qquad \text{for any } u,v \in H^1(\Omega)$$

is equivalent to the original scalar product of $H^1(\Omega)$.

Proof. Part (i) of the lemma is a well known result from classical trace theorems, see for example the book by [17]. Part (ii) can be obtained in a classical way by using the divergence formula and the Hölder-Young inequality. Finally, part (iii) is an immediate consequence of (i) and (ii). □

Thanks to Lemma 2.7 the space $H^1(\Omega)$ may be equivalently endowed with the scalar product

$$(u,v)_0 := \int_\Omega \nabla u \nabla v\,dx + \int_{\partial\Omega} uv\,dS \qquad \text{for any } u,v \in H^1(\Omega) \tag{9}$$

and the corresponding norm

$$\|u\|_0 := (u,u)_0^{1/2} \qquad \text{for any } u \in H^1(\Omega)\,. \tag{10}$$

2.3. **The functional setting.** Assume that $\Omega \subset \mathbb{R}^N$ $(N \ge 2)$ is a bounded domain of class $C^{0,1}$.

Similarly to [14], we introduce the following resolvent operator $S : H^1(\Omega) \to H^1(\Omega)$ associated with (1) which turns out to be a nonnegative self-adjoint compact operator.

In order to costruct the operator S, we first introduce the operator $T : H^1(\Omega) \to (H^1(\Omega))'$ defined by

$$_{(H^1(\Omega))'}\langle Tu, v\rangle_{H^1(\Omega)} := \int_\Omega \nabla u \nabla v\,dx + \int_{\partial\Omega} uv\,dS \qquad \text{for any } u,v \in H^1(\Omega)\,. \tag{11}$$

The operator T is clearly well-defined and continuous. Moreover by Lemma 2.7 (iii) and Lax-Milgram Theorem, we also deduce that T is invertible and T^{-1} is continuous.

Then we introduce the operator $J : H^1(\Omega) \to (H^1(\Omega))'$ defined by

$$_{(H^1(\Omega))'}\langle Ju, v\rangle_{H^1(\Omega)} := \int_{\partial\Omega} uv\,dS \qquad \text{for any } u,v \in H^1(\Omega)\,. \tag{12}$$

Since the trace map

$$\begin{aligned} H^1(\Omega) &\mapsto L^2(\partial\Omega) \\ u &\mapsto u_{|\partial\Omega} \end{aligned} \tag{13}$$

is well-defined and compact being $\partial\Omega$ Lipschitzian (see [17, Theorem 6.2, Chap. 2] for more details), then the operator J is also well-defined and compact.

We are ready to define the operator $S : H^1(\Omega) \to H^1(\Omega)$ as $S := T^{-1} \circ J$. Clearly S is a linear compact operator and moreover it is easy to see that it is also self-adjoint. Moreover one can show that $\mu \neq 0$ is an eigenvalue of S with corresponding eigenfunction u if and only if $\lambda := \frac{1}{\mu} - 1$ is an eigenvalue of (1) with corresponding eigenfunction u.

2.4. **Domain perturbations and construction of a connecting system.** Let $\mathcal{A}$ be an atlas. Let $\{\Omega_\varepsilon\}_{0<\varepsilon\le\varepsilon_0}$ be a family of domains of class $C^{0,1}(\mathcal{A})$ which converges to a fixed domain Ω of class $C^{0,1}(\mathcal{A})$ in a sense which will be specified below. For any $0<\varepsilon\le\varepsilon_0$ denote by

$$S_\varepsilon : H^1(\Omega_\varepsilon)\to H^1(\Omega_\varepsilon)$$

the resolvent operators associated with (1) in Ω_ε according with the definition given in Section 2.3.

Since our final purpose will be to apply the abstract results of Section 2.1, we need to define a family of operators E_ε, which satisfy condition (7). In the specific case under consideration, this means that we have to introduce linear operators $E_\varepsilon : H^1(\Omega)\to H^1(\Omega_\varepsilon)$ such that

$$\|E_\varepsilon u\|_\varepsilon \to \|u\|_0 \qquad \text{as } \varepsilon\to 0\,, \quad \text{for any } u\in H^1(\Omega)\,, \tag{14}$$

where

$$(u,v)_\varepsilon := \int_{\Omega_\varepsilon} \nabla u\nabla v\,dx + \int_{\partial\Omega_\varepsilon} uv\,dS, \qquad \text{for any } u,v\in H^1(\Omega_\varepsilon), \tag{15}$$

$$\|u\|_\varepsilon = (u,u)_\varepsilon^{1/2} \qquad \text{for any } u\in H^1(\Omega_\varepsilon)\,.$$

We recall that by Lemma 2.7 the norms $\|\cdot\|_\varepsilon$ and $\|\cdot\|_0$ are equivalent to the original norms of $H^1(\Omega_\varepsilon)$ and $H^1(\Omega)$ respectively. For this reason it will be convenient in the sequel, except when it is otherwise specified, to endow the spaces $H^1(\Omega_\varepsilon)$ and $H^1(\Omega)$ with the scalar products $(\cdot,\cdot)_\varepsilon$ and $(\cdot,\cdot)_0$ respectively.

In order to prove (14), we proceed ad in [5], see also [14]. Denote by $g_j, g_{\varepsilon,j}\in C^{0,1}(\overline{W}_j)$ the functions corresponding respectively to Ω and Ω_ε according to Definition 2.6.

Suppose that the following assumptions hold true: for any $j=1,\dots,s'$ and $k\in\{1,\dots,N-1\}$

$$\lim_{\varepsilon\to 0}\|g_{\varepsilon,j}-g_j\|_{L^\infty(W_j)}=0\,, \quad \left\|\frac{\partial g_{\varepsilon,j}}{\partial x_k}\right\|_{L^\infty(W_j)}=O(1), \qquad \text{as } \varepsilon\to 0 \tag{16}$$

and

$$\mathrm{Per}(\Omega_\varepsilon)\to\mathrm{Per}(\Omega), \qquad \text{as } \varepsilon\to 0, \tag{17}$$

where $\mathrm{Per}(\Omega_\varepsilon)$ and $\mathrm{Per}(\Omega)$ denote the perimeters of the domains Ω_ε and Ω respectively, i.e. the $(N-1)$-dimensional measures of $\partial\Omega_\varepsilon$ and $\partial\Omega$ respectively.

Remark 2.8. *We recall that condition* (17) *is a condition already used in* [8] *to prove convergence of the eigenvalues of the Steklov operator. Moreover by the proof of* [8, Proposition 3.2] *one can deduce that, assuming* (16)-(17)*, the following pointwise convergence*

$$\sqrt{1+|\nabla_{x'}g_{\varepsilon,j}|^2}\to\sqrt{1+|\nabla_{x'}g_j|^2} \quad \text{a.e. in } W_j \quad \text{as } \varepsilon\to 0\,, \tag{18}$$

holds true.

Following the construction introduced in [5] and used in [14], we are going to define the family of operators $\{E_\varepsilon\}_{0<\varepsilon\le\varepsilon_0}$ (up to shrink ε_0 if necessary) by using a partition of unity and pasting together suitable pull-back operators associated with local diffeomorphisms defined on each cuboid of the atlas $\mathcal{A}$. Note that in the simplified setting of one single cuboid, partition of unity would not be required and the operator E_ε would be simply defined as in Remark 2.9 below.

Let $\hat{k}$ be a fixed constant satisfying $\hat{k}>4$ whose meaning will be explained below. Let us define

$$\kappa_\varepsilon := \max_{1\le j\le s'}\|g_{\varepsilon,j}-g_j\|_{L^\infty(W_j)}\,, \qquad k_\varepsilon := \hat{k}\kappa_\varepsilon\,, \qquad \tilde{g}_{\varepsilon,j} := g_{\varepsilon,j}-k_\varepsilon\,,$$

and

$$K_{\varepsilon,j} := \{(x',x_N)\in W_j\times(a_{Nj},b_{Nj}) : a_{Nj}<x_N<\tilde{g}_{\varepsilon,j}(x')\} \qquad \text{for any } j=1,\dots,s'\,.$$

For any $1\le j\le s'$ we define the map $h_{\varepsilon,j} : r_j(\overline{\Omega_\varepsilon\cap V_j})\to\mathbb{R}$

$$h_{\varepsilon,j}(x',x_N) := \begin{cases} 0, & \text{if } a_{jN}\le x_N\le\tilde{g}_{\varepsilon,j}(x'),\\ (g_{\varepsilon,j}(x')-g_j(x'))\left(\frac{x_N-\tilde{g}_{\varepsilon,j}(x')}{g_{\varepsilon,j}(x')-\tilde{g}_{\varepsilon,j}(x')}\right)^2, & \text{if } \tilde{g}_{\varepsilon,j}(x')<x_N\le g_{\varepsilon,j}(x')\,. \end{cases} \tag{19}$$

We observe that $h_{\varepsilon,j} \in C^{0,1}(r_j(\overline{\Omega_\varepsilon \cap V_j}))$ and that the map $\Phi_{\varepsilon,j} : r_j(\overline{\Omega_\varepsilon \cap V_j}) \to r_j(\overline{\Omega \cap V_j})$ defined by $\Phi_{\varepsilon,j}(x', x_N) := (x', x_N - h_{\varepsilon,j}(x', x_N))$ is a homeomorphism of class $C^{0,1}$. When $s' + 1 \le j \le s$ we define $\Phi_{\varepsilon,j} : r_j(\overline{V_j}) \to r_j(\overline{V_j})$ as the identity map.

Consider now a partition of unity $\{\psi_j\}_{1\le j\le s}$ subordinate to the open cover $\{V_j\}_{1\le j\le s}$ of the compact set $\overline{\Omega \cup \bigcup_{\varepsilon\in(0,\varepsilon_0]} \Omega_\varepsilon}$, see [14, Page 12].

We also define the deformation $\Psi_{\varepsilon,j} : \overline{\Omega_\varepsilon \cap V_j} \to \overline{\Omega \cap V_j}$ by $\Psi_{\varepsilon,j} := r_j^{-1} \circ \Phi_{\varepsilon,j} \circ r_j$. In this way $\Psi_{\varepsilon,j}$ becomes a $C^{0,1}$ homeomorphism from $\overline{\Omega_\varepsilon \cap V_j}$ onto $\overline{\Omega \cap V_j}$ for any $j \in \{1, \dots, s\}$.

From the definition of $h_{\varepsilon,j}$ and the restriction $\hat{k} > 4$, we deduce that

$$\frac{1}{2} \le \det(D\Psi_{\varepsilon,j}(x)) \le \frac{3}{2} \qquad \text{for any } x \in \Omega_\varepsilon \cap V_j\,. \tag{20}$$

In order to show this, we observe that $\det(D\Psi_{\varepsilon,j}) = \det(D\Phi_{\varepsilon,j}) = 1 - \frac{\partial h_{\varepsilon,j}}{\partial x_N}(x', x_N)$.

Given $u \in H^1(\Omega)$ we put $u_j = \psi_j u$ for any $j \in \{1, \dots, s\}$ in such a way that $u = \sum_{j=1}^{s} u_j$. Then we define

$$E_\varepsilon u := \sum_{j=1}^{s'} \tilde{u}_{\varepsilon,j} + \sum_{j=s'+1}^{s} u_j \in H^1(\Omega_\varepsilon) \tag{21}$$

where

$$\tilde{u}_{\varepsilon,j}(x) = \begin{cases} u_j(\Psi_{\varepsilon,j}(x)), & \text{if } x \in \Omega_\varepsilon \cap V_j\,, \\ 0, & \text{if } x \in \Omega_\varepsilon \setminus V_j\,. \end{cases} \tag{22}$$

for any $j \in \{1, \dots, s'\}$.

Remark 2.9. *If Ω and Ω_ε are in the form*

$$\Omega = \{(x', x_N) \in \mathbb{R}^N : x' \in W \text{ and } a_N < x_N < g(x')\}\,,$$
$$\Omega_\varepsilon = \{(x', x_N) \in \mathbb{R}^N : x' \in W \text{ and } a_N < x_N < g_\varepsilon(x')\}\,,$$

where W is a cuboid or a bounded domain in $\mathbb{R}^{N-1}$ of class $C^{0,1}$ then the operator E_ε can be defined in the following simple way

$$E_\varepsilon u(x) = u(\Phi_\varepsilon(x)) \qquad \text{for any } x \in \Omega_\varepsilon$$

where $\Phi_\varepsilon(x', x_N) = (x', x_N - h_\varepsilon(x', x_N))$ and h_ε is defined by (19) *with g_ε and g in place of $g_{\varepsilon,j}$ and g_j respectively.*

We will show that the family of operators $\{E_\varepsilon\}_{0<\varepsilon\le\varepsilon_0}$ is really a connecting system in the sense of [21]. We first introduce some notations and state a preliminary result.

For any $y \in \Psi_{\varepsilon,j}(\partial\Omega_\varepsilon \cap V_i \cap V_j)$ we put $\Theta_{\varepsilon,i,j}(y) := \Psi_{\varepsilon,i}(\Psi_{\varepsilon,j}^{-1}(y))$ in order to define

$$\Theta_{\varepsilon,i,j} : \Psi_{\varepsilon,j}(\partial\Omega_\varepsilon \cap V_i \cap V_j) \to \Theta_{\varepsilon,i,j}(\Psi_{\varepsilon,j}(\partial\Omega_\varepsilon \cap V_i \cap V_j)) \tag{23}$$

as a diffeomorphism between two open subsets of the manifold $\partial\Omega$.

For any $j \in \{1, \dots, s'\}$ let $\Gamma_j : \partial\Omega \cap V_j \to W_j \subset \mathbb{R}^{N-1}$ be the maps defined by

$$\Gamma_j(y) := P(r_j(y)) \qquad \text{for any } j \in \{1, \dots, s'\} \tag{24}$$

where $P : \mathbb{R}^N \to \mathbb{R}^{N-1}$ is the projection $(x', x_N) \mapsto x'$. We observe that $\Gamma_j^{-1} : W_j \to \partial\Omega \cap V_j$ satisfies $\Gamma_j^{-1}(z') = r_j^{-1}((z', g_j(z')))$ for any $z' \in W_j$.

We now report below a result taken from [14, Lemma 7] which was stated, in that setting, with a $C^{1,1}$ regularity assumption on the domains Ω_ε but we observe that the arguments used in its proof work with a $C^{0,1}$ regularity assumption and our condition (16) as well.

Proposition 2.10. [14, Lemma 7] *Let $\mathcal{A}$ be an atlas. Let $\{\Omega_\varepsilon\}_{0<\varepsilon\leq\varepsilon_0}$ be a family of domains of class $C^{0,1}(\mathcal{A})$ and Ω a domain of class $C^{0,1}(\mathcal{A})$. Assume the validity of condition* (16).

Let $\{\omega_\varepsilon\}_{0<\varepsilon\leq\varepsilon_0} \subset L^2(\partial\Omega)$ be such that $\mathrm{supp}(\omega_\varepsilon) \subset \partial\Omega \cap V_i$ *for any $\varepsilon \in (0,\varepsilon_0]$, for some $i \in \{1,\dots,s'\}$. Suppose that there exists $\omega \in L^2(\partial\Omega)$ such that $\omega_\varepsilon \to \omega$ in $L^2(\partial\Omega)$ as $\varepsilon \to 0$. For $j \in \{1,\dots,s'\}$ let $\Theta_{\varepsilon,i,j}$ be as in* (23). *For any $\varepsilon \in (0,\varepsilon_0]$ define the function*

$$\tilde{\omega}_\varepsilon(y) := \begin{cases} \omega_\varepsilon(\Theta_{\varepsilon,i,j}(y)) & \text{if } y \in \Psi_{\varepsilon,j}(\partial\Omega_\varepsilon \cap V_i \cap V_j)\,, \\ 0 & \text{if } y \in \partial\Omega \setminus \Psi_{\varepsilon,j}(\partial\Omega_\varepsilon \cap V_i \cap V_j)\,. \end{cases}$$

Then $\tilde{\omega}_\varepsilon \to \omega\chi_{\partial\Omega\cap V_i\cap V_j}$ in $L^2(\partial\Omega)$ as $\varepsilon \to 0$.

We are ready to prove that the family of operators $\{E_\varepsilon\}$ is a connecting system.

Lemma 2.11. *Let $\mathcal{A}$ be an atlas. Let $\{\Omega_\varepsilon\}_{0<\varepsilon\leq\varepsilon_0}$ be a family of domains of class $C^{0,1}(\mathcal{A})$, Ω a domain of class $C^{0,1}(\mathcal{A})$ and for any $\varepsilon \in (0,\varepsilon_0]$ let E_ε be the map defined in* (21). *Assume the validity of* (16)-(17).

Then the following assertions hold true:

(i) *the map $E_\varepsilon : H^1(\Omega) \to H^1(\Omega_\varepsilon)$ is continuous for any $\varepsilon \in (0,\varepsilon_0]$;*

(ii) *the family of operators $\{E_\varepsilon\}_{0<\varepsilon\leq\varepsilon_0}$ satisfies* (14).

Proof. This lemma is essentially an adaptation of [14, Lemma 2] with the obvious changes. For this reason we only give an idea of the proof quoting the necessary references contained in the proof of [14, Lemma 2].

Since the proof of (i) easily follows from the definition of E_ε, it is left to the reader.

It remains to show the validity of (14). Let $u \in H^1(\Omega)$ and let $\tilde{u}_{\varepsilon,j}$ be the functions introduced in (22).

Note that in order to prove (14), it is sufficient to prove the following convergences:

$$\lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon} \frac{\partial\tilde{u}_{\varepsilon,i}}{\partial x_k}\frac{\partial\tilde{u}_{\varepsilon,j}}{\partial x_k}\,dx = \int_\Omega \frac{\partial u_i}{\partial x_k}\frac{\partial u_j}{\partial x_k}\,dx \quad \forall i,j \in \{1,\dots,s'\}\,, \forall\, k \in \{1,\dots,N\}; \tag{25}$$

$$\lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon} \frac{\partial\tilde{u}_{\varepsilon,i}}{\partial x_k}\frac{\partial u_j}{\partial x_k}\,dx = \int_\Omega \frac{\partial u_i}{\partial x_k}\frac{\partial u_j}{\partial x_k}\,dx \quad \forall i \in \{1,\dots,s'\}, \forall j \in \{s'+1,\dots,s\},\ \forall k \in \{1,\dots,N\}; \tag{26}$$

$$\lim_{\varepsilon\to 0}\int_{\partial\Omega_\varepsilon} \tilde{u}_{\varepsilon,i}\,\tilde{u}_{\varepsilon,j}\,dS = \int_{\partial\Omega} u_i\,u_j\,dS \quad \forall i,j \in \{1,\dots,s'\}\,, \forall\, k \in \{1,\dots,N\}; \tag{27}$$

$$\lim_{\varepsilon\to 0}\int_{\partial\Omega_\varepsilon} \tilde{u}_{\varepsilon,i}\,u_j\,dS = \int_{\partial\Omega} u_i\,u_j\,dS \quad \forall i \in \{1,\dots,s'\}, \forall j \in \{s'+1,\dots,s\},\ \forall k \in \{1,\dots,N\}\,. \tag{28}$$

We only give some details of the proof of (25) and (27) since (26) and (28) can be proved in a completely equivalent way, actually easier.

We first prove (25) in the case $i = j$. For any $x \in \Omega_\varepsilon \cap V_i$ we have

$$\frac{\partial\tilde{u}_{\varepsilon,i}}{\partial x_k}(x) = \sum_{l=1}^N \frac{\partial u_i}{\partial x_l}(\Psi_{\varepsilon,i}(x))\frac{\partial[(\Psi_{\varepsilon,i}(x))_l]}{\partial x_k}\,. \tag{29}$$

By (16) and the definitions of $\Psi_{\varepsilon,i}$, $\Phi_{\varepsilon,i}$ and $h_{\varepsilon,i}$ we infer

$$\left\|\frac{\partial[(\Psi_{\varepsilon,i}(x))_l]}{\partial x_k}\right\|_{L^\infty(V_i\cap\Omega_\varepsilon)} = O(1) \ \text{ as } \varepsilon \to 0\,, \qquad \text{for any } l,k \in \{1,\dots,N\}\,. \tag{30}$$

Exploiting (29)-(30), (20) and the fact that $|r_i(\Omega\cap V_i)\setminus K_{\varepsilon,i}| \to 0$ as $\varepsilon \to 0$ one gets

$$\lim_{\varepsilon\to 0}\int_{(\Omega_\varepsilon\cap V_i)\setminus r_i^{-1}(K_{\varepsilon,i})} \left(\frac{\partial\tilde{u}_{\varepsilon,i}}{\partial x_k}\right)^2 dx = 0\,. \tag{31}$$

On the other hand

$$\lim_{\varepsilon\to 0}\int_{r_i^{-1}(K_{\varepsilon,i})}\left(\frac{\partial \tilde u_{\varepsilon,i}}{\partial x_k}\right)^2 dx=\lim_{\varepsilon\to 0}\int_{r_i^{-1}(K_{\varepsilon,i})}\left(\frac{\partial u_i}{\partial x_k}\right)^2 dx=\int_{\Omega\cap V_i}\left(\frac{\partial u_i}{\partial x_k}\right)^2 dx=\int_{\Omega}\left(\frac{\partial u_i}{\partial x_k}\right)^2 dx. \tag{32}$$

Combining (31) and (32) we conclude the proof of (25) in the case $i=j$.

In order to prove (25) in the general case $i\neq j$ one can use a suitable decomposition of the domains of integrations like in [14, Identity (39)].

Let us proceed with the proof of (27). Combining (16)-(17) (and hence (18)) with Proposition 2.10, we obtain

$$\begin{aligned}\lim_{\varepsilon\to 0}\int_{\partial\Omega_\varepsilon}\tilde u_{\varepsilon,i}\,\tilde u_{\varepsilon,j}\,dS&=\lim_{\varepsilon\to 0}\int_{\partial\Omega_\varepsilon\cap V_i\cap V_j}u_i(\Psi_{\varepsilon,i}(x))\,u_j(\Psi_{\varepsilon,j}(x))\,dS\\&=\lim_{\varepsilon\to 0}\int_{\Psi_{\varepsilon,j}(\partial\Omega_\varepsilon\cap V_i\cap V_j)}u_i(\Psi_{\varepsilon,i}(\Psi_{\varepsilon,j}^{-1}(y)))\,u_j(y)W_{\varepsilon,j}(y)\,dS\\&=\int_{\partial\Omega\cap V_i\cap V_j}u_i(y)\,u_j(y)\,dS=\int_{\partial\Omega}u_i\,u_j\,dS\end{aligned}$$

where we used the change of variables $y=\Psi_{\varepsilon,j}(x)$ and where we put

$$W_{\varepsilon,j}(y):=\frac{\sqrt{1+|\nabla_{x'}g_{\varepsilon,j}(\Gamma_j(y))|^2}}{\sqrt{1+|\nabla_{x'}g_j(\Gamma_j(y))|^2}}\,. \tag{33}$$

Here $\Gamma_j:\partial\Omega\cap V_j\to W_j\subset\mathbb{R}^{N-1}$ are the maps defined in (24).

This completes the proof of the lemma. □

2.5. **Minimax characterization for the eigenvalues of** (1)**.** We denote by $\{u_n\}_{n\geq 0}$ an orthonormal system of eigenfunctions with respect to the scalar product of $L^2(\partial\Omega)$, i.e. $\int_{\partial\Omega}u_nu_m\,dS=\delta_{nm}$, where u_n is an eigenfunction of λ_n for any $n\geq 0$. In particular we have

$$(u_n,u_m)_0=\int_\Omega\nabla u_n\nabla u_m\,dx=\int_{\partial\Omega}u_nu_m\,dS=0\qquad\text{for any } n,m\geq 0,\ n\neq m \tag{34}$$

and

$$\int_\Omega|\nabla u_n|^2dx=\lambda_n\,,\qquad\int_{\partial\Omega}u_n^2\,dS=1\qquad\text{for any } n\geq 0\,. \tag{35}$$

Let $S:H^1(\Omega)\to H^1(\Omega)$ be the resolvent operator defined in Section 2.3. We observe that S is an operator which acts from $(H^1_0(\Omega))^\perp$ to itself, i.e.

$$S:(H^1_0(\Omega))^\perp\to(H^1_0(\Omega))^\perp$$

where orthogonality is meant with respect to the scalar product $(\cdot,\cdot)_0$.

Indeed, if we choose $w\in(H^1_0(\Omega))^\perp$, letting $u=Sw$, we have $Tu=Jw$, i.e.

$${}_{(H^1(\Omega))'}\langle Tu,v\rangle_{H^1(\Omega)}={}_{(H^1(\Omega))'}\langle Jw,v\rangle_{H^1(\Omega)}\qquad\text{for any } v\in H^1(\Omega)\,.$$

This is equivalent to

$$(u,v)_0=\int_{\partial\Omega}wv\,dS\qquad\text{for any } v\in H^1(\Omega)\,. \tag{36}$$

In particular, if we choose $v\in H^1_0(\Omega)$ in (36), we obtain that the right hand side of (36) vanishes and so it does its left hand side, thus proving that $u\in(H^1_0(\Omega))^\perp$.

The invariance property of the space $(H^1_0(\Omega))^\perp$ under the action of the self-adjoint compact operator S shows that the basis $\{u_n\}_{n\geq 0}$ defined before, is an orthogonal basis of $(H^1_0(\Omega))^\perp$.

A standard application of the classical minimax principle for eigenvalues allows to show that, for the eigenvalues of (1), the following characterization holds true:

$$\lambda_n = \min_{W \in \mathcal{W}_n} \max_{v \in W \setminus \{0\}} \frac{\int_\Omega |\nabla v|^2 dx}{\int_{\partial\Omega} v^2 dS} \tag{37}$$

where for any $n \geq 0$

$$\mathcal{W}_n := \{W \subseteq (H^1_0(\Omega))^\perp \text{ subspace} : \dim(W) = n+1\} . \tag{38}$$

Actually, if $w \in (H^1_0(\Omega))^\perp$ and $v \in H^1_0(\Omega)$, not only we have $(w, v)_0 = 0$ by definition of orthogonality, but we also have

$$\int_\Omega \nabla w \nabla v \, dx = \int_{\partial\Omega} wv \, dS = 0 . \tag{39}$$

We also prove that for any $n \geq 0$ the n-th eigenvalue λ_n admits the alternative inf-sup characterization

$$\lambda_n = \inf_{V \in \mathcal{V}_n} \sup_{v \in V \setminus H^1_0(\Omega)} \frac{\int_\Omega |\nabla v|^2 dx}{\int_{\partial\Omega} v^2 dS} \tag{40}$$

where

$$\mathcal{V}_n := \{V \subseteq H^1(\Omega) \text{ subspace} : \dim(V) = n+1 \text{ and } V \not\subseteq H^1_0(\Omega)\} .$$

Proposition 2.12. *The eigenvalues of* (1) *admit the variational characterization* (40).

Moreover we have:

(i) *if $V \in \mathcal{V}_n$ satisfies $V \cap H^1_0(\Omega) \neq \{0\}$ then*

$$\sup_{v \in V \setminus H^1_0(\Omega)} \frac{\int_\Omega |\nabla v|^2 dx}{\int_{\partial\Omega} v^2 dS} = +\infty ; \tag{41}$$

(ii) *if $V \in \mathcal{V}_n$ satisfies $V \cap H^1_0(\Omega) = \{0\}$ then the supremum in* (41) *is finite and it is achieved;*

(iii) *the infimum in* (40) *is achieved so that we may write*

$$\lambda_n = \min_{V \in \mathcal{V}_n} \sup_{v \in V \setminus H^1_0(\Omega)} \frac{\int_\Omega |\nabla v|^2 dx}{\int_{\partial\Omega} v^2 dS} . \tag{42}$$

Proof. We prove the three parts of the lemma separately.

Proof of (i). Let V be as in (i) and let $v \in V \setminus \left[H^1_0(\Omega) \cup (H^1_0(\Omega))^\perp\right]$.

Consider its orthogonal decomposition $v = v_0 + v_1 \in H^1_0(\Omega) \oplus (H^1_0(\Omega))^\perp$. Since $v \notin H^1_0(\Omega) \cup (H^1_0(\Omega))^\perp$ we clearly have that $v_0, v_1 \neq 0$. Let us use $v_t = tv_0 + v_1$, $t \in (0, +\infty)$, as a test function in the Rayleigh quotient appearing in (41). By (39) and the fact that v_0 has null trace on $\partial\Omega$, we have

$$\frac{\int_\Omega |\nabla v_t|^2 dx}{\int_{\partial\Omega} v_t^2 dS} = \frac{t^2 \int_\Omega |\nabla v_0|^2 dx + \int_\Omega |\nabla v_1|^2 dx}{\int_{\partial\Omega} v_1^2 dS} \to +\infty \qquad \text{as } t \to +\infty .$$

This completes the proof of (i).

Proof of (ii). First of all, if V is as in (ii) we have that $V \setminus H^1_0(\Omega) = V \setminus \{0\}$. Due to the homogeneity property of the Rayleigh quotient we clearly have that

$$\sup_{v \in V \setminus H^1_0(\Omega)} \frac{\int_\Omega |\nabla v|^2 dx}{\int_{\partial\Omega} v^2 dS} = \sup_{v \in V \setminus \{0\}} \frac{\int_\Omega |\nabla v|^2 dx}{\int_{\partial\Omega} v^2 dS} = \sup_{v \in V, \|v\|_0 = 1} \frac{\int_\Omega |\nabla v|^2 dx}{\int_{\partial\Omega} v^2 dS} = \max_{v \in V, \|v\|_0 = 1} \frac{\int_\Omega |\nabla v|^2 dx}{\int_{\partial\Omega} v^2 dS}$$

where the last equality follows from the compactness of the unit sphere in a finite dimensional space.

Proof of (iii). Let $I(\mathcal{W}_n)$ be the minimax value introduced in (37) and let $I(\mathcal{V}_n)$ be the inf-sup value defined in (40). By (i)-(ii) we clearly have that

$$I(\mathcal{W}_n) \geq I(\mathcal{V}_n) = \inf_{V \in \mathcal{V}_n, V \cap H^1_0(\Omega) = \{0\}} \sup_{v \in V \setminus \{0\}} \frac{\int_\Omega |\nabla v|^2 dx}{\int_{\partial\Omega} v^2 dS} \tag{43}$$

where the inequality above follows from the fact that $\mathcal{W}_n \subset \mathcal{V}_n$.

On the other hand, for any $V \in \mathcal{V}_n$ with $V \cap H_0^1(\Omega) = \{0\}$, let us consider the orthogonal projections $P : V \to P(V) \subset H_0^1(\Omega)$ and $Q : V \to Q(V) \subset (H_0^1(\Omega))^\perp$ so that

$$v = Pv + Qv \in H_0^1(\Omega) \oplus (H_0^1(\Omega))^\perp \qquad \text{for any } v \in V\,. \tag{44}$$

We claim that $\dim(Q(V)) = \dim(V) = n + 1$. By definition we have that the linear map Q is surjective. Let us prove that it is also injective. By contradiction suppose that Q is not injective so that, being Q linear, there exists $v \in V \setminus \{0\}$ such that $Qv = 0$ which inserted in (44) shows that $v = Pv \in H_0^1(\Omega)$ and this is in contradiction with the fact that $V \cap H_0^1(\Omega) = \{0\}$. Being Q an isomorphism between vector spaces the proof of the claim follows.

For any $v \in V \setminus \{0\}$ with $V \cap H_0^1(\Omega) = \{0\}$, let us put $v_0 = Pv$ and $v_1 = Qv$ in such a way that $v_1 \neq 0$. With this choice of V and v, by (39) we obtain

$$\frac{\int_\Omega |\nabla v|^2 dx}{\int_{\partial\Omega} v^2 dS} = \frac{\int_\Omega |\nabla v_0|^2 dx + \int_\Omega |\nabla v_1|^2 dx}{\int_{\partial\Omega} v_1^2 dS} \geq \frac{\int_\Omega |\nabla v_1|^2 dx}{\int_{\partial\Omega} v_1^2 dS}$$

so that

$$\max_{v \in V\setminus\{0\}} \frac{\int_\Omega |\nabla v|^2 dx}{\int_{\partial\Omega} v^2 dS} \geq \max_{w \in Q(V)\setminus\{0\}} \frac{\int_\Omega |\nabla w|^2 dx}{\int_{\partial\Omega} w^2 dS} \geq I(\mathcal{W}_n) \tag{45}$$

since $Q(V) \in \mathcal{W}_n$ being $Q(V) \subset (H_0^1(\Omega))^\perp$ and $\dim(Q(V)) = n + 1$.

Since (45) holds for any $V \in \mathcal{V}_n$ such that $V \cap H_0^1(\Omega) = \{0\}$, taking the infimum over V in (45), by the equality in the right hands side of (43), we conclude that $I(\mathcal{V}_n) \geq I(\mathcal{W}_n)$. This combined with the inequality in left hand side of (43) and with (37) proves that $I(\mathcal{V}_n) = I(\mathcal{W}_n) = \lambda_n$.

Finally, the fact that the infimum in (40) is achieved follows from $I(\mathcal{V}_n) = \lambda_n$ combined with the particular choice $V = \mathrm{span}\{u_0, \ldots, u_n\} \in \mathcal{V}_n$: indeed, in this way the inequality

$$\max_{v \in \mathrm{span}\{u_0,\ldots,u_n\}\setminus\{0\}} \frac{\int_\Omega |\nabla v|^2 dx}{\int_{\partial\Omega} v^2 dS} \leq \lambda_n$$

becomes an equality if one chooses $v = u_n$ and, in turn, the above maximum achieves the minimum in (40). □

3. Main results

We start with the following result on spectral convergence for problem (1):

Theorem 3.1. *Let $\mathcal{A}$ be an atlas. Let $\{\Omega_\varepsilon\}_{0<\varepsilon\leq\varepsilon_0}$ be a family of domains of class $C^{0,1}(\mathcal{A})$ and let Ω be a domain of class $C^{0,1}(\mathcal{A})$. Assume the validity of conditions* (16)-(17). *Then $S_\varepsilon \xrightarrow{C} S$ with respect to the operators E_ε defined in* (21). *In particular, the spectrum of* (1) *behaves continuously at $\varepsilon = 0$ in the sense of Theorem 2.5.*

In the next theorem we relax assumptions of Theorem 3.1 by replacing condition (17) with

$$\sqrt{1 + |\nabla_{x'} g_{\varepsilon,j}|^2} \rightharpoonup \gamma_j \qquad \text{weakly in } L^1(W_j) \quad \text{as } \varepsilon \to 0 \tag{46}$$

where $\gamma_j \in L^1(W_j)$ for any $j = 1, \ldots, s'$.

Looking at (46), it seems reasonable to define the function $\gamma : \partial\Omega \to \mathbb{R}$, locally given by

$$\gamma(y) = \frac{\gamma_j(\Gamma_j(y))}{\sqrt{1 + |\nabla_{x'} g_j(\Gamma_j(y))|^2}} \qquad \text{for any } y \in V_j \cap \partial\Omega \tag{47}$$

where Γ_j is the map defined in (24), for any $j = 1, \ldots, s'$.

We observe that, arguing as in [2, Lemma 5.1, Corollary 5.1], one can deduce that the function γ is well defined being its definition not depending on local charts.

We now introduce the following weighted Steklov problem

$$\begin{cases} \Delta u = 0, & \text{in } \Omega\,, \\ u_\nu = \lambda\, \gamma(x) u, & \text{on } \partial\Omega\,. \end{cases} \tag{48}$$

Let us denote by

$$0 = \lambda_0(\gamma) < \lambda_1(\gamma) \leq \lambda_2(\gamma) \leq \dots \leq \lambda_n(\gamma) \leq \dots \tag{49}$$

the eigenvalues of (48).

We now define on the space $H^1(\Omega)$ the scalar product

$$(u,v)_\gamma := \int_\Omega \nabla u \nabla v \, dx + \int_{\partial\Omega} \gamma \, uv \, dS \qquad \text{for any } u, v \in H^1(\Omega) \tag{50}$$

and the corresponding norm

$$\|u\|_\gamma := (u,u)_\gamma^{1/2} \qquad \text{for any } u \in H^1(\Omega)\,. \tag{51}$$

We observe that the boundary integral in (50) is well defined since by (16) and (46) we deduce that $\gamma \in L^\infty(\partial\Omega)$.

We observe that the norm (51) is equivalent to the usual norm of $H^1(\Omega)$: indeed, for one estimate we can combine the fact that $\gamma \in L^\infty(\partial\Omega)$ with the classical trace inequality and for the other one Lemma 2.7 (ii) and the fact that $\gamma \geq 1$, as one can verify looking at [2, Corollary 5.1].

We now construct the operators T_γ, J_γ simply by replacing the boundary integrals in (11) and (12) by $\int_{\partial\Omega} \gamma \, uv \, dS$. The operator S_γ is again defined by $T_\gamma^{-1} \circ J_\gamma$.

The family of operators $E_\varepsilon : H^1(\Omega) \to H^1(\Omega_\varepsilon)$ can be defined exactly as in Section 2.4.

It can be proved that

$$\|E_\varepsilon u\|_\varepsilon \to \|u\|_\gamma \qquad \text{as } \varepsilon \to 0\,, \qquad \text{for any } u \in H^1(\Omega) \tag{52}$$

thus showing that the family of operators $\{E_\varepsilon\}$ is still a connecting system in the sense of Section 2.1 provided that $H^1(\Omega)$ is endowed with the norm $\|\cdot\|_\gamma$.

The proof of (52) may be obtained by proceeding exactly as in the proof of Lemma 2.11 with the only difference that in the concluding part of the proof of the lemma, the function $W_{\varepsilon,j}$ weakly converges in $L^p(\partial\Omega \cap V_i \cap V_j)$ to the function $\frac{\gamma_j}{\sqrt{1+|\nabla_{x'} g_\varepsilon|^2}} \circ \Gamma_j$ for any $1 \leq p < \infty$, as a consequence of [2, Remark 2.1].

We are ready to state the following result.

Theorem 3.2. *Let $\mathcal{A}$ be an atlas. Let $\{\Omega_\varepsilon\}_{0<\varepsilon\leq\varepsilon_0}$ be a family of domains of class $C^{0,1}(\mathcal{A})$ and let Ω be a domain of class $C^{0,1}(\mathcal{A})$. Assume the validity of conditions* (16) *and* (46).

Then $S_\varepsilon \xrightarrow{C} S_\gamma$ with respect to the operators E_ε defined in (21). *In particular, for any $n \geq 1$ we have that*

$$\lambda_n^\varepsilon \to \lambda_n(\gamma) \qquad \text{as } \varepsilon \to 0$$

where, according with the notation used in the Introduction, by λ_n^ε we mean the eigenvalues of the Steklov problem in Ω_ε and by $\lambda_n(\gamma)$ the eigenvalues defined in (49).

We observe that Theorem 3.2 becomes an instability result whenever $\gamma \not\equiv 1$ on $\partial\Omega$.

We now state another instability result in which we consider a family of domains $\{\Omega_\varepsilon\}_{0<\varepsilon\leq\varepsilon_0}$ of class $C^{0,1}(\mathcal{A})$ and a fixed domain of class $C^{0,1}(\mathcal{A})$. We assume that the corresponding functions $g_{\varepsilon,j}$ satisfy

$$\lim_{\varepsilon\to 0} \|g_{\varepsilon,j} - g_j\|_{L^\infty(W_j)} = 0 \tag{53}$$

but differently from the statements of Theorem 3.1 and Thorem 3.2, we assume the following blow up condition of the surface element:

$$\lim_{\varepsilon\to 0} \mathcal{H}^{N-1}\left(\left\{x' \in W_j : \sqrt{1+|\nabla_{x'} g_{\varepsilon,j}(x')|^2} \leq t\right\}\right) = 0 \qquad \text{for any } t > 0\,, \tag{54}$$

where we have denoted by $\mathcal{H}^{N-1}$ the $(N-1)$-dimensional Lebesgue measure.

Theorem 3.3. *Let $\mathcal{A}$ be an atlas. Let $\{\Omega_\varepsilon\}_{0<\varepsilon\leq\varepsilon_0}$ be a family of domains of class $C^{0,1}(\mathcal{A})$ and let Ω be a domain of class $C^{0,1}(\mathcal{A})$. Assume that* (53) *and* (54) *hold true. Then for any $n \geq 1$ we have that*

$$\lambda_n^\varepsilon \to 0 \qquad \text{as } \varepsilon \to 0$$

where according with the notation used in the Introduction, λ_n^ε denote the eigenvalues of the Steklov problem in Ω_ε.

In order to better compare the assumptions of Theorems 3.1-3.3, we consider a special case in which every domain is covered by only one chart. Let W be a cuboid or a bounded domain in $\mathbb{R}^{N-1}$ of class $C^{0,1}$. Let us assume that Ω_ε is given by

$$\Omega_\varepsilon := \{(x', x_N) : x' \in W\,, -1 < x_N < g_\varepsilon(x')\} \tag{55}$$

where $g_\varepsilon(x') = \varepsilon^\alpha b(x'/\varepsilon)$ for any $x' \in W$ and the function b satisfies:

$$b \in C^{0,1}(\mathbb{R}^{N-1}), \quad b \geq 0 \ \text{ in } \mathbb{R}^{N-1}, \quad b \text{ is a } Y\text{-periodic function} \tag{56}$$

where $Y = \left(-\frac{1}{2}, \frac{1}{2}\right)^{N-1}$ is the unit cell in $\mathbb{R}^{N-1}$. We also assume that

$$\mathcal{H}^{N-1}\left(\left\{x' \in \mathbb{R}^{N-1} : |\nabla_{x'} b(x')| = 0\right\}\right) = 0\,. \tag{57}$$

We also put $\Omega = W \times (-1, 0)$.

The next result shows how the exponent α introduced in the definition of g_ε plays a crucial role in determining the validity of one of the three coupled conditions (16) and (17), (16) and (46), (53) and (54).

Proposition 3.4. *Let $\{\Omega_\varepsilon\}_{\varepsilon \geq 0}$ be a family of domains like in* (55) *with $g_\varepsilon(x') = \varepsilon^\alpha b(x'/\varepsilon)$ if $\varepsilon > 0$, b satisfying* (56)-(57)*, and with $\Omega_0 = \Omega = W \times (-1, 0)$. Then we have*

(i) *if $\alpha > 1$ then* (16)-(17) *hold true with $W_j = W$, g_ε in place of $g_{\varepsilon,j}$ and $g_j \equiv 0$, i.e.*

$$\lim_{\varepsilon \to 0} \|g_\varepsilon\|_{L^\infty(W)} = 0\,, \qquad \left\|\frac{\partial g_\varepsilon}{\partial x_k}\right\|_{L^\infty(W)} = O(1) \quad \textit{as } \varepsilon \to 0 \quad \textit{for any } k \in \{1, \dots, N\}\,,$$

$$\mathrm{Per}(\Omega_\varepsilon) \to \mathrm{Per}(\Omega) \quad \textit{as } \varepsilon \to 0\,;$$

(ii) *if $\alpha = 1$ then* (16)*,* (46) *hold true with $W_j = W$, g_ε in place of $g_{\varepsilon,j}$, $g_j \equiv 0$ and with the constant function $\int_Y \sqrt{1 + |\nabla_{x'} b(y')|^2} dy'$ in place of γ_j, i.e.*

$$\lim_{\varepsilon \to 0} \|g_\varepsilon\|_{L^\infty(W)} = 0\,, \qquad \left\|\frac{\partial g_\varepsilon}{\partial x_k}\right\|_{L^\infty(W)} = O(1) \quad \textit{as } \varepsilon \to 0 \quad \textit{for any } k \in \{1, \dots, N\}\,,$$

$$\sqrt{1 + |\nabla_{x'} g_\varepsilon|^2} \rightharpoonup \int_Y \sqrt{1 + |\nabla_{x'} b(y')|^2}\, dy' \qquad \textit{weakly in } L^1(W) \quad \textit{as } \varepsilon \to 0\,;$$

(iii) *if $0 < \alpha < 1$ then* (53)-(54) *hold true with $W_j = W$, g_ε in place of $g_{\varepsilon,j}$ and $g_j \equiv 0$, i.e.*

$$\lim_{\varepsilon \to 0} \|g_\varepsilon\|_{L^\infty(W)} = 0\,, \qquad \lim_{\varepsilon \to 0} \mathcal{H}^{N-1}\left(\left\{x' \in W : \sqrt{1 + |\nabla_{x'} g_\varepsilon(x')|^2} \leq t\right\}\right) = 0 \quad \textit{for any } t > 0\,.$$

Looking at the statement of Proposition 3.4 it becomes clear that, at least in the particular case of the family of domains defined in (55), the three couples of assumptions (16) and (17), (16) and (46), (53) and (54), becomes complementary.

We set now

$$\Gamma_\varepsilon := \{(x', g_\varepsilon(x')) : x' \in W\}\,, \qquad \Gamma := \{(x', 0) : x' \in W\}\,, \tag{58}$$

$$\Sigma_\varepsilon := \partial\Omega_\varepsilon \setminus \Gamma_\varepsilon\,, \qquad \Sigma := \partial\Omega \setminus \Gamma\,.$$

For any $\varepsilon \geq 0$ consider the following modified Steklov problem:

$$\begin{cases} \Delta u = 0\,, & \text{in } \Omega_\varepsilon\,, \\ u = 0\,, & \text{on } \Sigma_\varepsilon\,, \\ u_\nu = \lambda u\,, & \text{on } \Gamma_\varepsilon\,, \end{cases} \tag{59}$$

with the notation $\Omega_0 := \Omega$.

Let us denote by μ_n^ε the eigenvalues of (59) for any $n \in \mathbb{N} \cup \{0\}$ and $\varepsilon \geq 0$.

We also consider the weighted problem

$$\begin{cases} \Delta u = 0\,, & \text{in } \Omega\,, \\ u = 0\,, & \text{on } \Sigma\,, \\ u_\nu = \lambda C_b\, u\,, & \text{on } \Gamma\,, \end{cases} \tag{60}$$

where $C_b = \int_Y \sqrt{1 + |\nabla_{x'} b(y')|^2}\, dy'$.

It is clear that, if we denote by $\mu_n(b)$ the eigenvalues of (60) for any $n \in \mathbb{N} \cup \{0\}$, then $\mu_n(b) = \frac{\mu_n^0}{C_b}$.

Combining the arguments used in the proofs of Theorems 3.1-3.3 with the statement of Proposition 3.4, we obtain a trichotomy result which better emphasizes the complementarity of conditions assumed in the three main theorems.

Theorem 3.5. *Let $\{\Omega_\varepsilon\}_{\varepsilon \geq 0}$ be a family of domains like in* (55) *with $g_\varepsilon(x') = \varepsilon^\alpha b(x'/\varepsilon)$ if $\varepsilon > 0$, b satisfying* (56)-(57)*, and with $\Omega_0 = \Omega = W \times (-1, 0)$.*

Let μ_n^ε be the eigenvalues of (59) *for any $\varepsilon \geq 0$ and let $\mu_n(b)$ be the eigenvalues of* (60)*. Then the following statements hold true:*

(i) *if $\alpha > 1$ then $\mu_n^\varepsilon \to \mu_n^0$ as $\varepsilon \to 0$;*

(ii) *if $\alpha = 1$ then $\mu_n^\varepsilon \to \mu_n(b)$ as $\varepsilon \to 0$, i.e. $\mu_n^\varepsilon \to \dfrac{\mu_n^0}{C_b}$ as $\varepsilon \to 0$;*

(iii) *if $0 < \alpha < 1$ then $\mu_n^\varepsilon \to 0$ as $\varepsilon \to 0$.*

4. Proof of Theorem 3.1

Inspired by [14], we define a map which acts between the spaces $H^1(\Omega)$, $H^1(\Omega_\varepsilon)$ in a reversed way with respect to E_ε. For any $w \in H^1(\Omega_\varepsilon)$ we put

$$\widehat{w}_{\varepsilon,j}(x) = \begin{cases} w_j(\Psi_{\varepsilon,j}^{-1}(x)), & \text{if } x \in \Omega \cap V_j \\ 0, & \text{if } x \in \Omega \setminus V_j\,. \end{cases} \tag{61}$$

for any $j \in \{1, \dots, s'\}$ and $w_j := \psi_j w$ for any $j \in \{1, \dots, s\}$. We define

$$B_\varepsilon w := \sum_{j=1}^{s'} \widehat{w}_{\varepsilon,j} + \sum_{j=s'+1}^{s} w_j. \tag{62}$$

In this way we have constructed a map $B_\varepsilon : H^1(\Omega_\varepsilon) \to H^1(\Omega)$.

Next we prove the following

Lemma 4.1. *Let $\mathcal{A}$ be an atlas. Let $\{\Omega_\varepsilon\}_{0<\varepsilon\leq\varepsilon_0}$ be a family of domains of class $C^{0,1}(\mathcal{A})$ and Ω a domain of class $C^{0,1}(\mathcal{A})$. Assume the validity of conditions* (16)-(17)*. Let $w_\varepsilon \in H^1(\Omega_\varepsilon)$ with $0 < \varepsilon \leq \varepsilon_0$, and $w \in H^1(\Omega)$ be such that $w_\varepsilon \xrightarrow{E} w$. If we put $u_\varepsilon := S_\varepsilon w_\varepsilon$ and $u := Sw$, then $B_\varepsilon u_\varepsilon \rightharpoonup u$ in $H^1(\Omega)$ as $\varepsilon \to 0$.*

Proof. We divide the proof of the lemma into several steps. The argument used in this proof is essentially based on the one presented in the proof of [14, Lemma 8].

Step 1. In this step we prove that $\|u_\varepsilon\|_\varepsilon$ is uniformly bounded with respect to $\varepsilon \in (0, \varepsilon_0]$.

Indeed

$$\|u_\varepsilon\|_\varepsilon^2 = \int_{\partial\Omega_\varepsilon} w_\varepsilon u_\varepsilon\, dS \leq \left(\int_{\partial\Omega_\varepsilon} w_\varepsilon^2\, dS\right)^{1/2} \left(\int_{\partial\Omega_\varepsilon} u_\varepsilon^2\, dS\right)^{1/2} \leq \|w_\varepsilon\|_\varepsilon \|u_\varepsilon\|_\varepsilon \tag{63}$$

from which it follows that $\|u_\varepsilon\|_\varepsilon \leq \|w_\varepsilon\|_\varepsilon$. Now we observe that $\|w_\varepsilon\|_\varepsilon$ is uniformly bounded since

$$\|w_\varepsilon\|_\varepsilon \leq \|w_\varepsilon - E_\varepsilon w\|_\varepsilon + \|E_\varepsilon w\|_\varepsilon = O(1) \qquad \text{as } \varepsilon \to 0 \tag{64}$$

as one can deduce by Definition 2.1, (7), (14) and Lemma 2.11 (ii).

Step 2. We prove that $\{B_\varepsilon u_\varepsilon\}_{0<\varepsilon\le\varepsilon_0}$ is bounded in $H^1(\Omega)$ and, in particular, that it is weakly convergent in $H^1(\Omega)$ along a sequence $\varepsilon_n \downarrow 0$ as $n\to+\infty$. Indeed, by (61), (62), (16), (20) (30) and some computations, one can prove that, up to shrink ε_0 if necessary,

$$\int_\Omega |\nabla(B_\varepsilon u_\varepsilon)|^2\,dx \le C\int_{\Omega_\varepsilon}|\nabla u_\varepsilon|^2\,dx \qquad \text{as } \varepsilon\to 0 \tag{65}$$

for some constant C independent of ε. For the same reason one can prove that, up to shrink ε_0 if necessary,

$$\int_{\partial\Omega}(B_\varepsilon u_\varepsilon)^2\,dS \le C\int_{\partial\Omega_\varepsilon} u_\varepsilon^2\,dS \tag{66}$$

for some constant C independent of ε. Combining (65)-(66), we deduce that there exists a constant C independent of ε such that $\|B_\varepsilon u_\varepsilon\|_0 \le C\|u_\varepsilon\|_\varepsilon$ for any ε small enough. The boundedness of $\|B_\varepsilon u_\varepsilon\|_0$ now follows by Step 1.

Hence, we have that there exists $\tilde u\in H^1(\Omega)$ such that, along a sequence $\varepsilon_n\downarrow 0$, $B_{\varepsilon_n}u_{\varepsilon_n}\rightharpoonup \tilde u$ in $H^1(\Omega)$. For simplicity in the sequel we only write $B_\varepsilon u_\varepsilon \rightharpoonup \tilde u$ as $\varepsilon\to 0$ for denoting this convergence along the sequence $\{\varepsilon_n\}$. By passing to the limit in the following identity

$$(u_\varepsilon, E_\varepsilon\varphi)_\varepsilon = \int_{\partial\Omega_\varepsilon} w_\varepsilon\, E_\varepsilon\varphi\,dS \qquad \text{for any } \varphi\in H^1(\Omega)\,, \tag{67}$$

in the next steps we will show that $\tilde u = u$.

Step 3. In this step we pass to the limit in the left hand side of (67). We follow closely the argument contained in the proof of Step 3 in [14, Lemma 8].

We define $K_\varepsilon := \left(\bigcup_{j=1}^{s'} r_j^{-1}(K_{\varepsilon,j})\right)\cup\left(\bigcup_{j=s'+1}^{s} V_j\right)$ and we split the left hand side of (67) in the following way

$$(u_\varepsilon, E_\varepsilon\varphi)_\varepsilon = Q_{K_\varepsilon}(u_\varepsilon, E_\varepsilon\varphi) + Q_{\Omega_\varepsilon\setminus K_\varepsilon}(u_\varepsilon, E_\varepsilon\varphi) + \int_{\partial\Omega_\varepsilon} u_\varepsilon E_\varepsilon\varphi\,dS \tag{68}$$

where $K_{\varepsilon,j}$ denotes the set defined in Section 2.4 and, for any measurable set $A\subset\mathbb{R}^N$, $Q_A(\cdot,\cdot)$ is the bilinear form defined by

$$Q_A(u,v) := \int_A \nabla u\nabla v\,dx\,. \tag{69}$$

We also denote by $Q_A(\cdot)$ the quadratic form

$$Q_A(u) := Q_A(u,u) = \int_A |\nabla u|^2 dx\,. \tag{70}$$

Following the argument employed for proving [14, Estimate (82)] with Q_A replaced by our Q_A defined in (69) and (70), we obtain

$$Q_{K_\varepsilon}(u_\varepsilon, E_\varepsilon\varphi) = Q_{K_\varepsilon}(u_\varepsilon,\varphi) + o(1)\,. \tag{71}$$

Similarly one can prove that

$$Q_{K_\varepsilon}(B_\varepsilon u_\varepsilon,\varphi) = Q_{K_\varepsilon}(u_\varepsilon,\varphi) + o(1) \qquad \text{as } \varepsilon\to 0\,. \tag{72}$$

For more details about (72) see the proof of [14, Estimate (84)].

Combining (71) and (72) we obtain

$$Q_{K_\varepsilon}(u_\varepsilon, E_\varepsilon\varphi) = Q_{K_\varepsilon}(B_\varepsilon u_\varepsilon,\varphi) + o(1) \qquad \text{as } \varepsilon\to 0\,. \tag{73}$$

Now, proceeding as in [14, Estimates (86)-(87)], for the second term on the right hand side of (68), we have

$$Q_{\Omega_\varepsilon\setminus K_\varepsilon}(u_\varepsilon, E_\varepsilon\varphi) = o(1) \qquad \text{as } \varepsilon\to 0\,. \tag{74}$$

Similarly we also have

$$Q_{\Omega\setminus K_\varepsilon}(B_\varepsilon u_\varepsilon,\varphi) = o(1) \qquad \text{as } \varepsilon\to 0\,. \tag{75}$$

Combining (74)-(75) with (73) we infer

$$Q_{\Omega_\varepsilon}(u_\varepsilon, E_\varepsilon\varphi) = Q_{K_\varepsilon}(B_\varepsilon u_\varepsilon, \varphi) + Q_{\Omega\setminus K_\varepsilon}(B_\varepsilon u_\varepsilon, \varphi) + o(1) = Q_\Omega(B_\varepsilon u_\varepsilon, \varphi) + o(1) \qquad \text{as } \varepsilon \to 0 \,. \tag{76}$$

Let us consider now the third term in the right hand side of (68). We proceed as follows:

$$\begin{aligned} \int_{\partial\Omega_\varepsilon} u_\varepsilon E_\varepsilon \varphi \, dS &= \sum_{i,j=1}^{s'} \int_{\partial\Omega_\varepsilon \cap V_i \cap V_j} u_{\varepsilon,j}(x)\varphi_i(\Psi_{\varepsilon,i}(x))\, dS \\ &= \sum_{i,j=1}^{s'} \int_{\Psi_{\varepsilon,j}(\partial\Omega_\varepsilon\cap V_i\cap V_j)} \widehat{u}_{\varepsilon,j}(y)\, \varphi_i(\Psi_{\varepsilon,i}(\Psi_{\varepsilon,j}^{-1}(y)))\, W_{\varepsilon,j}(y)\, dS \end{aligned} \tag{77}$$

with $W_{\varepsilon,j}$ as in (33). By (17) and its consequence (18), we deduce that the trivial extension of $W_{\varepsilon,j}$ to the whole $\partial\Omega$ converges almost everywhere to the function $\chi_{\partial\Omega\cap V_j}$.

Therefore, by (16), (18) and Proposition 2.10, we obtain as $\varepsilon \to 0$

$$\begin{aligned} \int_{\partial\Omega_\varepsilon} u_\varepsilon E_\varepsilon \varphi \, dS &= \sum_{i,j=1}^{s'} \int_{\partial\Omega\cap V_i\cap V_j} \widehat{u}_{\varepsilon,j}(y)\, \varphi_i(y)\, dS + o(1) \\ &= \sum_{i=1}^{s'} \int_{\partial\Omega\cap V_i} B_\varepsilon u_\varepsilon\, \varphi_i\, dS + o(1) = \int_{\partial\Omega} B_\varepsilon u_\varepsilon\, \varphi\, dS + o(1)\,. \end{aligned} \tag{78}$$

Since $B_\varepsilon u_\varepsilon \rightharpoonup \tilde{u}$ in $H^1(\Omega)$, inserting (76) and (78) into (68) and exploiting the continuity of the trace map from $H^1(\Omega)$ into $L^2(\partial\Omega)$, we obtain

$$(u_\varepsilon, E_\varepsilon\varphi)_\varepsilon \to Q_\Omega(\tilde{u}, \varphi) + \int_{\partial\Omega} \tilde{u}\varphi\, dS = (\tilde{u}, \varphi)_0 \qquad \text{as } \varepsilon \to 0\,. \tag{79}$$

Step 4. The next purpose is to pass to the limit in the right hand side of (67).

First of all we observe that thanks to Lemma 2.11 and the fact that $w_\varepsilon \overset{E}{\longrightarrow} w$

$$\begin{aligned} \left| \int_{\partial\Omega_\varepsilon} w_\varepsilon\, E_\varepsilon\varphi\, dS - \int_{\partial\Omega_\varepsilon} E_\varepsilon w\, E_\varepsilon \varphi\, dS \right| &\le \left(\int_{\partial\Omega_\varepsilon} |w_\varepsilon - E_\varepsilon w|^2\, dS \right)^{1/2} \cdot \left(\int_{\partial\Omega_\varepsilon} |E_\varepsilon\varphi|^2 dS \right)^{1/2} \\ &\le \|w_\varepsilon - E_\varepsilon w\|_\varepsilon\, \|E_\varepsilon \varphi\|_\varepsilon = o(1) \qquad \text{as } \varepsilon \to 0\,. \end{aligned} \tag{80}$$

Proceeding as for the proof of (78) one can prove that

$$\int_{\partial\Omega_\varepsilon} E_\varepsilon w\, E_\varepsilon\varphi\, dS \to \int_{\partial\Omega} w\varphi\, dS \qquad \text{as } \varepsilon \to 0\,. \tag{81}$$

Combining (80) and (81), we conclude that

$$\int_{\partial\Omega_\varepsilon} w_\varepsilon\, E_\varepsilon\varphi\, dS \to \int_{\partial\Omega} w\varphi\, dS \qquad \text{as } \varepsilon \to 0\,. \tag{82}$$

Step 5. In this last step we complete the proof of the lemma.

Inserting (79) and (82) into (67) we deduce that

$$(\tilde{u}, \varphi)_0 = \int_{\partial\Omega} w\varphi\, dS \qquad \text{for any } \varphi \in H^1(\Omega)\,.$$

We have shown that $\tilde{u}$ and u are solutions of the same variational problem which admits a unique solution as the reader can easily check. This proves that $\tilde{u} = u$. In particular this means that the weak limit $\tilde{u}$ does not depend on the choice of the sequence $\varepsilon_n \downarrow 0$, thus proving that the convergence $B_\varepsilon u_\varepsilon \rightharpoonup u$ does not occur only along a special sequence but as $\varepsilon \to 0$ in the usual sense. This completes the proof of the lemma. □

Lemma 4.2. *Let $\mathcal{A}$ be an atlas. Let $\{\Omega_\varepsilon\}_{0<\varepsilon\le\varepsilon_0}$ be a family of domains of class $C^{0,1}(\mathcal{A})$ and Ω a domain of class $C^{0,1}(\mathcal{A})$. Assume the validity of conditions* (16), (17). *Let $w_\varepsilon \in H^1(\Omega_\varepsilon)$ with $0<\varepsilon\le\varepsilon_0$, and $w\in H^1(\Omega)$ be such that $w_\varepsilon \xrightarrow{E} w$. If we put $u_\varepsilon := S_\varepsilon w_\varepsilon$ and $u := Sw$ then $u_\varepsilon \xrightarrow{E} u$. In particular this implies that $S_\varepsilon \xrightarrow{EE} S$ as $\varepsilon\to 0$ in the sense of Definition 2.2.*

Proof. We use the notation for $Q_A(\cdot,\cdot)$ and $Q_A(\cdot)$ introduced in (69) and (70).
We write

$$\|u_\varepsilon - E_\varepsilon u\|_\varepsilon^2 = \|u_\varepsilon\|_\varepsilon^2 - 2(u_\varepsilon, E_\varepsilon u)_\varepsilon + \|E_\varepsilon u\|_\varepsilon^2 \,. \tag{83}$$

By (76), (78) and (79) and the fact that $B_\varepsilon u_\varepsilon \rightharpoonup u$ in $H^1(\Omega)$ as proved in Lemma 4.1, we obtain

$$(u_\varepsilon, E_\varepsilon u)_\varepsilon = (B_\varepsilon u_\varepsilon, u)_0 + o(1) \to (u,u)_0 = \|u\|_0^2 \,. \tag{84}$$

Moreover, by Lemma 2.11 we have

$$\|E_\varepsilon u\|_\varepsilon^2 \to \|u\|_0^2 \,. \tag{85}$$

We now prove that $\|u_\varepsilon\|_\varepsilon^2 \to \|u\|_0^2$. We write

$$\|u_\varepsilon\|_\varepsilon^2 = \int_{\partial\Omega_\varepsilon} w_\varepsilon\, u_\varepsilon\, dS = \int_{\partial\Omega_\varepsilon} E_\varepsilon w\, u_\varepsilon\, dS + o(1) \tag{86}$$

where the second identity can be obtained by proceeding exactly as in (80) and exploiting the fact that $\|u_\varepsilon\|_\varepsilon = O(1)$ as $\varepsilon\to 0$ as we have shown in Step 1 of Lemma 4.1.
We claim that

$$\int_{\partial\Omega_\varepsilon} E_\varepsilon w\, u_\varepsilon\, dS \to \int_{\partial\Omega} w\, u\, dS = (u,u)_0 = \|u\|_0^2 \,. \tag{87}$$

Once (87) is proved, combining (84)-(87) with (83) the proof of the lemma follows. Therefore, in order to complete the proof of the lemma, we only have to prove the validity of (87).
In order to estimate $\int_{\partial\Omega_\varepsilon} E_\varepsilon w\, u_\varepsilon\, dS$ we proceed as follows:

$$\begin{aligned}\int_{\partial\Omega_\varepsilon} E_\varepsilon w\, u_\varepsilon\, dS &= \sum_{i,j=1}^{s'} \int_{\partial\Omega_\varepsilon\cap V_i\cap V_j} w_i(\Psi_{\varepsilon,i}(x)) u_{\varepsilon,j}(x)\, dS \\ &= \sum_{i,j=1}^{s'} \int_{\Psi_{\varepsilon,j}(\partial\Omega_\varepsilon\cap V_i\cap V_j)} w_i(\Psi_{\varepsilon,i}(\Psi_{\varepsilon,j}^{-1}(y)))\widehat{u}_{\varepsilon,j}(y) W_{\varepsilon,j}(y)\, dS \,.\end{aligned} \tag{88}$$

with $W_{\varepsilon,j}$ as in (33).
Applying Proposition 2.10 to w, exploiting the fact that $B_\varepsilon u_\varepsilon \rightharpoonup u$ in $H^1(\Omega)$ and recalling that by (17), the trivial extension of $W_{\varepsilon,j}$ to the whole $\partial\Omega$ converges almost everywhere to the function $\chi_{\partial\Omega\cap V_j}$ as explained in Step 3 of the proof of Lemma 4.1, as $\varepsilon\to 0$, we have

$$\begin{aligned}\int_{\partial\Omega_\varepsilon} E_\varepsilon w\, u_\varepsilon\, dS &= \sum_{i,j=1}^{s'} \int_{\partial\Omega\cap V_i\cap V_j} w_i \widehat{u}_{\varepsilon,j}\, dS + o(1) \\ &= \sum_{i=1}^{s'} \int_{\partial\Omega\cap V_i} w_i B_\varepsilon u_\varepsilon\, dS + o(1) = \sum_{i=1}^{s'} \int_{\partial\Omega\cap V_i} w_i u\, dS + o(1) = \int_{\partial\Omega} wu\, dS + o(1)\,.\end{aligned}$$

This completes the proof of (87). □

Lemma 4.3. *Let $\mathcal{A}$ be an atlas. Let $\{\Omega_\varepsilon\}_{0<\varepsilon\le\varepsilon_0}$ be a family of domains of class $C^{0,1}(\mathcal{A})$ and Ω a domain of class $C^{0,1}(\mathcal{A})$. Assume the validity of conditions* (16), (17). *Let $w_\varepsilon\in H^1(\Omega_\varepsilon)$ with $0<\varepsilon\le\varepsilon_0$ be such that $\|w_\varepsilon\|_\varepsilon = 1$. Then $\{S_\varepsilon w_\varepsilon\}_{0<\varepsilon\le\varepsilon_0}$ is precompact in the sense of Definition 2.3. In particular, by Definition 2.4 and Lemma 4.2 we have that $S_\varepsilon \xrightarrow{C} S$.*

Proof. As in Lemma 4.1 we put $u_\varepsilon := S_\varepsilon w_\varepsilon$. Since $\|w_\varepsilon\|_\varepsilon = 1$, proceeding as in the proof of Lemma 4.1, one can show that $B_\varepsilon u_\varepsilon \rightharpoonup \widetilde{u}$ along a sequence, for some $\widetilde{u} \in H^1(\Omega)$. We divide the remaining part of the proof into four steps. We observe that, as in the proof of Lemma 4.1, u_ε satisfies (67) for any $\varepsilon \in (0, \varepsilon_0]$.

Step 1. In this step we pass to the limit in the right hand side of (67).

We proceed as follows:

$$\begin{aligned}(89)\qquad \int_{\partial\Omega_\varepsilon} w_\varepsilon E_\varepsilon \varphi \, dS &= \sum_{i,j=1}^{s'} \int_{\partial\Omega_\varepsilon \cap V_i \cap V_j} w_{\varepsilon,i}(x) \varphi_j(\Psi_{\varepsilon,j}(x)) \, dS \\ &= \sum_{i,j=1}^{s'} \int_{\Psi_{\varepsilon,j}(\partial\Omega_\varepsilon \cap V_i \cap V_j)} w_{\varepsilon,i}(\Psi_{\varepsilon,j}^{-1}(y)) \varphi_j(y) W_{\varepsilon,j}(y) \, dS \, .\end{aligned}$$

with $W_{\varepsilon,j}$ as in (33). As in the proofs of Lemmas 4.1-4.2, we have that the trivial extension of $W_{\varepsilon,j}$ to the whole $\partial\Omega$ converges almost everywhere to the function $\chi_{\partial\Omega \cap V_j}$ and it remains uniformly bounded as $\varepsilon \to 0$ thanks to (16).

Since $\|w_\varepsilon\|_\varepsilon = 1$, by (61) we deduce that

$$\|\widehat{w}_{\varepsilon,i}\|_0 = O(1) \qquad \text{as } \varepsilon \to 0 \, , \text{ for any } i \in \{1, \dots, s'\} \, ,$$

see Step 2 in the proof of Lemma 4.1 for more details.

Hence, by the classical trace inequality for the space $H^1(\Omega)$, we also have

$$\|\widehat{w}_{\varepsilon,i}\|_{H^{1/2}(\partial\Omega)} = O(1) \qquad \text{as } \varepsilon \to 0 \text{ for any } i \in \{1, \dots, s'\} \, .$$

Since the embedding $H^{1/2}(\partial\Omega) \subset L^2(\partial\Omega)$ is compact, $\{\widehat{w}_{\varepsilon,i}\}_{0<\varepsilon\le\varepsilon_0}$ is precompact in $L^2(\partial\Omega)$.

Then, along a sequence $\varepsilon_k \downarrow 0$, we may assume that $\widehat{w}_{\varepsilon_k,i} \to F_i$ in $L^2(\partial\Omega)$ as $k \to +\infty$. For simplicity, in the rest of the proof of the lemma we will omit the subindex k and we simply write $\widehat{w}_{\varepsilon,i} \to F_i$ in $L^2(\partial\Omega)$ as $\varepsilon \to 0$.

Letting $\Theta_{\varepsilon,i,j}$ be as in (23), we see that $\widehat{w}_{\varepsilon,i}(\Theta_{\varepsilon,i,j}(y)) = w_{\varepsilon,i}(\Psi_{\varepsilon,j}^{-1}(y))$ for any $y \in \Psi_{\varepsilon,j}(\partial\Omega_\varepsilon \cap V_i \cap V_j)$.

Then, applying Proposition 2.10 to $\widehat{w}_{\varepsilon,i}$, by (89) we obtain

$$(90)\qquad \int_{\partial\Omega_\varepsilon} w_\varepsilon E_\varepsilon \varphi \, dS \to \sum_{i,j=1}^{s'} \int_{\partial\Omega \cap V_i \cap V_j} F_i \varphi_j \, dS = \int_{\partial\Omega} F \, \varphi \, dS \qquad \text{as } \varepsilon \to 0$$

where we put $F = \sum_{i=1}^{s'} F_i$.

Step 2. In this step we pass to the limit in the left hand side of (67).

One can proceed as in the proof of Step 3 in Lemma 4.1, where that argument was only based on the fact that $\|u_\varepsilon\|_\varepsilon = O(1)$ as $\varepsilon \to 0$, as in the present case. Thus, (79) still holds true.

Combining (79), (90) with (67), we infer

$$(91)\qquad (\widetilde{u}, \varphi)_0 = \int_{\partial\Omega} F \varphi \, dS \qquad \text{for any } \varphi \in H^1(\Omega) \, .$$

Step 3. In this step we pass to the limit in the right hand side of the following identity

$$(92)\qquad (u_\varepsilon, u_\varepsilon)_\varepsilon = \int_{\partial\Omega_\varepsilon} w_\varepsilon u_\varepsilon \, dS \, .$$

As we did for (89), we write

$$(93)\qquad \int_{\partial\Omega_\varepsilon} w_\varepsilon u_\varepsilon \, dS = \sum_{i,j=1}^{s'} \int_{\Psi_{\varepsilon,j}(\partial\Omega_\varepsilon \cap V_i \cap V_j)} w_{\varepsilon,i}(\Psi_{\varepsilon,j}^{-1}(y)) \widehat{u}_{\varepsilon,j}(y) W_{\varepsilon,j}(y) \, dS$$

with $W_{\varepsilon,j}$ as in Step 1, and $\widehat{u}_{\varepsilon,j}$ as in (61).

Proceeding as in the proof of the validity of (65) and (66), we deduce that $\|\widehat{u}_{\varepsilon,j}\|_0 = O(1)$ as $\varepsilon \to 0$. Therefore, passing to the limit along a subsequence $\{\varepsilon_{k_n}\}$ of the sequence $\{\varepsilon_k\}$ introduced in Step 1, for any $j \in \{1, \dots, s'\}$, there exists a function $U_j \in H^1(\Omega)$ such that $\widehat{u}_{\varepsilon_{k_n},j} \rightharpoonup U_j$ in $H^1(\Omega)$ as $n \to +\infty$.

With the same notations of Step 1, we simply write $\varepsilon \to 0$ to denote the convergence along the subsequence $\{\varepsilon_{k_n}\}$. By (93) and the compactness argument of Step 1, we then have

$$\int_{\partial\Omega_\varepsilon} w_\varepsilon u_\varepsilon \, dS \to \sum_{i,j=1}^{s'} \int_{\partial\Omega\cap V_i\cap V_j} F_i U_j \, dS \qquad \text{as } \varepsilon \to 0\,. \tag{94}$$

Since $B_\varepsilon u_\varepsilon = \sum_{j=1}^{s'} \widehat{u}_{\varepsilon,j}$ on $\partial\Omega$, from compactness of the trace map from $H^1(\Omega)$ into $L^2(\partial\Omega)$ and the uniqueness of the strong limit in $L^2(\partial\Omega)$, one immediately obtains $\widetilde{u} = \sum_{j=1}^{s'} U_j$ on $\partial\Omega$, which inserted into (94) gives

$$\int_{\partial\Omega_\varepsilon} w_\varepsilon u_\varepsilon \, dS \to \int_{\partial\Omega} F\widetilde{u} \, dS \qquad \text{as } \varepsilon \to 0\,. \tag{95}$$

Step 4. In this step we conclude the proof of the lemma.

Choosing $\varphi = \widetilde{u}$ in (91) and combining this with (92), (95), we obtain as $\varepsilon \to 0$ along an appropriate sequence

$$\|u_\varepsilon\|_\varepsilon^2 = \int_{\partial\Omega_\varepsilon} w_\varepsilon u_\varepsilon \, dS \to \int_{\partial\Omega} F\widetilde{u} \, dS = \|\widetilde{u}\|_0^2\,. \tag{96}$$

On the other hand, by (67), (90), (91) with $\varphi = \widetilde{u}$, we obtain as $\varepsilon \to 0$ along the same sequence converging to zero,

$$(u_\varepsilon, E_\varepsilon\widetilde{u})_\varepsilon = \int_{\partial\Omega_\varepsilon} w_\varepsilon E_\varepsilon\widetilde{u} \, dS \to \int_{\partial\Omega} F\widetilde{u} \, dS = \|\widetilde{u}\|_0^2\,. \tag{97}$$

Combining (96) and (97) with Lemma 2.11 (ii), we obtain

$$\|u_\varepsilon - E_\varepsilon\widetilde{u}\|_\varepsilon^2 = \|u_\varepsilon\|_\varepsilon^2 - 2(u_\varepsilon, E_\varepsilon\widetilde{u})_\varepsilon + \|E_\varepsilon\widetilde{u}\|_\varepsilon^2 \to 0\,.$$

This proves that, along a sequence converging to zero, we have $u_\varepsilon \xrightarrow{E} \widetilde{u}$ or equivalently that $\{S_\varepsilon w_\varepsilon\}_{0<\varepsilon\le\varepsilon_0}$ is precompact in the sense of Definition 2.3. The proof of the lemma now follows from Lemma 4.2 and Definition 2.4. □

The proof of the theorem now follows combining Lemma 4.3, which states the validity of the compact convergence $S_\varepsilon \xrightarrow{C} S$, with the abstract result stated in Theorem 2.5.

As a bypass product of a number of results proved in this section, we have the following proposition which we believe has its own interest since it clarifies even more the meaning of E-convergence with respect to the operators E_ε used above.

Proposition 4.4. *Let $\mathcal{A}$ be an atlas. Let $\{\Omega_\varepsilon\}_{0<\varepsilon\le\varepsilon_0}$ be a family of domains of class $C^{0,1}(\mathcal{A})$, Ω a domain of class $C^{0,1}(\mathcal{A})$ and for any $\varepsilon \in (0,\varepsilon_0]$ let E_ε be the map defined in* (21). *Assume the validity of condition* (16)*,* (17)*. If $u_\varepsilon \in H^1(\Omega_\varepsilon)$, $u \in H^1(\Omega)$ is such that $u_\varepsilon \xrightarrow{E} u$ as $\varepsilon \to 0$ then*

$$\|u_\varepsilon - u\|_{H^1(\Omega_\varepsilon\cap\Omega)} \to 0 \ \text{ and } \ \|u_\varepsilon - E_\varepsilon u\|_{L^2(\partial\Omega_\varepsilon)} \to 0, \tag{98}$$

as $\varepsilon \to 0$.

Proof. Let $K_\varepsilon \subset \Omega_\varepsilon \cap \Omega$ be as in the proof of Lemma 4.1. Recall that $E_\varepsilon u = u$ on K_ε.

We focus the attention on the proof of the first convergence in (98) since the second one is a trivial consequence of the definition of $\|\cdot\|_\varepsilon$ and the definition of E-convergence, see (15) and Definition 2.1 respectively. In the rest of the proof, we denote by C positive constants independent of ε which may vary from line to line. Combining (16) with Lemma 2.7 (ii) and arguing as in the proof of Lemma 2.11 by exploiting the fact that $|(\Omega_\varepsilon \cap \Omega) \setminus K_\varepsilon| \to 0$ as $\varepsilon \to 0$, we obtain

$$\begin{aligned}
\|u_\varepsilon - u\|_{H^1(\Omega_\varepsilon\cap\Omega)} &\le \|u_\varepsilon - E_\varepsilon u\|_{H^1(\Omega_\varepsilon\cap\Omega)} + \|E_\varepsilon u - u\|_{H^1(\Omega_\varepsilon\cap\Omega)} \\
&\le \|u_\varepsilon - E_\varepsilon u\|_{H^1(\Omega_\varepsilon)} + \|E_\varepsilon u - u\|_{H^1((\Omega_\varepsilon\cap\Omega)\setminus K_\varepsilon)} \\
&\le C\|u_\varepsilon - E_\varepsilon u\|_\varepsilon + o(1) = o(1) \qquad \text{as } \varepsilon \to 0.
\end{aligned}$$

This completes the proof of the proposition. □

5. Proof of Theorem 3.2

The proof of Theorem 3.2 can be obtained with a slightly different approach if compared with the proof of Theorem 3.1. For simplicity we only mention the main differences.

One can show that Lemmas 4.1-4.3, with assumption (17) replaced by (46), still hold true with the appropriate changes: the difference consists on the fact that the operator S of Section 4 has to be replaced here by the operator S_γ and the scalar product $(\cdot,\cdot)_0$ in $H^1(\Omega)$ has to be replaced by the scalar product $(\cdot,\cdot)_\gamma$ defined in (50) (the same has to be done with the associated norms). For simplicity we do not write down in details the adaptations of the proofs of those three lemmas but we prefer to draw the attention only on the more delicate parts appearing in them.

We divide our explanation in three steps each of them corresponding to one of three lemmas mentioned above.

Step 1. In this step we state and prove the counterpart of Lemma 4.1: let $w_\varepsilon \xrightarrow{E} w$, $u_\varepsilon = S_\varepsilon w_\varepsilon$, $u = S_\gamma w$ and let B_ε be as in (62). We have to show that $B_\varepsilon u_\varepsilon \rightharpoonup u$ in $H^1(\Omega)$ as $\varepsilon \to 0$.

According with the notation used in the proof of Lemma 4.1, let $\widetilde{u}$ be the weak limit in $H^1(\Omega)$ of $B_\varepsilon u_\varepsilon$ along a sequence converging to zero.

Let us proceed with the adaptation of (77)-(78).

In the present setting we know that by (46)-(47), $W_{\varepsilon,j} \rightharpoonup \gamma\,\chi_{\partial\Omega\cap V_j}$ weakly in $L^1(\partial\Omega)$ but thanks to (16) and [2, Remark 2.1] we actually have

(99) $$W_{\varepsilon,j} \rightharpoonup \gamma\,\chi_{\partial\Omega\cap V_j} \qquad \text{weakly in } L^p(\partial\Omega) \quad \text{as } \varepsilon\to 0 \quad \text{for any } 1\le p<\infty\,.$$

Looking at (77) we have that, along a sequence,

(100) $$\widehat{u}_{\varepsilon,j} \quad \text{is strongly convergent in } L^q(\partial\Omega) \text{ for any } 1\le q<\tfrac{2(N-1)}{N-2}$$

since $\|\widehat{u}_{\varepsilon,j}\|_{H^1(\Omega)} = O(1)$ as $\varepsilon\to 0$, where $\frac{2(N-1)}{N-2}$ is the critical trace exponent or equivalently the critical Sobolev exponent for the embedding $H^{1/2}(\partial\Omega)\subset L^q(\partial\Omega)$ with the usual understanding that $\frac{2(N-1)}{N-2}=\infty$ for $N=2$. Finally we also have that $\varphi_i(\Psi_{\varepsilon,i}(\Psi_{\varepsilon,j}^{-1}(y))) \to \varphi_i\,\chi_{\partial\Omega\cap V_i\cap V_j}$ strongly in $L^2(\partial\Omega)$ thanks to Proposition 2.10.

Hence we may conclude that (78) has to be replaced by

$$\int_{\partial\Omega_\varepsilon} u_\varepsilon E_\varepsilon\varphi\,dS = \sum_{i,j=1}^{s'}\int_{\partial\Omega\cap V_i\cap V_j}\widehat{u}_{\varepsilon,j}(y)\,\varphi_i(y)W_{\varepsilon,j}(y)\,dS + o(1) = \int_{\partial\Omega}\widetilde{u}\varphi\,\gamma\,dS\,.$$

Indeed, we recall that $\sum_{j=1}^{s'}\widehat{u}_{\varepsilon,j} = B_\varepsilon u_\varepsilon$ on $\partial\Omega$ and that, in view of (100), the trace of $B_\varepsilon u_\varepsilon$ is convergent to the trace of $\widetilde{u}$ strongly in $L^q(\partial\Omega)$ for any $1\le q<\frac{2(N-1)}{N-2}$, being $B_\varepsilon u_\varepsilon \rightharpoonup \widetilde{u}$ weakly in $H^1(\Omega)$.

This means that (79) has to be replaced by

(101) $$(u_\varepsilon, E_\varepsilon\varphi)_\varepsilon \to (\widetilde{u},\varphi)_\gamma$$

as $\varepsilon\to 0$ along an appropriate sequence.

Similarly one can prove that (82) has to be replaced by

(102) $$\int_{\partial\Omega_\varepsilon} w_\varepsilon\,E_\varepsilon\varphi\,dS \to \int_{\partial\Omega}\gamma w\varphi\,dS$$

as $\varepsilon\to 0$ along an appropriate sequence.

Combining (101) and (102) we conclude that $\widetilde{u}$ satisfies

$$(\widetilde{u},\varphi)_\gamma = \int_{\partial\Omega}\gamma w\varphi\,dS$$

and hence it coincides with the function u. The independence of $\widetilde{u}=u$ on the sequence converging to 0 shows that $B_\varepsilon u_\varepsilon \rightharpoonup u$ in $H^1(\Omega)$ as $\varepsilon\to 0$ in the usual sense thus completing the proof of Step 1.

Step 2. In this step we state and prove the counterpart of Lemma 4.2. We have to show that $S_\varepsilon \stackrel{EE}{\longrightarrow} S_\gamma$ as $\varepsilon \to 0$. The main point is to prove the validity of the claim corresponding to (87):

$$\int_{\partial\Omega_\varepsilon} E_\varepsilon w\, u_\varepsilon \, dS \to \int_{\partial\Omega} \gamma w u \, dS = (u,u)_\gamma = \|u\|_\gamma^2 \tag{103}$$

where w_ε, w, u_ε and u are as in Step 1.

To this purpose one has to pass to the limit in (88). This can be done thanks to (99), (100) and the fact that $w_i(\Psi_{\varepsilon,i}(\Psi_{\varepsilon,j}^{-1}(y))) \to w_i\, \chi_{\partial\Omega\cap V_i\cap V_j}$ strongly in $L^2(\partial\Omega)$ thanks to Proposition 2.10. Following the proof of Lemma 4.2, we easily obtain (103) and consequently also the convergence $u_\varepsilon \stackrel{E}{\longrightarrow} u$ as $\varepsilon \to 0$. This proves that $S_\varepsilon \stackrel{EE}{\longrightarrow} S_\gamma$ as $\varepsilon \to 0$.

Step 3. In this step we state and prove the counterpart of Lemma 4.3. Let w_ε be such that $\|w_\varepsilon\|_\varepsilon = 1$ and let $u_\varepsilon = S_\varepsilon w_\varepsilon$. We have to prove the E-convergence of u_ε along a sequence.

We have to pass to the limit in (89) and (93) under the present assumptions.

Concerning (89) we have that $w_{\varepsilon,i}(\Psi_{\varepsilon,j}^{-1}(y)) \to F_i\, \chi_{\partial\Omega\cap V_i\cap V_j}$ strongly in $L^2(\partial\Omega)$, as explained in the proof of Lemma 4.3, so that by (99) we obtain

$$\int_{\partial\Omega_\varepsilon} w_\varepsilon\, E_\varepsilon \varphi \, dS \to \int_{\partial\Omega} \gamma\, F\varphi \, dS$$

as $\varepsilon \to 0$ along a sequence. The left hand side of (67) can be treated as in Step 1 thus giving rise to the identity

$$(\widetilde{u},\varphi)_\gamma = \int_{\partial\Omega} \gamma F \varphi \, dS \tag{104}$$

where $\widetilde{u}$ is as in Step 1.

The second crucial point in the adaptation of the proof of Lemma 4.3 is to pass to the limit in the right hand side of (92) or equivalently in (93). We have again the two factors $w_{\varepsilon,i}(\Psi_{\varepsilon,j}^{-1}(y))$ and $W_{\varepsilon,j}$ that can be treated as above but this time we have the term $\widehat{u}_{\varepsilon,j}$ in place of φ_j which however is strongly convergent in $L^q(\partial\Omega)$ for any $1 \le q < \frac{2(N-1)}{N-2}$ as explained in Step 1. We then conclude that

$$\|u_\varepsilon\|_\varepsilon^2 = \int_{\partial\Omega_\varepsilon} w_\varepsilon u_\varepsilon \, dS \to \int_{\partial\Omega} \gamma F \widetilde{u} \, dS = \|\widetilde{u}\|_\gamma^2$$

as $\varepsilon \to 0$ along a sequence where in the last identity we used (104) with $\varphi = \widetilde{u}$.

Proceeding as in the proof of Lemma 4.3, we infer that $u_\varepsilon \stackrel{E}{\longrightarrow} \widetilde{u}$ as $\varepsilon \to 0$ along a sequence thus proving that $\{S_\varepsilon w_\varepsilon\}_{0<\varepsilon\le\varepsilon_0}$ is precompact in the sense of Definition 2.3.

Combining the three steps above, we conclude that $S_\varepsilon \stackrel{C}{\longrightarrow} S_\gamma$ as $\varepsilon \to 0$ thus completing the proof of the theorem as a consequence of the abstract result Theorem 2.5.

6. Proof of Theorem 3.3

We first state and prove two preliminary results.

Lemma 6.1. *Let Ω and $\{\Omega_\varepsilon\}_{0<\varepsilon\le\varepsilon_0}$ be as in the statement of Theorem 3.3. Assume that* (53) *and* (54) *hold true. Let $\{v_\varepsilon\}_{0<\varepsilon\le\varepsilon_0} \subset H^1_{\rm loc}(\mathbb{R}^N)$ be a family of functions such that*

$$\int_D |\nabla v_\varepsilon|^2 dx + \int_D v_\varepsilon^2 \, dx = O(1), \qquad \text{as } \varepsilon \to 0,$$

for some bounded domain D satisfying $D \supset \Omega \cup \bigcup_{\varepsilon\in(0,\varepsilon_0]} \Omega_\varepsilon$.

Then we have:

(i) *there exist $\varepsilon_k \downarrow 0$ and $v \in H^1(\Omega)$ such that $(v_{\varepsilon_k})_{|\Omega} \rightharpoonup v$ weakly in $H^1(\Omega)$;*

(ii) *if $\int_{\partial\Omega_{\varepsilon_k}} v_{\varepsilon_k}^2 \, dS = O(1)$ as $k \to +\infty$ then $v \in H^1_0(\Omega)$.*

Proof. For simplicity, throughout the proof of the lemma, the restrictions $(v_\varepsilon)_{|\Omega_\varepsilon}$ and $(v_\varepsilon)_{|\Omega}$ will be simply denoted by v_ε.

We first observe that

$$\|v_\varepsilon\|_{H^1(\Omega)} = O(1) \qquad \text{as } \varepsilon \to 0 \tag{105}$$

being $\Omega \subseteq D$. The proof of (i) then follows by the reflexivity of $H^1(\Omega)$.

In order to prove (ii) we have to show that the function v introduced in (i) belongs to $H^1_0(\Omega)$. By compactness of the trace map from $H^1(\Omega)$ into $L^2(\partial\Omega)$ we deduce that

$$v_{\varepsilon_k} \to v \qquad \text{strongly in } L^2(\partial\Omega) \qquad \text{as } k \to +\infty\,. \tag{106}$$

Letting for any $j = 1, \dots, s'$

$$\Gamma_{\varepsilon,j} := r_j^{-1}\left(\{(x', x_N) : x' \in W_j \text{ and } x_N = g_{\varepsilon,j}(x')\}\right) \subseteq \partial\Omega_\varepsilon\,,$$

$$\Gamma_j := r_j^{-1}\left(\{(x', x_N) : x' \in W_j \text{ and } x_N = g_j(x')\}\right) \subseteq \partial\Omega\,,$$

$$w_\varepsilon(x) := v_\varepsilon(r_j^{-1}(x))\,, \quad w(x) = v(r_j^{-1}(x)) \qquad \text{for any } x \in r_j(\Omega)$$

$$W_{t,\varepsilon,j} := \left\{x' \in W_j : \sqrt{1 + |\nabla_{x'}\, g_{\varepsilon,j}(x')|^2} \le t\right\}\,, \qquad W^c_{t,\varepsilon,j} := W_j \setminus W_{t,\varepsilon,j}\,,$$

omitting for simplicity the subindex k we have

$$\begin{aligned} O(1) &= \int_{\partial\Omega_\varepsilon} v_\varepsilon^2\, dS \ge \int_{\Gamma_{\varepsilon,j}} v_\varepsilon^2\, dS = \int_{W_j} w_\varepsilon^2(x', g_{\varepsilon,j}(x'))\sqrt{1 + |\nabla_{x'} g_{\varepsilon,j}(x')|^2}\, dx' \\ &= \int_{W_{t,\varepsilon,j}} w_\varepsilon^2(x', g_{\varepsilon,j}(x'))\sqrt{1 + |\nabla_{x'} g_{\varepsilon,j}(x')|^2}\, dx' + \int_{W^c_{t,\varepsilon,j}} w_\varepsilon^2(x', g_{\varepsilon,j}(x'))\sqrt{1 + |\nabla_{x'} g_{\varepsilon,j}(x')|^2}\, dx'\,. \end{aligned} \tag{107}$$

Inspired by [12] we estimate the first term in the second line of (107):

$$\begin{aligned} &\left(\int_{W_{t,\varepsilon,j}} w_\varepsilon^2(x', g_{\varepsilon,j}(x'))\sqrt{1 + |\nabla_{x'} g_{\varepsilon,j}(x')|^2}\, dx'\right)^{1/2} \\ &\le t^{\frac12}\left[\left(\int_{W_j} |w_\varepsilon(x', g_{\varepsilon,j}(x')) - w_\varepsilon(x', g_j(x'))|^2 dx'\right)^{\frac12} + \left(\int_{W_j} |w_\varepsilon(x', g_j(x')) - w(x', g_j(x'))|^2 dx'\right)^{\frac12}\right. \\ &\qquad \left. + \left(\int_{W_{t,\varepsilon,j}} w^2(x', g_j(x'))\, dx'\right)^{\frac12}\right]\,. \end{aligned} \tag{108}$$

The second term in the right hand side of (108) converges to zero thanks to (106) and the third term does the same since the measure of $W_{t,\varepsilon,j}$ converges to zero thanks to (54). For what it concerns the first one we have

$$\begin{aligned} \int_{W_j} |w_\varepsilon(x', g_{\varepsilon,j}(x')) - w_\varepsilon(x', g_j(x'))|^2 dx' &\le \int_{W_j} |g_{\varepsilon,j}(x') - g_j(x')| \left|\int_{g_j(x')}^{g_{\varepsilon,j}(x')} \left|\tfrac{\partial w_\varepsilon}{\partial t}(x', t)\right|^2 dt\right| dx' \\ &\le \|g_{\varepsilon,j} - g_j\|_{L^\infty(W_j)} \int_D |\nabla v_\varepsilon|^2 dx = O(1) \cdot \|g_{\varepsilon,j} - g_j\|_{L^\infty(W_j)} \to 0 \end{aligned} \tag{109}$$

as $\varepsilon \to 0$ thanks to (53). This proves that the left hand side of (108) tends to zero as $\varepsilon \to 0$ along the prescribed sequence.

Combining this fact with (107), we infer that there exists a positive constant C independent of ε and t such that

(110)
$$C \geq \int_{W^c_{t,\varepsilon,j}} w^2_\varepsilon(x', g_{\varepsilon,j}(x'))\sqrt{1+|\nabla_{x'} g_{\varepsilon,j}(x')|^2}\, dx' + o(1) \geq t \int_{W^c_{t,\varepsilon,j}} w^2_\varepsilon(x', g_{\varepsilon,j}(x'))\, dx' + o(1)\,.$$

We claim that

(111)
$$\int_{W^c_{t,\varepsilon,j}} w^2_\varepsilon(x', g_{\varepsilon,j}(x'))\, dx' = \int_{W^c_{t,\varepsilon,j}} w^2(x', g_j(x'))\, dx' + o(1)\,.$$

Indeed by (106) and (109) we have

$$\left| \left(\int_{W^c_{t,\varepsilon,j}} w^2_\varepsilon(x', g_{\varepsilon,j}(x'))\, dx'\right)^{\frac12} - \left(\int_{W^c_{t,\varepsilon,j}} w^2(x', g_j(x'))\, dx'\right)^{\frac12}\right|$$

$$\leq \left(\int_{W^c_{t,\varepsilon,j}} |w_\varepsilon(x', g_{\varepsilon,j}(x')) - w(x', g_j(x'))|^2\, dx'\right)^{\frac12}$$

$$\leq \left(\int_{W_j} |w_\varepsilon(x', g_{\varepsilon,j}(x')) - w_\varepsilon(x', g_j(x'))|^2\, dx'\right)^{\frac12} + \left(\int_{W_j} |w_\varepsilon(x', g_j(x')) - w(x', g_j(x'))|^2\, dx'\right)^{\frac12} = o(1)\,,$$

thus proving the validity of (111).

Inserting (111) into (110) and taking into account that

$$\int_{W_{t,\varepsilon,j}} w^2(x', g_j(x'))\, dx' = o(1) \qquad \text{as } \varepsilon \to 0\,,$$

as explained after (108), we obtain

$$C \geq t\left(\int_{W^c_{t,\varepsilon,j}} w^2(x', g_j(x'))\, dx' + o(1)\right) + o(1) = t\left(\int_{W_j} w^2(x', g_j(x'))\, dx' + o(1)\right) + o(1)$$

as $\varepsilon \to 0$, which implies

$$\int_{W_j} w^2(x', g_j(x'))\, dx' \leq \frac{C + o(1)}{t} + o(1) \qquad \text{for any } t > 0\,,$$

as $\varepsilon \to 0$. Here the quantities denoted by $o(1)$ do not depend on t. Letting $t \to +\infty$, for any $j = 1, \dots, s'$, we infer that the trace of w on $r_j(\Gamma_j)$ vanishes identically which, in turn, implies that the trace of v vanishes identically on Γ_j. Since the vanishing of v on Γ_j occurs for any $j \in \{1, \dots, s'\}$, this shows that $v \in H^1_0(\Omega)$. □

Lemma 6.2. *For any integer $n \geq 0$ there exist $n+1$ functions $\varphi_0, \dots \varphi_n$ in $C^\infty(\mathbb{R}^N)$ such that, for any bounded domain $\Omega \subset \mathbb{R}^N$, $\varphi_0, \dots, \varphi_n$ are linearly independent when restricted to $\partial\Omega$ in the sense that if $a_0, \dots, a_n \in \mathbb{R}$ are such that*

(112)
$$a_0\varphi_0(x) + \dots + a_n\varphi_n(x) = 0 \qquad \textit{for any } x \in \partial\Omega\,,$$

then $a_0 = \dots = a_n = 0$.

Proof. We may construct explicitly the functions $\varphi_0, \dots, \varphi_n$ choosing for example

(113)
$$\varphi_0(x) = 1 \quad \text{for any } x \in \mathbb{R}^N\,,$$

$$\varphi_k(x) = x_1^k \quad \text{and for any } x = (x_1, \dots, x_N) \in \mathbb{R}^N, \text{ for any } k \in \{1, \dots, n\}\,.$$

Let us prove that this functions are linearly independent when restricted to $\partial\Omega$.

To this purpose, let $a_0, \ldots, a_n \in \mathbb{R}$ be such that (112) holds true and define the polynomial in one variable $P(t) = a_0 + \sum_{k=1}^{n} a_k t^k$. Consider also the corresponding function $\Phi : \mathbb{R}^N \to \mathbb{R}$ defined by $\Phi(x) = P(x_1)$ for any $x = (x_1, \ldots, x_N) \in \mathbb{R}^N$.

Suppose by contradiction that $(a_0, \ldots, a_n) \neq (0, \ldots, 0)$ so that P is not the null polynomial.

Since the degree of the polynomial P is at most n, let $t_1, \ldots, t_m$, $0 \le m \le n$, be its real roots with the convention that when $m = 0$ the polynomial P has no real roots. Then we have that

$$\{x \in \mathbb{R}^N : \Phi(x) = 0\} = \bigcup_{k=1}^{m} \Pi_k \tag{114}$$

with $\Pi_k = \{x = (x_1, \ldots, x_N) \in \mathbb{R}^N : x_1 = t_k\}$.

By (112), (113) and (114), we deduce that $\partial\Omega \subset \bigcup_{k=1}^{m} \Pi_k$. This means that the boundary of the domain Ω is contained in a finite union of parallel hyperplanes of $\mathbb{R}^N$ and this contradicts the fact that Ω is bounded. Indeed, there exists $c \in \mathbb{R} \setminus \{t_1, \ldots, t_m\}$ such that the hyperplane $\Pi_c := \{x = (x_1, \ldots, x_N) : x_1 = c\}$ intersects the set $\overline{\Omega}$, hence also $\partial\Omega$ if Ω is bounded, but this contradicts the condition $\partial\Omega \subset \bigcup_{k=1}^{m} \Pi_k$ found above. □

End of the proof of Theorem 3.3. Let $n \ge 1$ and consider the corresponding eigenvalues λ_n^ε. Let $\varphi_0, \ldots, \varphi_n$ be $n+1$ functions as in Lemma 6.2. We introduce the following family of subspaces of $H^1(\Omega_\varepsilon)$

$$V_\varepsilon := \operatorname{span}\{(\varphi_0)_{|\Omega_\varepsilon}, \ldots, (\varphi_n)_{|\Omega_\varepsilon}\}\,.$$

By Lemma 6.2 we know that $\varphi_0, \ldots, \varphi_n$ are linearly independent as functions defined over $\partial\Omega_\varepsilon$ and in particular as functions defined over Ω_ε. This implies $\dim(V_\varepsilon) = n+1$ and $V_\varepsilon \cap H^1_0(\Omega_\varepsilon) = \{0\}$ for any $\varepsilon \in (0, \varepsilon_0]$.

Therefore by (ii)-(iii) in Proposition 2.12, we infer

$$\lambda_n^\varepsilon \le \max_{v \in V_\varepsilon \setminus \{0\}} \frac{\int_{\Omega_\varepsilon} |\nabla v|^2 dx}{\int_{\partial\Omega_\varepsilon} v^2 dS}\,. \tag{115}$$

Let $a_0^\varepsilon, \ldots, a_n^\varepsilon \in \mathbb{R}$ be such that $v_\varepsilon := \sum_{j=0}^{n} a_j^\varepsilon \varphi_j$ satisfies

$$\frac{\int_{\Omega_\varepsilon} |\nabla v_\varepsilon|^2 dx}{\int_{\partial\Omega_\varepsilon} v_\varepsilon^2 dS} = \max_{v \in V_\varepsilon \setminus \{0\}} \frac{\int_{\Omega_\varepsilon} |\nabla v|^2 dx}{\int_{\partial\Omega_\varepsilon} v^2 dS}\,. \tag{116}$$

It is not restrictive, up to normalization, to assume that $\sum_{j=0}^{n} (a_j^\varepsilon)^2 = 1$.

Let D be a bounded domain such that $D \supset \Omega \cup \bigcup_{\varepsilon \in (0, \varepsilon_0]} \Omega_\varepsilon$ whose existence easily follows by (53).

First of all, by the Cauchy-Schwarz inequality we have

$$\int_D |\nabla v_\varepsilon|^2 dx \le \sum_{j=0}^{n} \int_D |\nabla \varphi_j|^2 dx = O(1) \qquad \text{as } \varepsilon \to 0\,, \tag{117}$$

$$\int_D v_\varepsilon^2 dx \le \sum_{j=0}^{n} \int_D \varphi_j^2\, dx = O(1) \qquad \text{as } \varepsilon \to 0\,.$$

We claim that

$$\int_{\partial\Omega_\varepsilon} v_\varepsilon^2 dS \to +\infty \qquad \text{as } \varepsilon \to 0\,. \tag{118}$$

Suppose by contradiction that there exists a sequence $\varepsilon_k \downarrow 0$ such that $\int_{\partial\Omega_{\varepsilon_k}} v_{\varepsilon_k}^2 dS$ remains bounded as $k \to +\infty$.

Then, by (117) and Lemma 6.1, we deduce that there exists $v \in H^1(\Omega)$ such that

$$v_{\varepsilon_k} \rightharpoonup v \qquad \text{weakly in } H^1(\Omega) \tag{119}$$

and moreover $v \in H^1_0(\Omega)$. Actually the weak convergence in (119) is strong since the sequence of functions $\{(v_{\varepsilon_k})_{|\Omega}\}$ lives in a finite dimensional space. By possibly passing to a subsequence, we may assume that there exist $a_0, \dots, a_n \in \mathbb{R}$ such that

(120) $$a_j^{\varepsilon_k} \to a_j \qquad \text{as } k \to +\infty\,, \ \text{for any } j \in \{0, \dots, n\}\,.$$

Clearly the normalization condition $\sum_{j=0}^n (a_j^{\varepsilon_k})^2 = 1$ implies

(121) $$\sum_{j=0}^n (a_j)^2 = 1\,.$$

Combining (119) and (120) we deduce that $v = \sum_{j=0}^n a_j \varphi_j$. Since $v \in H^1_0(\Omega)$, we have that condition (112) on $\partial\Omega$ is satisfied and hence we infer

$$a_0 = \cdots = a_n = 0$$

thus contradicting (121). This completes the proof of claim (118).

The proof of the theorem now follows immediately combining (115), (116), (117) and (118).

7. Proof of Proposition 3.4

Proof of (i). It can be proved with a simple computation.

Proof of (ii). The fact that g_ε converges uniformly to zero as $\varepsilon \to 0$ and that the first order partial derivatives remain uniformly bounded as $\varepsilon \to 0$, follows immediately after a simple computation. It remains to prove the weak L^1-convergence of the surface element of $\partial\Omega_\varepsilon$. i.e. $\sqrt{1 + |\nabla_{x'} b(x'/\varepsilon)|^2}$. This is a standard fact for which we refer e.g., to [16]. However, since we need to introduce some notation for the proof of part (iii) and, having that notation, the proof takes only a few lines, we include a proof here for the convenience of the reader.

We actually prove the weak convergence in $L^p(W)$ for any $1 < p < \infty$, i.e.

(122) $$\int_W \sqrt{1 + |\nabla_{x'} b(x'/\varepsilon)|^2}\, \varphi(x')\, dx' \to C_b \int_W \varphi(x')\, dx' \qquad \text{for any } \varphi \in L^{\frac{p}{p-1}}(W)$$

where $C_b := \int_Y \sqrt{1 + |\nabla_{x'} b(y')|^2}\, dy'$.

By density of $C^\infty_c(W)$ in $L^p(W)$ it is enough to prove (122) for any $\varphi \in C^\infty_c(W)$.

For any $\varepsilon > 0$ let us define the family $\mathcal{Q}_\varepsilon$ of all $(N-1)$-dimensional closed cubes where every $Q \in \mathcal{Q}_\varepsilon$ is in the form $Q = \overline{\varepsilon(Y + z)}$ with $z \in \mathbb{Z}^{N-1}$. For every $\varepsilon > 0$ let n_ε be the number of cubes in $\mathcal{Q}_\varepsilon$ contained in W; let us denote these cubes by $Q_{\varepsilon,1}, \dots, Q_{\varepsilon,n_\varepsilon}$ so that $Q_{\varepsilon,j} = \overline{\varepsilon(Y + z_{\varepsilon,j})}$ with $z_{\varepsilon,j} \in \mathbb{Z}^{N-1}$ for any $j \in \{1, \dots, n_\varepsilon\}$.

Put $F_\varepsilon = \cup_{j=1}^{n_\varepsilon} Q_{\varepsilon,j}$; from this definition it easily follows that for every $\varepsilon_1 > 0$ there exists $0 < \varepsilon_2 < \varepsilon_1$ such that $F_\varepsilon \supseteq F_{\varepsilon_1}$ for any $0 < \varepsilon < \varepsilon_2$ and moreover $\cup_{\varepsilon>0} F_\varepsilon = W$. This implies

(123) $$\lim_{\varepsilon \to 0} \mathcal{H}^{N-1}(F_\varepsilon) = \mathcal{H}^{N-1}(W)\,.$$

On the other hand, being $Q_{\varepsilon,j}$ cubes with disjoint interiors, we deduce that

$$\mathcal{H}^{N-1}(F_\varepsilon) = \sum_{j=1}^{n_\varepsilon} \mathcal{H}^{N-1}(Q_{\varepsilon,j}) = n_\varepsilon\, \varepsilon^{N-1}$$

which combined with (123) gives

(124) $$\lim_{\varepsilon \to 0} n_\varepsilon\, \varepsilon^{N-1} = \mathcal{H}^{N-1}(W)\,.$$

Being $\varphi \in C^\infty_c(W)$, it is not restrictive to assume that $\operatorname{supp}(\varphi) \subset F_\varepsilon$ for all ε small enough.

Using a change of variables, exploiting the Y-periodicity of b, the Lipschitzianity of φ, as a consequence of its smoothness, and (124), we obtain

$$\int_W \sqrt{1+|\nabla_{x'} b(x'/\varepsilon)|^2}\,\varphi(x')\,dx' = \sum_{j=1}^{n_\varepsilon} \int_{Q_{\varepsilon,j}} \sqrt{1+|\nabla_{x'} b(x'/\varepsilon)|^2}\,\varphi(x')\,dx'$$

$$= \sum_{j=1}^{n_\varepsilon} \int_Y \sqrt{1+|\nabla_{x'} b(y')|^2}\,\varphi(\varepsilon y' + \varepsilon z_{\varepsilon,j})\,\varepsilon^{N-1}\,dy'$$

$$= \sum_{j=1}^{n_\varepsilon} \int_Y \sqrt{1+|\nabla_{x'} b(y')|^2}\,\varphi(\varepsilon z_{\varepsilon,j})\,\varepsilon^{N-1}\,dy' + o(1)$$

$$= \left[\int_Y \sqrt{1+|\nabla_{x'} b(y')|^2}\,dy' \cdot \sum_{j=1}^{n_\varepsilon} \varphi(\varepsilon z_{\varepsilon,j})\,\mathcal{H}^{N-1}(Q_{\varepsilon,j})\right] + o(1)\,.$$

Taking into account that $\sum_{j=1}^{n_\varepsilon} \varphi(\varepsilon z_{\varepsilon,j})\,\mathcal{H}^{N-1}(Q_{\varepsilon,j})$ represents a Cauchy-Riemann sum for $\int_W \varphi(x')\,dx'$, letting $\varepsilon \to 0$, the proof of (122) easily follows.

Clearly, the validity of the weak convergence in $L^p(W)$ for any $1 < p < \infty$ implies the validity of the weak convergence in $L^1(W)$ thus completing the proof of (ii).

Proof of (iii). The validity (53) follows immediately from the definition of g_ε. It remains to prove the validity of (54).

Let $\mathcal{Q}_\varepsilon$, n_ε, $Q_{\varepsilon,1}, \dots, Q_{\varepsilon,n_\varepsilon}$ and F_ε be as in the proof of (ii).

By (123), (124) and the Y-periodicity of b we infer that

$$\mathcal{H}^{N-1}\left(\{x' \in W : |\nabla_{x'} g_\varepsilon(x')| \le t\}\right) = \mathcal{H}^{N-1}\left(\{x' \in W : |\nabla_{x'}\, b(x'/\varepsilon)| \le \varepsilon^{1-\alpha} t\}\right) \tag{125}$$

$$\le \mathcal{H}^{N-1}(W \setminus F_\varepsilon) + \mathcal{H}^{N-1}\left(\{x' \in F_\varepsilon : |\nabla_{x'}\, b(x'/\varepsilon)| \le \varepsilon^{1-\alpha} t\}\right)$$

$$= o(1) + \sum_{j=1}^{n_\varepsilon} \mathcal{H}^{N-1}\left(\{x' \in Q_{\varepsilon,j} : |\nabla_{x'}\, b(x'/\varepsilon)| \le \varepsilon^{1-\alpha} t\}\right)$$

$$= o(1) + \sum_{j=1}^{n_\varepsilon} \varepsilon^{N-1}\,\mathcal{H}^{N-1}\left(\{y' \in \varepsilon^{-1} Q_{\varepsilon,j} : |\nabla_{x'}\, b(y')| \le \varepsilon^{1-\alpha} t\}\right)$$

$$= o(1) + \sum_{j=1}^{n_\varepsilon} \varepsilon^{N-1}\,\mathcal{H}^{N-1}\left(\{y' \in Y : |\nabla_{x'}\, b(y')| \le \varepsilon^{1-\alpha} t\}\right)$$

$$= o(1) + \varepsilon^{N-1} n_\varepsilon\,\mathcal{H}^{N-1}\left(\{y' \in Y : |\nabla_{x'}\, b(y')| \le \varepsilon^{1-\alpha} t\}\right)$$

$$= o(1) + \left(\mathcal{H}^{N-1}(W) + o(1)\right)\,\mathcal{H}^{N-1}\left(\{y' \in Y : |\nabla_{x'}\, b(y')| \le \varepsilon^{1-\alpha} t\}\right) \to 0$$

as $\varepsilon \to 0$ since $0 < \alpha < 1$ and

$$\lim_{\delta \to 0} \mathcal{H}^{N-1}\left(\{y' \in Y : |\nabla_{x'}\, b(y')| \le \delta\}\right) = 0$$

being $\mathcal{H}^{N-1}(\{y' \in Y : |\nabla_{x'}\, b(y')| = 0\}) = 0$ by (57).

By (125) and the fact that for any $t > 0$ we have

$$\left\{x' \in W : \sqrt{1+|\nabla_{x'}\, g_\varepsilon(x')|^2} \le t\right\} \subseteq \left\{x' \in W : |\nabla_{x'}\, g_\varepsilon(x')| \le t\right\}\,,$$

the validity of (54) follows immediately.

8. PROOF OF THEOREM 3.5

Proof of (i). Since $\alpha > 1$ it is sufficient to observe that thanks to Proposition 3.4 (i), we may proceed exactly as in the proof of Theorem 3.1 with the advantage that here we have to deal with an atlas with only one chart.

Proof of (ii). Similarly to (i), since $\alpha = 1$, we may exploit Proposition 3.4 (ii) and proceed as in the proof of Theorem 3.2.

Proof of (iii). The proof can be obtained essentially like the one of Theorem 3.3 once we exploit Proposition 3.4 (iii).

We first observe that under the assumptions of Theorem 3.5 the conclusions of Lemma 6.1 still hold true. Then we need a revised version of Lemma 6.2 which takes into account the homogeneous Dirichlet boundary conditions on Σ_ε appearing in (59).

Lemma 8.1. *Let $\{\Omega_\varepsilon\}_{\varepsilon\geq 0}$ be as in Theorem 3.5 (iii) and let Σ_ε the set defined in* (58). *For any integer $n \geq 0$ there exist $n+1$ functions $\psi_0, \dots, \psi_n$ in $C^\infty(\mathbb{R}^N)$ (independent of ε) with null traces on Σ_ε such that, for any $\varepsilon \geq 0$, $\psi_0, \dots \psi_n$ are linearly independent when restricted to $\partial\Omega_\varepsilon$ in the sense that if $a_0, \dots, a_n \in \mathbb{R}$ are such that*

$$a_0\psi_0(x) + \cdots + a_n\psi_n(x) = 0 \qquad \text{for any } x \in \partial\Omega_\varepsilon\,, \tag{126}$$

then $a_0 = \cdots = a_n = 0$.

Proof. Let $\varphi_0, \dots, \varphi_n$ be the functions defined in (113). We now introduce the two following cutoff functions denoted by η_1 and η_2 respectively.

Let W be as in (55). We choose $\eta_1 \in C_c^\infty(W)$, $\eta_1 \geq 0$ in W, $\eta_1 \equiv 1$ in B' where $B' \subset W$ is an open ball of $\mathbb{R}^{N-1}$. We then choose $\eta_2 \in C^\infty(\mathbb{R})$ with $\eta_2(t) = 0$ for any $t \leq -1$ and $\eta_2(t) = 1$ for any $t \geq -\frac{1}{2}$. Now, for any $k \in \{0, \dots, n\}$ we define

$$\psi_k(x', x_N) := \varphi_k(x', x_N)\,\eta_1(x')\,\eta_2(x_N) \qquad \text{for any } (x', x_N) \in \mathbb{R}^N\,. \tag{127}$$

Clearly $\psi_0, \dots, \psi_n$ vanish on Σ_ε.

It remains to prove the linear independence of the functions $\psi_0, \dots, \psi_n$ on $\partial\Omega_\varepsilon$. Let $a_0, \dots, a_n \in \mathbb{R}$ be as in (126) and as in the proof of Lemma 6.2 we define $P(t) = a_0 + \sum_{k=1}^n a_k t^k$.

Denote by S the segment $B' \cap \{(x_1, 0, \dots, 0) : x_1 \in \mathbb{R}\}$. Then by (126) and (127) we have

$$\begin{aligned} 0 &= \eta_1(x')\,\eta_2(x_N) \sum_{k=0}^n a_k\varphi_k(x', x_N) = \sum_{k=0}^n a_k\varphi_k(x', x_N) \\ &= a_0 + \sum_{k=1}^n a_k x_1^k \qquad \text{for any } x \in \{(x', g_\varepsilon(x')) : x' \in S\}\,. \end{aligned} \tag{128}$$

Let I be the open interval defined by $I = \{x_1 \in \mathbb{R} : (x_1, 0, \dots, 0) \in S\}$. Hence by (128) the polynomial P vanishes on the interval I which means that P is the null polynomial and in particular $a_0 = \cdots = a_n = 0$. This completes the proof of the lemma. □

We now observe that the eigenvalues of (59) in Ω_ε admit the following variational characterization

$$\lambda_n(\varepsilon) = \min_{V \in \mathcal{V}_{\varepsilon,n}} \sup_{v \in V \setminus H_0^1(\Omega_\varepsilon)} \frac{\int_{\Omega_\varepsilon} |\nabla v|^2 dx}{\int_{\partial\Omega_\varepsilon} v^2\, dS} \tag{129}$$

where

$$\mathcal{V}_{\varepsilon,n} := \{V \subseteq H^1_{0,\Sigma_\varepsilon}(\Omega_\varepsilon) : \dim(V) = n+1 \text{ and } V \not\subseteq H_0^1(\Omega_\varepsilon)\}\,,$$

$$H^1_{0,\Sigma_\varepsilon}(\Omega_\varepsilon) := \{v \in H^1(\Omega_\varepsilon) : v = 0 \text{ on } \Sigma_\varepsilon\}\,,$$

as one can verify by proceeding as we did for Proposition 2.12.

Once we have the variational characterization (129), we can proceed as in the proof of Theorem 3.3 using here as V_ε the space $V_\varepsilon := \mathrm{span}\{(\psi_0)_{|\Omega_\varepsilon}, \dots, (\psi_n)_{|\Omega_\varepsilon}\}$, where $\psi_0, \dots, \psi_n$ are the functions introduced in the statement of Lemma 8.1. The proof of (iii) then follows.

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REFERENCES

[1] J. M. Arrieta, S. Bruschi, *Very rapidly varying boundaries in equations with nonlinear boundary conditions. The case of a non uniformly Lipschitz deformation.* Discrete Contin. Dyn. Syst. Ser. B 14 (2010), no. 2, 327-351.

[2] J. M. Arrieta, S. Bruschi, *Rapidly varying boundaries in equations with nonlinear boundary conditions. The case of a Lipschitz deformation*, Math. Models Methods Appl. Sci. **17**, 1555-1585, (2007).

[3] J. M. Arrieta, S. Bruschi, *Boundary oscillations and nonlinear boundary conditions*. C. R. Math. Acad. Sci. Paris 343 (2006), no. 2, 99-104.

[4] J. M. Arrieta, A. N. Carvalho, G. Losada-Cruz, *Dynamics in dumbell domains I. Continuity of the set of equilibria*, J. Diff. Eq. **231**, 551-597, (2006).

[5] J. M. Arrieta, P. D. Lamberti, *Higher order elliptic operators on variable domains. Stability results and boundary oscillations for intermediate problems*, J. Differential Equations **263**, no. 7, 4222-4266, (2017).

[6] J.M. Arrieta, M. López-Fernández, E. Zuazua, *Enrique Approximating travelling waves by equilibria of non-local equations*, Asymptot. Anal. **78**, no. 3, 145-186, (2012).

[7] S. Bögli, *Convergence of sequences of linear operators and their spectra*, Integral Equations Operator Theory **88**, no. 4, 559-599, (2017).

[8] B. Bogosel, *The Steklov Spectrum on Moving Domains*, Appl. Math. Optim. **75**, 1-25, (2017).

[9] D. Bucur, A. Henrot, M. Michetti, *Asymptotic behaviour of the Steklov problem on dumbbell domains*, arXiv:2007.04844, July 2020.

[10] V. Burenkov, P. D. Lamberti, *Sharp spectral stability estimates via the Lebesgue measure of domains for higher order elliptic operators*, Rev. Mat. Complut. **25**, 435-457, (2012).

[11] A. Carvalho, S. Piskarev, A general approximation scheme for attractors of abstract parabolic problems. Numer. Funct. Anal. Optim. **27**, no. 7-8, 785-829, (2006).

[12] E. N. Dancer, D. Daners, *Domain perturbation for elliptic equations subject to Robin boundary conditions*, J. Differential Equations **138**, no. 1, 86-132, (1997).

[13] F. Ferraresso, P.D. Lamberti, *On a Babuška paradox for polyharmonic operators: spectral stability and boundary homogenization for intermediate problems*, Integral Equations Operator Theory 91 (2019), no. 6, Paper No. 55, 42 pp.

[14] A. Ferrero, P. D. Lamberti, *Spectral stability for a class of fourth order Steklov problems under domain perturbations*, Calc. Var. Partial Differential Equations **58**, no. 1, Art. 33, 57 pp, (2019).

[15] A. Girouard, I. Polterovich, *Spectral geometry of the Steklov problem*, J. Spectr. Theory **7**, no. 2, 321-359, (2017).

[16] D. Lukkassen, P. Wall, *On weak convergence of locally periodic functions*. J. Nonlinear Math. Phys. 9 (2002), no. 1, 42-57.

[17] J. Necas, *Direct Methods in the Theory of Elliptic Equations*, Springer-Verlag, (2012).

[18] W. Stekloff, *Sur les problèmes fondamentaux de la physique mathématique*, Ann. Sci. Ecol. Norm. Sup. **19**, 455-490, (1902).

[19] F. Stummel, *Perturbation of domains in elliptic boundary-value problems*, in: Lecture Notes Math., Vol. 503, Springer-Verlag, Berlin-Heidelberg-New York (1976), pp. 110-136.

[20] A. F. M. ter Elst, E.M. Ouhabaz, *Convergence of the Dirichlet-to-Neumann operator on varying domains*. Operator semigroups meet complex analysis, harmonic analysis and mathematical physics, 147-154, Oper. Theory Adv. Appl., 250, Birkhäuser/Springer, Cham, 2015.

[21] G. M. Vainikko, *Regular convergence of operators and the approximate solution of equations*, Mathematical analysis **16** (Russian), 5-53, (1979).

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