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DETERMINISTIC MEAN FIELD GAMES WITH CONTROL ON THE ACCELERATION AND STATE CONSTRAINTS*

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Abstract. We consider deterministic mean field games in which the agents control their acceleration and are constrained to remain in a region of $\mathbb{R}^n$. We study relaxed equilibria in the Lagrangian setting; they are described by a probability measure on trajectories. The main results of the paper concern the existence of relaxed equilibria under suitable assumptions. A difficulty in the proof of existence comes from the fact that the optimal trajectories of the related optimal control problem do not form a compact set. The proof requires closed graph properties of the map which associates to initial conditions the set of optimal trajectories.

Key words. deterministic mean field games, double integrator, state constraints, Lagrangian formulation

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1. Introduction.

The theory of mean field games (MFGs) has been investigated more and more since the pioneering works [19, 20, 21] of Lasry and Lions: it aims at studying the asymptotic behavior of differential games (Nash equilibria) as the number of agents tends to infinity. The dynamics of the agents can be either stochastic or deterministic. Concerning the latter case, we refer the reader to [14] for a detailed study of deterministic MFGs in which the interactions between the agents are modeled by a nonlocal regularizing operator acting on the distribution of the states of the agents. They are described by a system of PDEs coupling a continuity equation for the density of the distribution of states (forward in time) and a Hamilton–Jacobi (HJ) equation for the optimal value of a representative agent (backward in time). If the interaction cost depends locally on the density of the distribution (and hence is not regularizing), then, in the deterministic case, the available theory mostly deals with so-called variational MFGs; see [15].

The major part of the literature on deterministic MFGs addresses situations when the dynamics of a given agent is strongly controllable: for example, in crowd motion models, this happens if the control of a given agent is her velocity. Under the strong

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controllability assumption, it is possible to study realistic models in which the agents are constrained to remain in a given region K of the space state, i.e., state constrained deterministic MFGs. An important difficulty in state constrained deterministic MFGs is that nothing prevents the agents from concentrating on the boundary ∂K of the state space; let us call $m(t)$ the distribution of states at time t. Even if $m(0)$ is absolutely continuous, there may exist some $t > 0$, such that $m(t)$ has a singular part supported on ∂K and the absolute continuous part of $m(t)$ with respect to Lebesgue measure blows up near ∂K. This was first observed in some applications of MFGs to macroeconomics; see [1, 2]. From the theoretical viewpoint, the main issue is that, as we have already said, the distribution of states is generally not absolutely continuous with respect to Lebesgue measure; this makes it difficult to characterize the state distribution by means of PDEs. These theoretical difficulties have been addressed in [11]: following ideas contained in [8, 9, 16], the authors of [11] introduce a weak or relaxed notion of equilibrium, which is defined in a Lagrangian setting rather than with PDEs. Because there may be several optimal trajectories starting from a given point in the state space, the solutions of the relaxed MFG are probability measures defined on a set of admissible trajectories. Once the existence of a relaxed equilibrium is ensured, it is then possible to investigate the regularity of solutions and give a meaning to the system of PDEs and the related boundary conditions; this was done in [12].

On the other hand, if the agents control their acceleration instead of their velocity, the strong controllability property is lost. In [3], we have studied deterministic MFGs in the whole space $\mathbb{R}^n$ with finite time horizon T in which the dynamics of a generic agent is controlled by the acceleration. In [3], the state variable is the pair $(x, v) \in \mathbb{R}^n \times \mathbb{R}^n$, where x and v respectively stand for the position and the velocity; the dynamics of a given agent is described by a *double integrator*

$$\begin{cases} \xi'(s) &= \eta(s), \qquad s \in (t,T), \\ \eta'(s) &= \alpha(s), \qquad s \in (t,T), \\ \xi(t) &= x, \\ \eta(t) &= v. \end{cases}$$

The control α (the acceleration) is a measurable function of time with values in $\mathbb{R}^n$. At the equilibrium, a generic agent chooses her strategy by minimizing at time t the cost

$$\begin{aligned} J_t(\xi,\eta,\alpha) = &\int_t^T \left(L(\xi(s),\eta(s)) + \frac{|\alpha^2(s)|}{2} + F[m(s)](\xi(s),\eta(s)) \right) ds \\ &+ G[m(T)](\xi(T),\eta(T)), \end{aligned}$$

where $m(s)$ is the distribution of states at time s. The initial distribution is a given probability measure on $\mathbb{R}^{2n}$, assumed to be absolutely continuous with respect to Lebesgue measure; its density m_0 may be assumed to be continuous and compactly supported. Here,

- $(x, v) \mapsto L(x, v)$ is a bounded from below and C^2 function defined on $\mathbb{R}^{2n}$;
- F is a Lipschitz continuous map from the set of probability measures on $\mathbb{R}^{2n}$ with finite first moment to bounded C^2 functions defined on $\mathbb{R}^{2n}$ with bounded first and second order derivatives;
- G is a continuous map from the set of probability measures on $\mathbb{R}^{2n}$ with finite first moment to bounded C^2 functions defined on $\mathbb{R}^{2n}$ with bounded first and second order derivatives.

Note that in the optimal control problem described above, the running cost has the form $\ell(\xi(s), \eta(s), s) + 1/2|\alpha(s)|^2$, where

$$\ell(x, v, s) = L(x, v) + F[m(s)](x, v).$$

Similarly, the terminal cost is $g(\xi(T), \eta(T))$ where $g(x, v) = G[m(T)](x, v)$. In the literature, the coupling costs F and G are said to be strongly regularizing because they map probability measures to regular functions defined on $\mathbb{R}^{2n}$.

The MFG leads to the following system of forward-backward PDEs:

(1.1)
$$\begin{cases} -\partial_t u - v \cdot D_x u + H(x, v, D_v u) - F[m(t)](x, v) = 0 & \text{in } \mathbb{R}^{2n} \times (0, T), \\ \partial_t m + v \cdot D_x m - \operatorname{div}_v(D_{p_v} H(x, v, D_v u)m) = 0 & \text{in } \mathbb{R}^{2n} \times (0, T), \\ m(x, v, 0) = m_0(x, v), \quad u(x, v, T) = G[m(T)](x, v) & \text{on } \mathbb{R}^{2n}. \end{cases}$$

The unknowns are the value function of a generic agent u and the density of the distribution of states m. Here, $H : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is defined as a Fenchel–Legendre transform:

$$H(x, v, p_v) = \max_{\alpha \in \mathbb{R}^n} \left(-\alpha \cdot p_v - \frac{|\alpha|^2}{2} \right) - L(x, v) = \frac{|p_v|^2}{2} - L(x, v). \tag{1.2}$$

The Hamiltonian $(x, v, p_x, p_v) \mapsto -v \cdot p_x + H(x, v, p_v)$ is neither strictly convex nor coercive with respect to $p = (p_x, p_v)$. Hence the available results on the regularity of the value function u of the associated optimal control problem [13, 14] and on the existence of a solution of the MFG system [14] cannot be applied. In [3], the existence of a weak solution of the MFG system (1.1) was proved via a vanishing viscosity method; the distribution of states was characterized as the image of the initial distribution by the flow associated with the optimal control.

In traffic theory and also in economics, the models may require that the positions of the agents belong to a given compact subset $\overline{\Omega}$ of $\mathbb{R}^n$, and state constrained MFGs with control on the acceleration must be considered. In the present paper, we wish to investigate some examples of such MFGs and address the first step of the program followed by the authors of [11] in the strongly controllable case: we wish to prove the existence of a relaxed mean field equilibrium in the Lagrangian setting under suitable assumptions (see Definition 3.5 below). We leave for the future the next natural steps, i.e., to further investigate the regularity of the solutions, and then, if possible, to study a related boundary value problem with PDEs as we did in [3].

1.1. Our program. Most of the paper is devoted to the case when the acceleration α can take its values in the whole space and when the running cost depends separately on the acceleration and on the other variables. More precisely, the running cost and the terminal costs will be respectively of the form $L(x, v, t) + F[m(t)](x, v) + \frac{|\alpha|^2}{2}$ and $G[m(T)](x, v)$. The assumptions on F, G, and L will be specified later; at this point, we just need to say that F and G will be assumed to be bounded and L will be bounded from below. The admissible trajectories are pairs of functions (ξ, η), $\xi \in C^1([0, T]; \overline{\Omega})$, $\eta \in W^{1,1}(0, T; \mathbb{R}^n)$ and $\xi' = \eta$. An example of state constrained MFGs in which the acceleration takes its values in a compact subset of $\mathbb{R}^n$ (the optimal value may therefore take the value $+\infty$ in the interior of the x-domain) will be studied in a forthcoming work.

In view of the applications to traffic models, we will deal with the cases when

1. Ω is a bounded domain of $\mathbb{R}^n$ with a smooth boundary; we will also briefly discuss the case when Ω is a convex polygonal domain of $\mathbb{R}^2$;

2. $n = 1$ and Ω is a bounded straight line segment.

In the one-dimensional case, the simplicity of the geometry will allow us to obtain accurate information on the optimal trajectories, and in turn to get a more general existence result for the MFG, yet under an additional assumption on the running cost.

Recall that the admissible states are the pairs $(x, v) \in \overline{\Omega} \times \mathbb{R}^n$, where Ω is a bounded region of $\mathbb{R}^n$. At first glance, we see that some restrictions will have to be imposed on the initial distribution of states: indeed, for $x \in \partial\Omega$ and v pointing outward Ω at x, there is no admissible trajectory taking the value (x, v) at $t = 0$; hence the optimal value $u(x, v, 0)$ is $+\infty$; the definition of the mean field equilibrium would then be unclear if the probability that the initial state takes such values (x, v) was not zero.

As in [11], the aim is to prove the existence of relaxed MFG equilibria which are described by probability measures defined on a set of admissible trajectories (see Definition 3.5 below). We stress the fact that this definition does not involve any system of PDEs. Strongly related to this definition of equilibria in the Lagrangian setting is the notion of *mild solution of the MFG*; see Definition 3.7 below; by and large, a mild solution is a pair (u, m) for which there exists a mean field equilibrium in the sense of Definition 3.5 (i.e., a probability measure μ on admissible trajectories (ξ, η)), such that for each time t and each Borel subset A of the state space,

$$m(t)(A) = \mu\left(\{(\xi, \eta) \,:\, (\xi(t), \eta(t)) \in A\}\right),$$

and u is a value function defined as an infimum on admissible trajectories. Again, note that, for lack of smoothness, the definition of mild solutions is not written in terms of PDEs; see Remark 3.3 below for a discussion on the relationship with a boundary value problem of the form (1.1).

The proof of existence of an equilibrium in the Lagrangian setting involves Kakutani's fixed point theorem (see [17]) applied to a multivalued map defined on a convex and compact set of probability measures on a suitable set of admissible trajectories (itself endowed with the $C^1([0, T]; \mathbb{R}^n) \times C^0([0, T]; \mathbb{R}^n)$-topology). Difficulties in applying Kakutani's fixed point theorem will arise from the fact that all the optimal trajectories do not form a compact subset of $C^1([0, T]; \mathbb{R}^n) \times C^0([0, T]; \mathbb{R}^n)$ (due to the lack of strong controllability). This explains why we shall need additional assumptions, either on the support of the initial distribution of states, or, in some cases, on the running cost.

Assumptions on the support of the initial distribution of states. Note that if a set of trajectories is a compact metric space, then probability measures on this set form a compact set, as required by Kakutani's theorem. Therefore, a natural strategy is to identify a compact set of trajectories that contains the optimal trajectories whose initial value belongs to the support of the initial distribution of states. In such a strategy, we therefore need to identify a modulus of continuity common to all the velocity laws of the optimal trajectories; since the running cost is quadratic in the acceleration, the more natural idea is to look for a uniform bound on the $W^{1,2}$ norms of the velocity laws of the optimal trajectories. But, due to the lack of strong controllability, if x and v respectively belong to $\partial\Omega$ and to the boundary of the tangent cone to $\overline{\Omega}$ at x (the optimal value $u(x, v, 0)$ is finite), there exist sequences $(x_i, v_i)_{i\in\mathbb{N}}$ tending to (x, v) such that the optimal value $u(x_i, v_i, 0)$ blows up when $i \to \infty$; in other words, the cost of preventing the trajectories with initial value (x_i, v_i) from exiting the domain tends to $+\infty$ as $i \to \infty$. Hence, to get uniform bounds on the $W^{1,2}$ norms of the velocity law, the support of the initial distribution of states must not contain such sequences (x_i, v_i). Sufficient conditions on the support of the initial distribution will be given.

Furthermore, Kakutani's fixed point theorem requires a closed graph property for the multivalued map which maps a given point (x, v) to the set of optimal trajectories starting from (x, v). An important part of our work is therefore devoted to proving a closed graph property for the latter map. Note that this issue is of interest in its own right in optimal control theory, independently of MFGs.

Assumptions on the running cost. For $n = 1$, we will be able to get rid of the above-mentioned restrictions on the support of the initial distribution of states if an additional assumption is made on the running cost—namely that it does not favor the trajectories that exit the domain. The existence of equilibria is then proved by approximating the initial distribution m_0 by a sequence $m_{0,k}$ for which Kakutani's theorem can be applied, and by passing to the limit. To pass to the limit, accurate information on the optimal trajectories is needed. We managed to obtain this for $n = 1$ only.

1.2. Organization of the paper. The paper is organized as follows: Section 2 is devoted to state constrained optimal control problems in a bounded region of $\mathbb{R}^n$ with a smooth boundary, and in particular to the closed graph properties of the above-mentioned multivalued map. Although this issue seems to be important in several applications, we were not able to find any relevant result in the available literature. Then, section 3 deals with an existence result for a related mean field equilibrium in the Lagrangian setting, under sufficient conditions on the support of the initial distribution of states. A variant with a nonquadratic cost will be investigated as well. We also shortly address the case of a convex polygonal region in $\mathbb{R}^2$; in the present paper, we will skip the details for brevity, and we will refer to the extended version; see the preprint [4]. In section 4, we address the case when the dynamics takes place in a bounded straight line segment ($n = 1$): under a natural additional assumption on the running cost, we are able to prove the existence of mean field equilibria without any restriction on the initial distribution of states; the proof requires a quite careful study of the optimal trajectories.

2. State constrained optimal control problems in a region of $\mathbb{R}^n$.

2.1. Setting and notation. Let Ω be a bounded domain of $\mathbb{R}^n$ with a boundary $\partial\Omega$ of class C^2. For $x \in \partial\Omega$, let $n(x)$ be the unitary vector normal to $\partial\Omega$ pointing outward Ω. We will use the signed distance to $\partial\Omega$, $d : \mathbb{R}^n \to \mathbb{R}$,

$$d(x) = \begin{cases} \min_{y\in\partial\Omega} |x - y| & \text{if} \quad x \notin \Omega, \\ -\min_{y\in\partial\Omega} |x - y| & \text{if} \quad x \in \Omega. \end{cases}$$

Since $\partial\Omega$ is C^2, the function d is C^2 near $\partial\Omega$. In particular, for all $x \in \partial\Omega$, $\nabla d(x) = n(x)$.

Given a time horizon T and a pair $(x, v) \in \overline{\Omega} \times \mathbb{R}^n$, we are interested in optimal control problems for which the dynamics is of the form

$$\begin{cases} \xi'(s) = \eta(s), & s \in (0, T), \\ \eta'(s) = \alpha(s), & s \in (0, T), \\ \xi(0) = x, \\ \eta(0) = v. \end{cases} \tag{2.1}$$

The state space is $\Xi = \overline{\Omega} \times \mathbb{R}^n$. The optimal control problem consists of minimizing the cost

$$J(\xi,\eta,\alpha)=\int_0^T\left(\ell(\xi(s),\eta(s),s)+\frac{1}{2}|\alpha|^2(s)\right)ds+g(\xi(T),\eta(T)) \tag{2.2}$$

on the dynamics given by (2.1) and staying in Ξ.

ASSUMPTION 2.1. *Here, $\ell:\Xi\times[0,T]\to\mathbb{R}$ is a continuous function, bounded from below. The terminal cost $g:\Xi\to\mathbb{R}$ is also assumed to be continuous and bounded from below. Set*

$$M=\|g_-\|_{L^\infty(\Xi)}+\|\ell_-\|_{L^\infty(\Xi\times[0,T])}, \tag{2.3}$$

where we use the notation $\zeta_-=\max(-\zeta,0)$.

Remark 2.1. The results contained in the present section will be useful for studying the MFGs described in the introduction: in section 3 below, they will be applied to $\ell(x,v,t)=L(x,v,t)+F[m(t)](x,v)$ and $g(x,v)=G[m(T)](x,v)$.

It is convenient to define the set of admissible trajectories as follows:

$$\Gamma=\left\{\ (\xi,\eta)\in C^1([0,T];\mathbb{R}^n)\times AC([0,T];\mathbb{R}^n):\ \left|\ \begin{array}{ll}\xi'(s)=\eta(s) & \forall s\in[0,T]\\ (\xi(s),\eta(s))\in\Xi & \forall s\in[0,T]\end{array}\right.\right\}. \tag{2.4}$$

For any $(x,v)\in\Xi$, set

$$\Gamma[x,v]=\{(\xi,\eta)\in\Gamma:\ \xi(0)=x,\ \eta(0)=v\}. \tag{2.5}$$

Then, $\Gamma^{\rm opt}[x,v]$ is the set of all $(\xi,\eta)\in\Gamma[x,v]$ such that $\eta\in W^{1,2}(0,T,\mathbb{R}^n)$ and (ξ,η,η') achieves the minimum of J in $\Gamma[x,v]$.

Note that $\Gamma[x,v]=\emptyset$ if $x\in\partial\Omega$ and v points outward Ω. This is the reason why we introduce $\Xi^{\rm ad}$ as follows:

$$\Xi^{\rm ad}=\{(x,v):x\in\overline{\Omega},\ v\cdot n(x)\le 0 \text{ if } x\in\partial\Omega\}\subset\Xi. \tag{2.6}$$

LEMMA 2.1. *For all $(x,v)\in\Xi^{\rm ad}$, the optimal value*

$$u(x,v)=\inf_{(\xi,\eta)\in\Gamma[x,v]}J(\xi,\eta,\eta') \tag{2.7}$$

is finite. The function u is lower semicontinuous on $\Xi^{\rm ad}$.

Proof. Let us consider $(x,v)\in\Xi^{\rm ad}$. We distinguish two cases:

1. $x\in\Omega$ or $x\in\partial\Omega$ and $v\cdot n(x)<0$: in this case, for $\bar t$ small enough, the trajectory (ξ,η) defined by

$$\begin{cases}\eta(s) = (1-\frac{s}{\bar t})v \quad\text{and}\quad \xi(s)= x+(s-\frac{s^2}{2\bar t})v & \text{if}\quad 0\le s\le\bar t,\\ \eta(s) = 0 \quad\text{and}\quad \xi(s)= x+\frac{\bar t}{2}v & \text{if}\quad \bar t\le s\le T\end{cases}$$

is admissible and $J(\xi,\eta,\eta')$ is finite.
2. $x\in\partial\Omega$ and $v\cdot n(x)=0$. We make a simple observation that will also be used in the proof of Lemma 2.3 below: for all $x\in\partial\Omega$, there exist an open neighborhood V_x of x in $\mathbb{R}^n$, a positive number R_x, and a C^2-diffeomorphism Φ_x from V_x onto $B(0,R_x)$ such that for all $y\in V_x$, the nth coordinate of $\Phi_x(y)$ is $d(y)$, i.e., $\Phi_{x,n}(y)=d(y)$. Hence, $\Phi_x|_{V_x\cap\Omega}$ is a C^2-diffeomorphism from $V_x\cap\Omega$ onto $B_-(0,R_x)=B(0,R_x)\cap$

$\{x_n < 0\}$, and $\Phi_x|_{V_x \cap \partial\Omega}$ is a C^2-diffeomorphism from $V_x \cap \partial\Omega$ onto $B(0, R_x) \cap \{x_n = 0\}$. Let us also call Ψ_x the inverse of Φ_x, which is a C^2-diffeomorphism from $B(0, R_x)$ onto V_x. Note that

$$\nabla d(y) = D\Phi_x^T(y) e_n \quad \text{for all } y \in V_x, \tag{2.8}$$

where e_n is the nth vector of the canonical basis. In particular, $n(x) = D\Phi_x^T(x)e_n$. In the present case, let us set $\hat{x} = \Phi_x(x)$ and $\hat{v} = D\Phi_x(x)v$. It is easy to see that $\hat{x}_n = 0$ and $\hat{v}_n = 0$. Then, for $\bar{t}$ small enough, the trajectory (ξ, η) defined by $\xi(s) = \Psi_x(\hat{\xi}(s))$, $\eta(s) = \frac{d\xi}{ds}(s) = D\Psi_x(\hat{\xi}(s))\hat{\eta}(s)$ for all $s \in [0, T]$, with

$$\begin{cases} \hat{\eta}(s) = (1 - \frac{s}{\bar{t}})\hat{v} & \text{and} \quad \hat{\xi}(s) = \hat{x} + (s - \frac{s^2}{2\bar{t}})\hat{v} & \text{if} \quad 0 \le s \le \bar{t}, \\ \hat{\eta}(s) = 0 & \text{and} \quad \hat{\xi}(s) = \hat{x} + \frac{\bar{t}}{2}\hat{v} & \text{if} \quad \bar{t} \le s \le T, \end{cases}$$

is admissible and $J(\xi, \eta, \eta')$ is finite.

The lower semicontinuity of u on Ξ^{ad} stems from standard arguments in the calculus of variations. ∎

2.2. Closed graph properties. An important feature of the optimal control problem described above is the closed graph property.

PROPOSITION 2.2. *Consider a closed subset Θ of Ξ^{ad}. Assume that for all sequences $(x^i, v^i)_{i\in\mathbb{N}}$ such that for all $i \in \mathbb{N}$, $(x^i, v^i) \in \Theta$, and $\lim_{i\to+\infty}(x^i, v^i) = (x, v) \in \Theta$, the following holds: if $x \in \partial\Omega$, then*

$$\left((v^i \cdot \nabla d(x^i))_+\right)^3 = o\left(\left|d(x^i)\right|\right) \tag{2.9}$$

(note that (2.9) is meaningful for i large enough because d is C^1 near $\partial\Omega$); then the graph of the multivalued map

$$\Gamma^{\mathrm{opt}}: \quad \begin{array}{l} \Theta \rightrightarrows \Gamma, \\ (x, v) \mapsto \Gamma^{\mathrm{opt}}[x, v] \end{array}$$

is closed, which means that for any sequence $(y^i, w^i)_{i\in\mathbb{N}}$ such that for all $i \in \mathbb{N}$, $(y^i, w^i) \in \Theta$ with $(y^i, w^i) \to (y, w)$ as $i \to \infty$; consider a sequence $(\xi^i, \eta^i)_{i\in\mathbb{N}}$ such that for all $i \in \mathbb{N}$, $(\xi^i, \eta^i) \in \Gamma^{\mathrm{opt}}[y^i, w^i]$; if (ξ^i, η^i) tends to (ξ, η) uniformly, then $(\xi, \eta) \in \Gamma^{\mathrm{opt}}[y, w]$.

Remark 2.2. In Proposition 2.2, the condition (2.9) is restrictive only for sequences $(x^i, v^i) \in \Theta$ which tend to $(x, v) \in \Theta$ such that $x \in \partial\Omega$ and v is tangent to $\partial\Omega$ at x. We will see that this assumption makes it possible to control the cost associated to the optimal trajectories starting from (x^i, v^i).

Remark 2.3. In section 4.1 below, we will see that in dimension one (Ω is then a bounded straight line), and under stronger assumptions on the running cost, the closed graph properties hold for $\Theta = \Xi^{\mathrm{ad}}$.

Remark 2.4. In the context of MFGs (see section 3 below), the assumptions in Proposition 2.2 will yield sufficient conditions on the support of the initial distribution for the existence of relaxed mean field equilibria.

The proof of Proposition 2.2 relies on Lemmas 2.3 and 2.5 below.

LEMMA 2.3. *Consider $(x, v) \in \Xi^{\mathrm{ad}}$, $(\xi, \eta) \in \Gamma[x, v]$ such that $\eta \in W^{1,2}(0, T; \mathbb{R}^n)$ and a sequence $(x^i, v^i)_{i\in\mathbb{N}}$ such that for all $i \in \mathbb{N}$, $(x^i, v^i) \in \Xi^{\mathrm{ad}}$ and $\lim_{i\to\infty}(x^i, v^i) = (x, v)$.*

Assume that one of the following conditions is true:

1. *$x \in \Omega$;*
2. *$x \in \partial\Omega$ and $v \cdot n(x) < 0$ (hence $v^i \cdot \nabla d(x^i) < 0$ for i large enough);*
3. *$x \in \partial\Omega$, $v \cdot n(x) = 0$ and one of the following properties is true:*
   (a) *for i large enough, $v^i \cdot \nabla d(x^i) \le 0$;*
   (b) *for i large enough, $v^i \cdot \nabla d(x^i) > 0$ (hence $d(x^i) < 0$) and*

$$\lim_{i\to\infty} \frac{(v^i \cdot \nabla d(x^i))^3}{|d(x^i)|} = 0. \tag{2.10}$$

Then there exists a sequence $(\xi^i, \eta^i)_{i\in\mathbb{N}}$ such that $(\xi^i, \eta^i) \in \Gamma[x^i, v^i]$, $\eta^i \in W^{1,2}(0, T; \mathbb{R}^n)$, and (ξ^i, η^i) tends to (ξ, η) in $W^{2,2}(0, T; \mathbb{R}^n)\times W^{1,2}(0, T; \mathbb{R}^n)$, hence uniformly in $[0, T]$.

Before proving Lemma 2.3, let us define a family of third order polynomials with values in $\mathbb{R}^n$.

Definition 2.4. *Given $t > 0$ and $x, v, y, w \in \mathbb{R}^n$, let $Q_{t,x,v,y,w}$ be the unique third order polynomial with value in $\mathbb{R}^n$ such that*

$$Q_{t,x,v,y,w}(0) = x, \quad Q'_{t,x,v,y,w}(0) = v, \quad Q_{t,x,v,y,w}(t) = y, \quad Q'_{t,x,v,y,w}(t) = w.$$

It is given by

$$Q_{t,x,v,y,w}(s) = x + vs + \left(3\frac{y-x-vt}{t^2} - \frac{w-v}{t}\right)s^2 + \left(-2\frac{y-x-vt}{t^3} + \frac{w-v}{t^2}\right)s^3. \tag{2.11}$$

Proof of Lemma 2.3. We are going to see that each of the three conditions mentioned in the statement makes it possible to explicitly construct families of admissible trajectories fulfilling all the desired properties (in particular with a finite energy or cost). The trickier situations will arise when $x \in \partial\Omega$ and $v^i \cdot \nabla d(x^i) > 0$ for i large enough, in which case the restrictive condition (2.10) will be needed. Since the construction is different in each of the three cases mentioned in Lemma 2.3, we discuss each case separately.

1. If $x \in \Omega$, then there exist $\bar t \in (0, T]$ and $c > 0$ such that $d(\xi(s)) < -c$ for all $s \in [0, \bar t]$. We construct the sequence $(\xi^i, \eta^i)_{i\in\mathbb{N}}$ as follows:

$$\xi^i(s) = \begin{cases} \xi(s) + Q_{\bar t,\delta x^i,\delta v^i,0,0}(s) & \text{if} \quad 0 \le s \le \bar t, \\ \xi(s) & \text{if} \quad \bar t \le s \le T, \end{cases}$$

   where $\delta x^i = x^i - x$ and $\delta v^i = v^i - v$; see Definition 2.4 for the third order polynomial $Q_{\bar t,\delta x^i,\delta v^i,0,0}$. It is clear that for i large enough, $\xi^i(s) \in \overline{\Omega}$ for all $s \in [0, T]$; hence $(\xi^i, \eta^i) \in \Gamma[x^i, v^i]$ and $\eta^i \in W^{1,2}(0, T; \mathbb{R}^n)$. On the other hand, it can be easily checked that

$$\lim_{i\to+\infty} \int_0^T \left|\frac{d\eta^i}{dt}(s) - \frac{d\eta}{dt}(s)\right|^2 ds = 0. \tag{2.12}$$

   This achieves the proof in the first case.
2. $x \in \partial\Omega$ and $v \cdot n(x) < 0$; hence for i large enough, $v^i \cdot \nabla d(x^i) < 0$. We can always assume that the latter property holds for all i.

   Notation. We use the same geometric arguments as in the proof of Lemma 2.1: for the neighborhood V_x mentioned there, there exists $\hat T$, $0 < \hat T \le T$, such that $\xi(s) \in V_x \cap \overline{\Omega}$ for all $s \in [0, \hat T]$. Consider the local chart Φ_x introduced in the proof of Lemma 2.1 and call Ψ_x the inverse map from $B(0,R_x)$ onto V_x. For $t\in[0,\hat T]$, let us set $\hat\xi(t)=\Phi_x(\xi(t))$, $\hat\eta(t)=\frac{d\hat\xi}{dt}(t)=D\Phi_x(\xi(t))\frac{d\xi}{dt}(t)$, $\hat x=\Phi_x(x)$, and $\hat v=\hat\eta(0)=D\Phi_x(x)v$. It is clear that $\hat x_n=0$ and that $\hat v_n<0$. We also set $\hat x^i=\Phi_x(x^i)$ and $\hat v^i=D\Phi_x(x^i)v^i$.

Since $\eta\in W^{1,2}(0,T)$, there exists $\bar t\in(0,\hat T)$ such that for all $s\in[0,\bar t]$,

$$\frac{3}{2}\hat v_n\le\hat\eta(s)\cdot e_n\le\frac{1}{2}\hat v_n,\tag{2.13}$$

$$\frac{3s}{2}\hat v_n\le(\hat\xi(s)-\hat x)\cdot e_n\le\frac{s}{2}\hat v_n.\tag{2.14}$$

For $t_i\in[0,\bar t]$, we set

$$\hat\xi^i(s)=\begin{cases} Q_{t_i,\hat x^i,\hat v^i,\hat\xi(t_i),\hat\eta(t_i)}(s) & \text{if}\quad s\in[0,t_i],\\ \hat\xi(s) & \text{if}\quad s\in[t_i,\hat T],\end{cases}$$

and $\hat\eta^i(s)=\frac{d\hat\xi^i}{dt}(s)$ for $s\in[0,\hat T]$. Then, we define ξ^i as follows:

$$\xi^i(s)=\begin{cases}\Psi_x\left(\hat\xi^i(s)\right) & \text{if}\quad s\in[0,\hat T],\\ \xi(s) & \text{if}\quad s\in[\hat T,T],\end{cases}$$

and $\eta^i=\frac{d\xi^i}{dt}$. Let us first see why $(\xi^i,\eta^i)\in\Gamma[x_i,v_i]$ for t_i small enough and i large enough. A straightforward calculation shows that for $s\le t_i$,

$$\begin{aligned}\hat\xi^i(s)-\hat x=&(\hat x^i-\hat x)\left(1-3\frac{s^2}{t_i^2}+2\frac{s^3}{t_i^3}\right)+s\hat v^i\left(1-2\frac{s}{t_i}+\frac{s^2}{t_i^2}\right)\\ &+(\hat\xi(t_i)-\hat x)\left(3\frac{s^2}{t_i^2}-2\frac{s^3}{t_i^3}\right)+s\hat\eta(t_i)\left(-\frac{s}{t_i}+\frac{s^2}{t_i^2}\right).\end{aligned}\tag{2.15}$$

Let us focus on $(\hat\xi^i(s)-\hat x)\cdot e_n=\hat\xi^i(s)\cdot e_n$: from the formula above, we see that $\hat\xi^i(s)\cdot e_n$ is the sum of four terms, the first three of them being nonpositive and the last one being nonnegative for all $s\in[0,t_i]$. Let us consider the sum of the last three terms, namely

$$A(s)=s\hat v^i\cdot e_n\left(1-2\frac{s}{t_i}+\frac{s^2}{t_i^2}\right)+\hat\xi(t_i)\cdot e_n\left(3\frac{s^2}{t_i^2}-2\frac{s^3}{t_i^3}\right)+s\hat\eta(t_i)\cdot e_n\left(-\frac{s}{t_i}+\frac{s^2}{t_i^2}\right);$$

from (2.13) and (2.14), we see that

$$\begin{aligned}&\hat\xi(t_i)\cdot e_n\left(3\tfrac{s^2}{t_i^2}-2\tfrac{s^3}{t_i^3}\right)+s\hat\eta(t_i)\cdot e_n\left(-\tfrac{s}{t_i}+\tfrac{s^2}{t_i^2}\right)\\ \le\;& s\hat v\cdot e_n\left(\tfrac12\left(3\tfrac{s^2}{t_i^2}-2\tfrac{s^3}{t_i^3}\right)+\tfrac32\left(-\tfrac{s}{t_i}+\tfrac{s^2}{t_i^2}\right)\right)\\ =\;& s\hat v\cdot e_n\left(-\tfrac32\tfrac{s}{t_i}+3\tfrac{s^2}{t_i^2}-\tfrac{s^3}{t_i^3}\right)\end{aligned}$$

for all $s\in[0,t_i]$. On the other hand, since $\lim_{i\to\infty}\hat v^i=\hat v$, we see that for i large enough, $\hat v^i\cdot e_n\le 3\hat v\cdot e_n/4$. Hence,

$$A(s)\le s\hat v\cdot e_n\left(\frac34\left(1-\frac{s}{t_i}\right)^2-\frac32\frac{s}{t_i}+3\frac{s^2}{t_i^2}-\frac{s^3}{t_i^3}\right)=s\hat v\cdot e_n\left(\frac34-3\frac{s}{t_i}+\frac{15}{4}\frac{s^2}{t_i^2}-\frac{s^3}{t_i^3}\right).$$

It is easy to check that the function $\theta \mapsto \frac{3}{4} - 3\theta + \frac{15}{4}\theta^2 - \theta^3$ is positive for $\theta \in [0,1]$, which implies that $A(s)$ is negative for $s \in [0, t_i]$.
Hence, for i large enough, $\hat{\xi}^i(s) \cdot e_n \leq 0$ for all $0 \leq s \leq t_i \leq \bar{t}$.
On the other hand, since $\lim_{i\to+\infty} \left(|\hat{x}^i - \hat{x}| + |\hat{v}^i - v|\right) = 0$, $\hat{\xi}$ and $\hat{\eta}$ are continuous, and (2.15) implies that there exist $I > 0$ and $\tilde{t} \in (0, \bar{t}\,]$ such that, if $i \geq I$ and $t_i \in (0, \tilde{t})$, then $\hat{\xi}^i(s) \in \overline{B_-(0, R_x)}$ for all $s \in [0, t_i]$. Hence, for $i \geq I$ and $t_i \in (0, \tilde{t})$, $(\xi^i, \eta^i) \in \Gamma[x^i, v^i]$.

Let us now turn to $\|\frac{d\eta^i}{dt}\|_{L^2(0,t_i)}$: straightforward calculus shows that

$$\frac{d\eta^i}{dt}(t) = D\Psi_x(\hat{\xi}^i(t))\frac{d\hat{\eta}^i}{dt}(t) + \left(D^2\Psi_x(\hat{\xi}^i(t))\hat{\eta}^i(t)\right)\hat{\eta}^i(t).$$

This implies that

$$\left\|\frac{d\eta^i}{dt}\right\|^2_{L^2(0,t_i)} \leq C\left(\left\|\frac{d\hat{\eta}^i}{dt}\right\|^2_{L^2(0,t_i)} + \|\hat{\eta}^i\|^4_{L^4(0,t_i)}\right) \tag{2.16}$$

for a constant C independent of $x \in \partial\Omega$. Hereafter, C may vary from line to line.
First, we focus on $\|\frac{d\hat{\eta}^i}{dt}\|^2_{L^2(0,t_i)}$:

$$\begin{aligned}
&\left\|\frac{d\hat{\eta}^i}{dt}\right\|^2_{L^2(0,t_i)} \\
&\quad = 4\int_0^{t_i}\left|3\frac{\hat{\xi}(t_i) - \hat{v}^i t_i - \hat{x}^i}{t_i^2} - \frac{\hat{\eta}(t_i) - \hat{v}^i}{t_i} + \frac{3s}{t_i}\left(-2\frac{\hat{\xi}(t_i) - \hat{v}^i t_i - \hat{x}^i}{t_i^2} + \frac{\hat{\eta}(t_i) - \hat{v}^i}{t_i}\right)\right|^2 ds \\
&\quad \leq 2I_1 + 2I_2,
\end{aligned}$$

where

$$I_1 = \int_0^{t_i}\left|2\left(3\frac{\hat{\xi}(t_i) - \hat{v}t_i - \hat{x}}{t_i^2} - \frac{\hat{\eta}(t_i) - \hat{v}}{t_i}\right) + \frac{6s}{t_i}\left(-2\frac{\hat{\xi}(t_i) - \hat{v}t_i - \hat{x}}{t_i^2} + \frac{\hat{\eta}(t_i) - \hat{v}}{t_i}\right)\right|^2 ds$$

and

$$I_2 = \int_0^{t_i}\left|2\left(3\frac{\hat{x} - \hat{x}^i}{t_i^2} + 2\frac{\hat{v} - \hat{v}^i}{t_i}\right) + \frac{6s}{t_i}\left(-2\frac{\hat{x} - \hat{x}^i}{t_i^2} - \frac{\hat{v} - \hat{v}^i}{t_i}\right)\right|^2 ds.$$

Standard arguments yield that

$$I_1 \leq C\left\|\frac{d\hat{\eta}}{dt}\right\|^2_{L^2(0,t_i)}$$

for an absolute constant $C > 0$. Therefore, given $\epsilon > 0$, there exists $\hat{t}:\ 0 < \hat{t} < \bar{t}$ such that $2I_1 < \epsilon/2$ for all $t_i < \hat{t}$.
On the other hand,

$$I_2 \leq C_1\left(\frac{|\hat{x} - \hat{x}^i|^2}{t_i^3} + \frac{|\hat{v} - \hat{v}^i|^2}{t_i}\right) \leq C\left(\frac{|x - x^i|^2}{t_i^3} + \frac{|v - v^i|^2}{t_i}\right).$$

It is possible to choose the sequence t_i such that

- $\lim_{i\to\infty} t_i = 0$,

- $\lim_{i\to\infty} \frac{|x-x^i|^2}{t_i^3} + \frac{|v-v^i|^2}{t_i} = 0$,
- $(\xi^i, \eta^i) \in \Gamma[x^i, v^i]$ for i large enough.

Such a choice of t_i yields that $\lim_{i\to\infty} \|\frac{d\tilde{\eta}^i}{dt}\|^2_{L^2(0,t_i)} = 0$. On the other hand, the choice made on t_i also implies that $\lim_{i\to\infty} \frac{|x-x^i|}{t_i} = 0$, and in turn that $\|\hat{\eta}^i\|_{L^\infty(0,t_i)}$ is uniformly bounded with respect to i; therefore, the quantity $\|\hat{\eta}^i\|^4_{L^4(0,t_i)}$ tends to 0 as i tends to ∞; using (2.16), we have proved that $\lim_{i\to\infty} \|\frac{d\eta^i}{dt}\|_{L^2(0,t_i)} = 0$. Therefore, it is possible to choose a sequence $t_i > 0$ such that the trajectories (ξ^i, η^i) are admissible for i large enough and $\lim_{i\to\infty} \|\frac{d\eta^i}{dt} - \frac{d\eta}{dt}\|_{L^2(0,T)} = 0$. This achieves the proof in case 2.

3. (a) $x \in \partial\Omega$, $v \cdot n(x) = 0$ and $v^i \cdot \nabla d(x^i) \le 0$ at least for i large enough. We may assume that $v^i \cdot \nabla d(x^i) \le 0$ for all i. Using the same notation as in case 2, we see from (2.8) that $\hat{v}^i \cdot e_n = v^i \cdot D\Phi_x^T(x^i)e_n = v^i \cdot \nabla d(x^i) \le 0$.

In the present case, the approximate trajectories will have three successive phases; see Remarks 2.5 and 2.6 for explanations on these different phases.

Given $t_{i,1} \in (0, \hat{T})$, we define $(\hat{y}^i, \hat{w}^i)$ as follows:

$$\begin{aligned} \hat{y}^i &= \hat{x}_n e_n + \pi_{e_n^\perp}\left(\hat{x}^i + \hat{v}^i t_{i,1}\right), \\ \hat{w}^i &= \hat{v}_n e_n + \pi_{e_n^\perp}\left(\hat{v}^i\right) = \pi_{e_n^\perp}\left(\hat{v}^i\right), \end{aligned}$$

where $\pi_{e_n^\perp}$ stands for the orthogonal projector on $e_n^\perp = \mathbb{R}^{n-1} \times \{0\}$, and we set

$$\hat{\xi}^i(s) = Q_{t_{i,1},\hat{x}^i,\hat{v}^i,\hat{y}^i,\hat{w}^i}(s) \quad \text{and} \quad \hat{\eta}^i(s) = \frac{d\hat{\xi}^i}{ds}(s) \qquad \text{for} \quad 0 \le s \le t_{i,1}.$$

Remark 2.5. In this first phase of the approximate trajectory, i.e., for $s \in [0, t_{i,1}]$, $\pi_{e_n^\perp}(Q''_{t_{i,1},\hat{x}^i,\hat{v}^i,\hat{y}^i,\hat{w}^i}(s)) = 0$. The effort only lies in driving the nth components of $\hat{\xi}^i(s)$ and $\hat{\eta}^i(s)$ so that they match those of $\hat{x}$ and $\hat{v}$ at $s = t_{i,1}$.

As above, we first check that for $t_{i,1}$ small enough and i large enough, $\hat{\xi}^i(s) \in \overline{B_-(0, R_x)}$ for all $s \in [0, t_{i,1}]$: from the definition of $Q_{t_{i,1},\hat{x}^i,\hat{v}^i,\hat{y}^i,\hat{w}^i}$, we see that

$$\hat{\xi}^i(s) \cdot e_n = \left(\hat{x}^i \cdot e_n \left(1 + 2\frac{s}{t_{i,1}}\right) + s\hat{v}^i \cdot e_n\right)\left(1 - \frac{s}{t_{i,1}}\right)^2 \tag{2.17}$$

is nonpositive for $s \in [0, t_{i,1}]$. On the other hand, we see that there exist $I > 0$ and $0 < \tilde{t} \le \hat{T}$ such that if $i \ge I$ and $0 < t_{i,1} < \tilde{t}$, then for all $s \in [0, t_{i,1}]$, $\hat{\xi}^i(s) \in \overline{B_-(0, R_x)}$.

As in case 2, we need to focus on $\|\frac{d\hat{\eta}^i}{dt}\|_{L^2(0,t_{i,1})}$: straightforward calculus shows that

$$\begin{aligned} \left\|\frac{d\hat{\eta}^i}{dt}\right\|^2_{L^2(0,t_{i,1})} &\le C\left(\frac{|\hat{x}^i \cdot e_n|^2}{t_{i,1}^3} + \frac{|(\hat{v} - \hat{v}^i) \cdot e_n|^2}{t_{i,1}}\right) \\ &= C\left(\frac{d^2(x^i)}{t_{i,1}^3} + \frac{|v^i \cdot \nabla d(x^i)|^2}{t_{i,1}}\right), \end{aligned}$$

and we see as above that there exists a sequence $t_{i,1}$ such that

- $\lim_{i\to\infty} t_{i,1} = 0$,
- $\lim_{i\to\infty} \frac{d^2(x^i)}{t_{i,1}^3} + \frac{|v^i \cdot \nabla d(x^i)|^2}{t_{i,1}} = 0$,

- $\hat{\xi}^i(s) \in \overline{B_-(0,R_x)}$ for all $0 \le s \le t_{i,1}$.

Taking the derivative of $\hat{\xi}^i$ and arguing as in case 2, we also see that $\lim_{i\to\infty} \|\hat{\eta}^i\|_{L^4(0,t_{i,1})} = 0$ because $\lim_{i\to\infty} \frac{d(x^i)}{t_{i,1}} = 0$.

Next, for $t_{i,1} < t_{i,2} < \hat{T}$, we set

$$\hat{\xi}^i(s) = \begin{cases} Q_{t_{i,1},\hat{x}^i,\hat{v}^i,\hat{y}^i,\hat{w}^i}(s) & \text{if} \quad s \le t_{i,1}, \\ Q_{t_{i,2}-t_{i,1},\hat{y}^i-\hat{x},\hat{w}^i-\hat{v},0,0}(s-t_{i,1}) + \hat{\xi}(s-t_{i,1}) & \text{if} \quad t_{i,1} \le s \le t_{i,2}, \end{cases} \tag{2.18}$$

and

$$\xi^i(s) = \begin{cases} \Psi_x\left(\hat{\xi}^i(s)\right) & \text{if} \quad s \le t_{i,2}, \\ \xi(s-t_{i,1}) & \text{if} \quad t_{i,2} \le s \le T. \end{cases} \tag{2.19}$$

As above, $\hat{\eta}^i(s) = \frac{d\hat{\xi}^i}{dt}(s)$ for $0 \le s \le t_{i,2}$ and $\eta^i(s) = \frac{d\xi^i}{dt}(s)$ for $0 \le s \le T$.

Remark 2.6. In the second phase of the approximate trajectory, i.e., for $s \in [t_{i,1}, t_{i,2}]$, the components of $\hat{\xi}^i(s)$ and $\hat{\xi}(s-t_{i,1})$ parallel to e_n coincide, i.e., $Q_{t_{i,2}-t_{i,1},\hat{y}^i-\hat{x},\hat{w}^i-\hat{v},0,0}(s-t_{i,1}) \cdot e_n = 0$. The effort only consists of driving the projections of $\hat{\xi}^i(s)$ and $\hat{\eta}^i(s)$ on $e_n^\perp$ such that they match those of $\hat{\xi}(s-t_{i,1})$ and $\hat{\eta}(s-t_{i,1})$ at $s = t_{i,2}$. We will see that it is not necessary to have $t_{i,2}$ tend to zero, because from the choice of $t_{i,1}$, the distance between $(\hat{\xi}^i(t_{i,1}), \hat{\eta}^i(t_{i,1}))$ and $(\hat{x}, \hat{v})$ tends to 0 as $i \to +\infty$.

It is possible to choose the sequence $t_{i,2}$ bounded from below by a positive constant which depends on (x,v) but not on i such that $\hat{\xi}^i(s)$ stays in $\overline{B_-(0,R_x)}$ for $s \in [t_{i,1}, t_{i,2}]$. Hence, $(\xi^i, \eta^i) \in \Gamma[x^i, v^i]$.
Moreover, since $t_{i,2}$ is bounded away from 0 and $\lim_{i\to\infty}\left(|\hat{y}^i - \hat{x}| + |\hat{w}^i - \hat{v}|\right) = 0$, it is not difficult to check that $\lim_{i\to\infty} \|\frac{d\eta^i}{dt} - \frac{d\eta}{dt}\|_{L^2(0,T)} = 0$; this achieves the proof in subcase 3(a).

(b) $x \in \partial\Omega$, $v \cdot n(x) = 0$, $v^i \cdot \nabla d(x^i) > 0$ for all i (or for i large enough), and (2.10) holds.
The trajectory ξ^i is constructed as in (2.18)–(2.19), but a further restriction on $t_{i,1}$ is needed in order to guarantee that the trajectory is admissible. Using (2.17), we see that the trajectory is admissible if

$$\frac{\hat{v}^i \cdot e_n}{|\hat{x}^i \cdot e_n|} \le \frac{1}{s} + \frac{2}{t_{i,1}} \qquad \text{for all } 0 \le s \le t_{i,1}.$$

This happens if and only if

$$t_{i,1} \le \frac{3|\hat{x}^i \cdot e^n|}{\hat{v}^i \cdot e_n} = \frac{3|d(x^i)|}{v^i \cdot \nabla d(x_i)},$$

which should be supplemented with the other two conditions as in subcase 3(a):

$$\lim_{i\to\infty} t_{i,1} = 0,$$
$$\lim_{i\to\infty} \frac{|d(x^i)|^2}{t_{i,1}^3} + \frac{|v^i \cdot \nabla d(x^i)|^2}{t_{i,1}} = 0.$$

If (2.10) holds, then it is possible to choose such a sequence $t_{i,1}$. Then, as in subcase 3(a), it is possible to choose the sequence $t_{i,2}$ bounded from below by a positive constant independent of i such that $(\xi^i, \eta^i) \in \Gamma[x^i, v^i]$; the last part of the proof is identical to subcase 3(a). ∎

LEMMA 2.5. *Consider* $(x, v) \in \Xi^{\mathrm{ad}}$ *and a sequence* $(x^i, v^i)_{i\in\mathbb{N}}$ *such that for all* $i \in \mathbb{N}$, $(x^i, v^i) \in \Xi^{\mathrm{ad}}$ *and* $(x^i, v^i) \to (x, v)$ *as* $i \to \infty$. *Suppose that Assumption* 2.1 *and one of the three conditions in Lemma* 2.3 *are satisfied. Let a sequence* $(\xi^i, \eta^i)_{i\in\mathbb{N}}$ *be such that for all* $i \in \mathbb{N}$, $(\xi^i, \eta^i) \in \Gamma^{\mathrm{opt}}[x^i, v^i]$. *If* (ξ^i, η^i) *tends to* (ξ, η) *uniformly in* $[0, T]$, *then* $\eta \in W^{1,2}(0, T; \mathbb{R}^n)$ *and* $(\xi, \eta) \in \Gamma^{\mathrm{opt}}[x, v]$.

Proof. We need to prove that for any $(\widetilde{\xi}, \widetilde{\eta}) \in \Gamma[x, v]$ such that $\widetilde{\eta} \in W^{1,2}(0, T; \mathbb{R}^n)$,

$$J(\xi, \eta, \eta') \le J(\widetilde{\xi}, \widetilde{\eta}, \widetilde{\eta}'). \tag{2.20}$$

From Lemma 2.3 applied to $(\widetilde{\xi}, \widetilde{\eta})$, there exists a sequence $(\widetilde{\xi}^i, \widetilde{\eta}^i)_{i\in\mathbb{N}}$, with $(\widetilde{\xi}^i, \widetilde{\eta}^i) \in \Gamma[x^i, v^i]$, such that $(\widetilde{\xi}^i, \widetilde{\eta}^i) \to (\widetilde{\xi}, \widetilde{\eta})$ uniformly on $[0, T]$ as $i \to \infty$, and

$$\lim_{i\to\infty} \int_0^T \left| \frac{d\widetilde{\eta}^i}{dt}(s) \right|^2 ds = \int_0^T \left| \frac{d\widetilde{\eta}}{dt}(s) \right|^2 ds.$$

On the other hand, the optimality of (ξ^i, η^i) yields that

$$J\left(\xi^i, \eta^i, \frac{d\eta^i}{dt}\right) \le J\left(\widetilde{\xi}^i, \widetilde{\eta}^i, \frac{d\widetilde{\eta}^i}{dt}\right). \tag{2.21}$$

From the properties of $(\widetilde{\xi}^i, \widetilde{\eta}^i)$, the right-hand side of (2.21) converges to $J(\widetilde{\xi}, \widetilde{\eta}, \frac{d\widetilde{\eta}}{dt})$. The left-hand side of (2.21) is thus bounded. Combining this fact with the uniform convergence of (ξ^i, η^i) to (ξ, η) in $[0, T]$, we obtain that the sequence $\int_0^T |\frac{d\eta^i}{dt}(s)|^2 ds$ is bounded. This implies $\frac{d\eta^i}{dt} \rightharpoonup \frac{d\eta}{dt}$ in $L^2(0, T; \mathbb{R}^n)$ weakly and $\liminf_{i\to\infty} \int_0^T |\frac{d\eta^i}{dt}(s)|^2 ds \ge \int_0^T |\frac{d\eta}{dt}(s)|^2 ds$. We deduce that

$$J\left(\xi, \eta, \frac{d\eta}{dt}\right) \le \liminf_{i\to\infty} J\left(\xi^i, \eta^i, \frac{d\eta^i}{dt}\right).$$

Combining the information obtained above, we obtain (2.20), which achieves the proof. ∎

Proof of Proposition 2.2. Consider $(y, w) \in \Theta$ and a sequence $(y^i, w^i)_{i\in\mathbb{N}}$ such that for all $i \in \mathbb{N}$, $(y^i, w^i) \in \Theta$ and $(y^i, w^i) \to (y, w)$ as $i \to \infty$. Consider a sequence $(\xi^i, \eta^i)_{i\in\mathbb{N}}$ such that for all $i \in \mathbb{N}$, $(\xi^i, \eta^i) \in \Gamma^{\mathrm{opt}}[y^i, w^i]$ and that (ξ^i, η^i) tends to (ξ, η) uniformly. Thanks to the assumption made in the statement of Proposition 2.2, possibly after the extraction of a subsequence, we may suppose that one of the three conditions in Lemma 2.3 holds. Then the conclusion follows from Lemma 2.5. ∎

2.3. Bounds related to optimal trajectories.

DEFINITION 2.6. *For a positive number* C, *let us set*

$$K_C = \{(x, v) \in \Xi : \ |v| \le C\}, \tag{2.22}$$

$$\Gamma_C = \left\{ (\xi, \eta) \in \Gamma : \left| \begin{array}{l} (\xi(t), \eta(t)) \in K_C \quad \forall t \in [0, T], \\ \left\| \frac{d\eta}{dt} \right\|_{L^2(0,T;\mathbb{R}^n)} \le C \end{array} \right. \right\}. \tag{2.23}$$

PROPOSITION 2.7. *Given $r > 0$, let us define*

$$\Theta_r = \Theta \cap K_r, \tag{2.24}$$

where K_r is defined by (2.22) *and Θ is a closed subset of Ξ^{ad} which satisfies the assumption in Proposition* 2.2.

Under Assumption 2.1*, the value function u defined in* (2.7) *is continuous on Θ_r. There exists a positive number $C = C(r, M)$ such that if $(x, v) \in \Theta_r$, then $\Gamma^{\mathrm{opt}}[x, v] \subset \Gamma_C$.*

Remark 2.7. The set of trajectories Γ_C is a compact subset of Γ. In the context of MFGs (see section 3), the existence of relaxed equilibria will be obtained by applying Kakutani's fixed point theorem to a multivalued map defined on a closed set of probability measures on Γ_C.

Proof. Take $(x, v) \in \Theta_r$ and a sequence $(x^i, v^i)_{i\in\mathbb{N}}$, $(x^i, v^i) \in \Theta_r$, such that $\lim_{i\to\infty}(x^i, v^i) = (x, v)$.

From Lemma 2.1 we know that $u(x, v)$ is finite, and from Assumption 2.1 we know that the infimum in (2.7) is achieved by a trajectory $(\xi, \eta) \in \Gamma^{\mathrm{opt}}[x, v]$.

Possibly after the extraction of a subsequence, we may assume that (x^i, v^i) satisfies one of the three points in Lemma 2.3. Then, there exists a sequence $(\xi^i, \eta^i)_{i\in\mathbb{N}}$ such that for all $i \in \mathbb{N}$, $(\xi^i, \eta^i) \in \Gamma[x^i, v^i]$, $\eta^i \in W^{1,2}(0, T; \mathbb{R}^n)$, and (ξ^i, η^i) tends to (ξ, η) in $W^{2,2}(0, T; \mathbb{R}^n) \times W^{1,2}(0, T; \mathbb{R}^n)$, and hence uniformly in $[0, T]$. Hence,

$$\lim_{i\to\infty} J\left(\xi^i, \eta^i, \frac{d\eta^i}{dt}\right) = u(x, v).$$

On the other hand,

$$J\left(\xi^i, \eta^i, \frac{d\eta^i}{dt}\right) \geq u(x^i, v^i).$$

The latter two observations yield that

$$\limsup_{i\to\infty} u(x^i, v^i) \leq u(x, v).$$

This proves that u is upper-semicontinuous on Θ_r. From Lemma 2.1, u is continuous on Θ_r. Since Θ_r is a compact subset of $\mathbb{R}^n \times \mathbb{R}^n$, u is bounded on Θ_r.

Then from the definitions of J and u and the boundedness of u on Θ_r, it is clear that there exists a constant $C = C(r, M)$ such that $\Gamma^{\mathrm{opt}}[x, v] \subset \Gamma_C$ for any $(x, v) \in \Theta_r$. ∎

3. A mean field game with control on the acceleration and state constraints.

3.1. Setting and notation.

The bounded domain Ω of $\mathbb{R}^n$ and the sets Ξ and Ξ^{ad} have been introduced in section 2.1. Let $\mathcal{P}(\Xi)$ be the set of probability measures on Ξ.

Let $C^0_b(\Xi; \mathbb{R})$ denote the space of bounded and continuous real-valued functions defined on Ξ, and let $F, G : \mathcal{P}(\Xi) \to C^0_b(\Xi; \mathbb{R})$ be bounded and continuous maps (the continuity is with respect to the narrow convergence in $\mathcal{P}(\Xi)$). Let L be a real-valued, continuous, and bounded from below function defined on $\Xi \times [0, T]$.

Let $F[m]$ and $G[m]$ denote the images by F and G of $m \in \mathcal{P}(\Xi)$. Set

$$M = \max\left(\sup_{(x,v,s)\in\Xi\times[0,T]} L_-(x,v,s) + \sup_{m\in\mathcal{P}(\Xi)} \|F[m]\|_{L^\infty(\Xi)}, \sup_{m\in\mathcal{P}(\Xi)} \|G[m]\|_{L^\infty(\Xi)} \right). \tag{3.1}$$

Let Γ be the set of admissible trajectories given by (2.4). It is a metric space with the distance $d((\xi,\eta),(\tilde{\xi},\tilde{\eta})) = \|\xi - \tilde{\xi}\|_{C^1([0,T];\mathbb{R}^n)}$. Let $\mathcal{P}(\Gamma)$ be the set of probability measures on Γ.

For $t \in [0,T]$, the evaluation map $e_t : \Gamma \to \Xi$ is defined by $e_t(\xi,\eta) = (\xi(t),\eta(t))$ for all $(\xi,\eta) \in \Gamma$.

For any $\mu \in \mathcal{P}(\Gamma)$, let the Borel probability measure $m^\mu(t)$ on Ξ be defined by $m^\mu(t) = e_t\sharp\mu$. It is possible to prove that if $\mu \in \mathcal{P}(\Gamma)$, then $t \mapsto m^\mu(t)$ is continuous from $[0,T]$ to $\mathcal{P}(\Xi)$ for the narrow convergence in $\mathcal{P}(\Xi)$. Hence, for all $(\xi,\eta) \in \Gamma$, $t \mapsto F[m^\mu(t)](\xi(t),\eta(t))$ is continuous and bounded by the constant M in (3.1).

With $\mu \in \mathcal{P}(\Gamma)$, we associate the cost

$$J^\mu(\xi,\eta) = \left(\begin{array}{l} \displaystyle\int_0^T \left(F[m^\mu(s)](\xi(s),\eta(s)) + L(\xi(s),\eta(s),s) + \frac{1}{2}\left|\frac{d\eta}{dt}(s)\right|^2 \right) ds \\ + G[m^\mu(T)](\xi(T),\eta(T)) \end{array} \right). \tag{3.2}$$

Remark 3.1. It is clear from (3.1) that given $\mu \in \mathcal{P}(\Gamma)$, the running and terminal costs $\ell(y,w,s) = F[m^\mu(s)](y,w) + L(y,w,s)$ and $g(y,w) = G[m^\mu(T)](y,w)$ satisfy Assumption 2.1, and that the constant arising in (2.3) can be chosen uniformly with respect to $\mu \in \mathcal{P}(\Gamma)$. Hence, for all $\mu \in \mathcal{P}(\Gamma)$, Propositions 2.2 and 2.7 hold for the state constrained control problem related to J^μ, and the constants arising in these propositions can be chosen uniformly with respect to $\mu \in \mathcal{P}(\Gamma)$.

ASSUMPTION 3.1. *There exists a positive number r such that the initial distribution of states is a probability measure m_0 on Ξ supported in Θ_r, where Θ_r is a closed subset of Ξ^{ad} as in* (2.24).

Let $C = C(r,M)$ be the constant appearing in Proposition 2.7 (uniform with respect to μ) and Γ_C be the compact subset of Γ defined by (2.23); clearly, Γ_C is a Radon metric space. From the Prokhorov theorem (see [6, Theorem 5.1.3]), the set $\mathcal{P}(\Gamma_C)$ is compact for the narrow convergence of measures.

Let $\mathcal{P}_{m_0}(\Gamma)$ (resp., $\mathcal{P}_{m_0}(\Gamma_C)$) denote the set of probability measures μ on Γ (resp., Γ_C) such that $e_0\sharp\mu = m_0$.

Hereafter, we identify $\mathcal{P}(\Gamma_C)$ with a subset of $\mathcal{P}(\Gamma)$ by extending $\mu \in \mathcal{P}(\Gamma_C)$ by 0 outside Γ_C. Similarly, we may consider $\mathcal{P}_{m_0}(\Gamma_C)$ as a subset of $\mathcal{P}_{m_0}(\Gamma)$.

Note that for all $\mu \in \mathcal{P}(\Gamma_C)$ and for all $t \in [0,T]$, $m^\mu(t)$ is supported in K_C, where K_C is defined as in (2.22).

Remark 3.2. Note that Γ_C (endowed with the metric of the $C^1 \times C^0$-convergence of (ξ,η)) is a Polish space (because it is compact). The multivalued map $\widetilde{\Gamma}^{\mathrm{opt}}$, related, for instance, to $F \equiv 0$ and $G \equiv 0$, maps Θ_r to nonempty and closed subsets of Γ_C (the closedness can be checked by usual arguments of the calculus of variations). Since the graph of $\widetilde{\Gamma}^{\mathrm{opt}}$ is closed, $\widetilde{\Gamma}^{\mathrm{opt}}$ is measurable. Therefore, there exists a measurable selection $j : \Theta_r \to \Gamma_C$ from the Kuratowski and Ryll-Nardzewski theorem [18]. Then $j\sharp m_0$ belongs to $\mathcal{P}_{m_0}(\Gamma_C)$. The set $\mathcal{P}_{m_0}(\Gamma_C)$ is not empty.

3.2. Existence of a mean field game equilibrium.

LEMMA 3.1. *Let a sequence of probability measures $(\mu_i)_{i\in\mathbb{N}}$, $\mu_i \in \mathcal{P}(\Gamma)$, be narrowly convergent to $\mu \in \mathcal{P}(\Gamma)$. For all $t \in [0,T]$, $(m^{\mu_i}(t))_{i\in\mathbb{N}}$ is narrowly convergent to $m^\mu(t)$.*

Proof. For all $f \in C_b^0(\Xi;\mathbb{R})$,

$$\begin{aligned}\int_\Xi f(x,v)dm^{\mu_i}(t)(x,v) = \int_\Gamma f(\xi(t),\eta(t))d\mu_i(\xi,\eta) &\to \int_\Gamma f(\xi(t),\eta(t))d\mu(\xi,\eta)\\ &= \int_\Xi f(x,v)dm^\mu(t)(x,v). \qquad \square\end{aligned}$$

An easy consequence of Lemma 3.1 is that for $C = C(r,M)$ as in Proposition 2.7, $\mathcal{P}_{m_0}(\Gamma_C)$ is a closed subset of $\mathcal{P}(\Gamma_C)$ and is therefore compact.

LEMMA 3.2. *If $\mu \in \mathcal{P}(\Gamma_C)$, the map $t \mapsto m^\mu(t)$ is 1/2-Hölder continuous from $[0,T]$ to $\mathcal{P}(K_C)$ (K_C is defined in (2.22) and $\mathcal{P}(K_C)$ is endowed with the Kantorovitch–Rubinstein distance).*

Proof. Let ϕ be any Lipschitz function defined on K_C with a Lipschitz constant not larger than 1.

$$\begin{aligned}&\int_{K_C} \phi(x,v)\left(dm^\mu(t_2)(x,v) - dm^\mu(t_1)(x,v)\right)\\ =&\int_{\Gamma_C} \left(\phi(\xi(t_2),\eta(t_2)) - \phi(\xi(t_1),\eta(t_1))\right) d\mu(\xi,\eta)\\ \le&\int_{\Gamma_C} \left(|\xi(t_2)-\xi(t_1)| + |\eta(t_2)-\eta(t_1)|\right) d\mu(\xi,\eta)\\ \le& C\int_{\Gamma_C} \left(|t_2-t_1| + |t_2-t_1|^{\frac12}\right) d\mu(\xi,\eta)\\ \le&\tilde{C}\,|t_2-t_1|^{\frac12}\end{aligned}$$

for a constant $\tilde{C}$ which depends only on C and T. $\square$

It is useful to recall the disintegration theorem.

THEOREM 3.3. *Let X and Y be Radon metric spaces, $\pi : X \to Y$ be a Borel map, and μ be a probability measure on X. Set $\nu = \pi\sharp\mu$. There exists a ν-almost everywhere uniquely defined Borel measurable family of probability measures $(\mu_y)_{y\in Y}$ on X such that*

$$\mu_y(X \setminus \pi^{-1}(y)) = 0 \quad \text{for } \nu\text{-almost all } y \in Y,$$

and for every Borel function $f : X \to [0,+\infty]$,

$$\int_X f(x)d\mu(x) = \int_Y \left(\int_X f(x)d\mu_y(x)\right)d\nu(y) = \int_Y \left(\int_{\pi^{-1}(y)} f(x)d\mu_y(x)\right)d\nu(y).$$

Recall that $(\mu_y)_{y\in Y}$ is a Borel family of probability measures if for any Borel subset B of X, $Y \ni y \mapsto \mu_y(B)$ is a Borel function from Y to $[0,1]$.

It is possible to apply Theorem 3.3 with $X = \Gamma_C$, $Y = \Theta_r$, $\pi = e_0$, and $\nu = m_0$ (identifying m_0 and its restriction to Θ_r): for any $\mu \in \mathcal{P}_{m_0}(\Gamma_C)$, there exists an m_0-almost everywhere uniquely defined Borel measurable family of probability measures

$(\mu_{(x,v)})_{(x,v)\in\Theta_r}$ on Γ_C such that

$$\mu_{(x,v)}(\Gamma_C \setminus e_0^{-1}(x,v)) = 0 \quad \text{for } m_0\text{-almost all } (x,v) \in \Theta_r, \tag{3.3}$$

and for every Borel function $f : \Gamma_C \to [0,+\infty]$,

$$\begin{aligned}\int_{\Gamma_C} f(\xi,\eta)d\mu(\xi,\eta) &= \int_{\Theta_r}\left(\int_{\Gamma_C} f(\xi,\eta)d\mu_{(x,v)}(\xi,\eta)\right)dm_0(x,v)\\ &= \int_{\Theta_r}\left(\int_{e_0^{-1}(x,v)} f(\xi,\eta)d\mu_{(x,v)}(\xi,\eta)\right)dm_0(x,v).\end{aligned} \tag{3.4}$$

For $(x,v) \in \Theta_r$, m_0 supported in Θ_r, and $\mu \in \mathcal{P}_{m_0}(\Gamma_C)$ (where $C = C(r,M)$ is the constant appearing in Proposition 2.7), let us set

$$\Gamma^{\mu,\mathrm{opt}}[x,v] = \left\{(\xi,\eta) \in \Gamma[x,v] \ : \ J^\mu(\xi,\eta) = \min_{(\widetilde{\xi},\widetilde{\eta})\in\Gamma[x,v]} J^\mu(\widetilde{\xi},\widetilde{\eta})\right\},$$

where J^μ is defined as in (3.2). Standard arguments from the calculus of variations yield that for each $\mu \in \mathcal{P}_{m_0}(\Gamma_C)$ and $(x,v) \in \Xi^{\mathrm{ad}}$, $\Gamma^{\mu,\mathrm{opt}}[x,v]$ is not empty. Moreover, from Proposition 2.7, $\Gamma^{\mu,\mathrm{opt}}[x,v] \subset \Gamma_C$ for all $(x,v) \in \Theta_r$.

PROPOSITION 3.4. *Under the assumptions made on L, F, and G in section* 3.1, *and under Assumption* 3.1, *let $C = C(r,M)$ be chosen as in Proposition* 2.7.

Let a sequence of probability measures $(\mu_i)_{i\in\mathbb{N}}$, $\mu_i \in \mathcal{P}_{m_0}(\Gamma_C)$, be narrowly convergent to $\mu \in \mathcal{P}(\Gamma_C)$. Let $(x^i,v^i)_{i\in\mathbb{N}}$ be a sequence with $(x^i,v^i) \in \Theta_r$ which converges to (x,v). Consider a sequence $(\xi^i,\eta^i)_{i\in\mathbb{N}}$ such that for all $i \in \mathbb{N}$, $(\xi^i,\eta^i) \in \Gamma^{\mu_i,\mathrm{opt}}[x^i,v^i]$. If $(\xi^i,\eta^i)_{i\in\mathbb{N}}$ tends to (ξ,η) uniformly, then $(\xi,\eta) \in \Gamma^{\mu,\mathrm{opt}}[x,v]$. In other words, the multivalued map $(x,v,\mu) \mapsto \Gamma^{\mu,\mathrm{opt}}[x,v]$ has closed graph.

Proof. First, from Lemma 3.1, $\mu \in \mathcal{P}_{m_0}(\Gamma_C)$ and for all $t \in [0,T]$, $(m^{\mu_i}(t))_{i\in\mathbb{N}}$ is narrowly convergent to $m^\mu(t)$. From the continuity assumptions made on F and G and the dominated convergence theorem, we deduce that

$$\begin{aligned}\int_0^T F[m^{\mu_i}(t)](\xi^i(t),\eta^i(t))dt \quad &\to \quad \int_0^T F[m^\mu(t)](\xi(t),\eta(t))dt,\\ G[m^{\mu_i}(T)](\xi^i(T),\eta^i(T)) \to \quad &G[m^\mu(T)](\xi(T),\eta(T)).\end{aligned}$$

The last part of the proof is completely similar to the proof of Proposition 2.2. It makes use of Assumption 2.1 and Lemma 3.2. ∎

DEFINITION 3.5. *The probability measure $\mu \in \mathcal{P}_{m_0}(\Gamma)$ is a constrained MFG equilibrium associated with the initial distribution m_0 if*

$$\mathrm{supp}(\mu) \subset \bigcup_{(x,v)\in\mathrm{supp}(m_0)} \Gamma^{\mu,\mathrm{opt}}[x,v]. \tag{3.5}$$

THEOREM 3.6. *Under the assumptions made on L, F, and G at the beginning of section* 3.1 *and Assumption* 3.1, *let $C = C(r,M)$ be chosen as in Proposition* 2.7. *There exists a constrained MFG equilibrium $\mu \in \mathcal{P}_{m_0}(\Gamma_C)$; see Definition* 3.5. *Moreover, $t \mapsto e_t\sharp\mu \in C^{1/2}([0,T];\mathcal{P}(K_C))$ (K_C is defined as in* (2.22) *and $\mathcal{P}(K_C)$ is endowed with the Kantorovitch–Rubinstein distance).*

Proof. The proof follows that of Cannarsa and Capuani in [11]. Define the multivalued map E from $\mathcal{P}_{m_0}(\Gamma_C)$ to $\mathcal{P}_{m_0}(\Gamma_C)$ as follows: for any $\mu \in \mathcal{P}_{m_0}(\Gamma_C)$,

$$E(\mu) = \left\{\hat{\mu} \in \mathcal{P}_{m_0}(\Gamma_C) \ : \ \mathrm{supp}(\hat{\mu}_{(x,v)}) \subset \Gamma^{\mu,\mathrm{opt}}[x,v] \text{ for } m_0\text{-almost all } (x,v) \in \Xi\right\},$$

where $(\hat{\mu}_{(x,v)})_{(x,v)\in\Xi}$ is the m_0-almost everywhere uniquely defined Borel measurable family of probability measures which disintegrates $\hat{\mu}$; see the lines after Theorem 3.3. Then the measure $\mu \in \mathcal{P}_{m_0}(\Gamma_C)$ is a constrained MFG equilibrium if and only if $\mu \in E(\mu)$. This leads us to apply the Kakutani fixed point theorem to the multivalued map E; see [5, 17]. Several steps are needed in order to check that the assumptions of the Kakutani theorem are satisfied. First, we recall that $\mathcal{P}_{m_0}(\Gamma_C)$ is compact.

Step 1. For any $\mu \in \mathcal{P}_{m_0}(\Gamma_C)$, $E(\mu)$ is a nonempty convex set.

First, we have already seen that $\Gamma^{\mu,\mathrm{opt}}[x,v] \neq \emptyset$ and that the map $(x,v) \mapsto \Gamma^{\mu,\mathrm{opt}}[x,v]$ has closed graph. Therefore, from [7], $(x,v) \mapsto \Gamma^{\mu,\mathrm{opt}}[x,v]$ has a Borel measurable selection $(x,v) \mapsto (\xi^{\mu}_{(x,v)}, \eta^{\mu}_{(x,v)})$. The measure $\hat{\mu}$ defined by

$$\hat{\mu}(B) = \int_{\Theta_r} \delta_{(\xi^{\mu}_{(x,v)},\eta^{\mu}_{(x,v)})}(B)dm_0(x,v) \quad \text{for all Borel subsets } B \text{ of } \Gamma_C$$

belongs to $E(\mu)$; indeed, the total mass of $\hat{\mu}$ is one because m_0 is supported in Θ_r and $C = C(r,M)$ as in Proposition 2.7 so $E(\mu)$ is nonempty.

Second, take μ^1, μ^2 in $E(\mu)$ and $\lambda \in [0,1]$. We wish to prove that $\lambda\mu_1 + (1-\lambda)\mu_2 \in E(\mu)$. It is clear that $\lambda\mu^1 + (1-\lambda)\mu^2$ belongs to $\mathcal{P}_{m_0}(\Gamma_C)$. On the other hand, since μ^1 belongs to $E(\mu)$, there exist an m_0-almost everywhere uniquely defined Borel measurable family $(\hat{\mu}^1_{(x,v)})_{(x,v)\in\Theta_r}$ of probability measures which disintegrates μ^1 and a subset A^1 of Θ_r such that $m_0(A^1) = 0$ and $\mathrm{supp}(\mu^1_{(x,v)}) \subset \Gamma^{\mu,\mathrm{opt}}[x,v]$ for all $(x,v) \in \Theta_r \setminus A^1$. Similarly, μ^2 can be disintegrated into an m_0-almost everywhere uniquely defined Borel measurable family $(\hat{\mu}^2_{(x,v)})_{(x,v)\in\Theta_r}$ of probability measures, and there exists a subset A^2 of Θ_r such that $m_0(A^2) = 0$ and $\mathrm{supp}(\mu^2_{(x,v)}) \subset \Gamma^{\mu,\mathrm{opt}}[x,v]$ for all $(x,v) \in \Theta_r \setminus A^2$. Therefore, $\lambda\mu^1 + (1-\lambda)\mu^2$ can be disintegrated as follows: for any Borel function f defined on Γ_C,

$$\begin{aligned}
&\int_{\Gamma_C} f(\xi,\eta)d\left(\lambda\mu^1 + (1-\lambda)\,\mu^2\right)(\xi,\eta) \\
&= \int_{\Theta_r}\left(\int_{\Gamma_C} f(\xi,\eta)d\left(\lambda\mu^1_{(x,v)} + (1-\lambda)\mu^2_{(x,v)}\right)(\xi,\eta)\right)dm_0(x,v), \\
&\mathrm{supp}\left(\lambda\mu^1_{(x,v)} + (1-\lambda)\mu^2_{(x,v)}\right) \subset \Gamma^{\mu,\mathrm{opt}}[x,v] \qquad \forall (x,v) \in \Theta_r \setminus (A^1 \cup A^2),
\end{aligned}$$

and $m_0(A^1 \cup A^2) = 0$. Hence, $\lambda\mu^1 + (1-\lambda)\mu^2 \in E(\mu)$, so $E(\mu)$ is convex.

Step 2. The multivalued map E has closed graph.

Consider a sequence $(\mu^i)_{i\in\mathbb{N}}$, $\mu^i \in \mathcal{P}_{m_0}(\Gamma_C)$, narrowly convergent to $\mu \in \mathcal{P}_{m_0}(\Gamma_C)$. Let a sequence $(\hat{\mu}^i)_{i\in\mathbb{N}}$, $\hat{\mu}^i \in E(\mu^i)$, be narrowly convergent to $\hat{\mu} \in \mathcal{P}_{m_0}(\Gamma_C)$. We claim that $\hat{\mu} \in E(\mu)$.

First, there exists an m_0-almost everywhere uniquely defined Borel measurable family of probability measures $(\hat{\mu}_{(x,v)})_{(x,v)}$ on Γ_C such that (3.3) and (3.4) hold for $\hat{\mu}$ and $\hat{\mu}_{(x,v)}$. In particular, there exists a subset A of Θ_r with $m_0(A) = 0$ such that for $(x,v) \in \Theta_r \setminus A$, $\hat{\mu}_{(x,v)}(\Gamma_C \setminus e_0^{-1}(x,v)) = 0$.

Take $(x,v) \in \Theta_r \setminus A$ and $(\hat{\xi}, \hat{\eta}) \in \mathrm{supp}(\hat{\mu}_{(x,v)})$.

The Kuratowski convergence theorem applied to $(\hat{\mu}^i)_i$, $\hat{\mu}$ (see [10]) implies that there exists a sequence $(\hat{\xi}^i, \hat{\eta}^i)_{i\in\mathbb{N}}$, $(\hat{\xi}^i, \hat{\eta}^i) \in \mathrm{supp}(\hat{\mu}^i)$, which converges to $(\hat{\xi}, \hat{\eta})$

uniformly in $[0,T]$. Set $(x^i,v^i) = (\hat{\xi}^i(0), \hat{\eta}^i(0)) \in \Theta_r$. Since $\hat{\mu}^i \in E(\mu^i)$, it holds that $(\hat{\xi}^i, \hat{\eta}^i) \in \Gamma^{\mu^i,\mathrm{opt}}[x^i,v^i]$. From Proposition 3.4, we see that $(\hat{\xi}, \hat{\eta}) \in \Gamma^{\mu,\mathrm{opt}}[x,v]$. Since (x,v) is any point in $\Theta_r \setminus A$, this implies that $\hat{\mu} \in E(\mu)$.

All the assumptions of the Kakutani theorem are satisfied; hence, there exists $\mu \in \mathcal{P}_{m_0}(\Gamma_C)$ such that $\mu \in E(\mu)$. This achieves the proof. ∎

Definition 3.7. *A pair (u,m), where u is a measurable function defined on $\Xi \times [0,T]$ and $m \in C^0([0,T]; \mathcal{P}(\Xi))$, is called a* mild solution of the MFG *if there exists a constrained MFG equilibrium μ for m_0 (see Definition 3.5) such that*

(i) $m(t) = e_t \sharp \mu$;

(ii) $\forall (x,v) \in \Xi^{\mathrm{ad}}$, *$u(x,v,t)$ is given by*

$$u(x,v,t) = \inf_{(\xi,\eta,\alpha)\in\Gamma[x,v,t]} \left(\begin{array}{l} \displaystyle\int_t^T \left(F[m(s)](\xi(s),\eta(s)) + L(\xi(s),\eta(s),s) + \frac{1}{2}|\alpha(s)|^2 \right) ds \\ + G[m(T)](\xi(T),\eta(T)) \end{array} \right)$$

where $\Gamma[x,v,t]$ is the set of admissible trajectories starting from (x,v) at $s=t$.

Remark 3.3. It is tempting to say that a mild solution (u,m) is a very weak solution of a boundary value problem related to a system of PDEs posed in $\Omega \times [0,T]$, of the form (1.1) with $H(x,v,p_v)$ as in (1.2). However, to do so, we should also write boundary conditions on $\partial\Omega \times (0,T)$, which is tricky because u blows up on some part of the boundary.

A corollary of Theorem 3.6 is as follows.

Corollary 3.8. *Under the assumptions of Theorem 3.6, there exists a mild solution (u,m). Moreover, $m \in C^{\frac{1}{2}}([0,T]; \mathcal{P}(K_C))$.*

Remark 3.4. Under classical monotonicity assumptions for F and G (see, e.g., [11]), the mild solution is unique.

3.3. Nonquadratic running costs. It is possible to generalize the results of sections 2 and 3 to costs of the form

$$J(\xi,\eta,\alpha) = \int_0^T \left(\ell(\xi(s),\eta(s),s) + \frac{1}{p}|\alpha|^p(s) \right) ds + g(\xi(T),\eta(T)), \tag{3.6}$$

where $1 < p$ for dynamics given by (2.1) and staying in Ξ.

For brevity, we restrict ourselves to the closed graph result, whose proof is completely similar to that of Proposition 2.2. The generalization of Theorem 3.6 is then possible.

Proposition 3.9. *Consider a closed subset Θ of Ξ^{ad}. Assume that for any sequence $(x^i,v^i)_{i\in\mathbb{N}}$ such that for all $i \in \mathbb{N}$, $(x^i,v^i) \in \Theta$ and $\lim_{i\to+\infty}(x^i,v^i) = (x,v) \in \Theta$, the following holds: if $x \in \partial\Omega$, then*

$$((v^i \cdot \nabla d(x^i))_+)^{2p-1} = o\left(\left| d(x^i) \right|^{p-1} \right);$$

then the graph of the multivalued map $\Gamma^{\mathrm{opt}} : \Theta \rightrightarrows \Gamma$, $(x,v) \mapsto \Gamma^{\mathrm{opt}}[x,v]$ is closed in the sense given in Proposition 2.2.

3.4. The case when Ω is a convex polygonal region of $\mathbb{R}^2$. Let Ω be a bounded and convex domain of $\mathbb{R}^2$ with a polygonal boundary $\partial\Omega$. For $x \in \overline{\Omega}$, the tangent cone to Ω at x is defined by

$$T_\Omega(x) = \left\{ v \in \mathbb{R}^2 : \ x + tv \in \overline{\Omega}, \text{ for } t > 0 \text{ small enough} \right\}.$$

Note that $T_\Omega(x) = \mathbb{R}^2$ if $x \in \Omega$. A vector $v \in \mathbb{R}^2$ points outward Ω at $x \in \partial\Omega$ if $v \notin T_\Omega(x)$.

Let $(\nu_i)_{0\le i<N}$ be the vertices of $\partial\Omega$, labeled in such a way that $\partial\Omega = \bigcup_{i=0}^{N-1} \gamma_i$, where $\gamma_i = [\nu_i, \nu_{i+1}]$ and $\nu_N = \nu_0$. We may assume that three successive vertices are not aligned. We are going to use the notation (ν_i, ν_{i+1}) for the open straight line segment between ν_i and ν_{i+1}. For $i \in \{0, \dots, N-1\}$, let n_i be the unitary normal vector to γ_i pointing outward Ω. It is easy to see that $T_\Omega(\nu_i) = \{x \in \mathbb{R}^2 \ : \ n_i \cdot x \le 0 \text{ and } n_{i-1} \cdot x \le 0\}$, setting $n_{-1} = n_{N-1}$. Since Ω is convex, $\overline{\Omega}$ coincides locally near ν_i with $\nu_i + T_\Omega(\nu_i)$.

The optimal control problem is set exactly as in section 2: it consists of minimizing $J(\xi, \eta, \eta')$ given by (2.2) on the dynamics given by (2.1) and staying in $\Xi = \overline{\Omega} \times \mathbb{R}^2$. We set $\Xi^{\text{ad}} = \{(x, v) : x \in \overline{\Omega},\ v \in T_\Omega(x)\}$.

For brevity, we focus hereafter on the closed graph properties of Γ^{opt} and refer the reader to the extended version of the present paper (see the preprint [4]) for the analysis of relaxed mean field equilibria.

3.4.1. Closed graph properties. The closed graph result given in Proposition 3.10 below is similar to that contained in Proposition 2.2, but special conditions are needed near the vertices of $\partial\Omega$.

PROPOSITION 3.10. *Consider a closed subset Θ of Ξ^{ad}. Assume that for any sequence $(x^i, v^i)_{i\in\mathbb{N}}$ such that for all $i \in \mathbb{N}$, $(x^i, v^i) \in \Theta$ and $\lim_{i\to+\infty}(x^i, v^i) = (x, v) \in \Theta$, the following hold:*

1. *If $x \in (\nu_j, \nu_{j+1})$ for some $j \in \{0, \dots, N-1\}$ (recall that $\nu_N = \nu_0$), then $(v^i \cdot n_j)_+^3 = o((x - x^i) \cdot n_j)$;*
2. *if $x = \nu_j$ for some $j \in \{0, \dots, N-1\}$ and $v \neq 0$, then $(v^i \cdot n_k)_+^3 = o((x - x^i) \cdot n_k)$ for $k = j-1, j$, recalling that $n_{-1} = n_{N-1}$;*
3. *if $x = \nu_j$ for some $j \in \{0, \dots, N-1\}$ and $v = 0$, then $(v^i \cdot n_k)_+ (|x - x^i|^{\frac{2}{3}} + |v^i|^2) = o((x - x^i) \cdot n_k)$ for $k = j-1, j$.*

Then the graph of the multivalued map $\Gamma^{\text{opt}} :\ \Theta \rightrightarrows \Gamma$, $(x, v) \mapsto \Gamma^{\text{opt}}[x, v]$ is closed in the sense given in Proposition 2.2.

Proof. For brevity, we skip the proof, which is given in the extended version of the present paper (see [4]); it essentially consists of proving the counterpart of Lemma 2.3. ∎

4. One-dimensional problems: Refined results. In dimension one and for a running cost quadratic in α, it is possible to obtain refined results under a slightly stronger assumption on the running cost, namely that it does not favor the trajectories which exit the domain. In particular, the closed graph property can be proved to hold on the whole set Ξ^{ad}, and concerning MFGs, no assumptions are needed on the support of m_0 by contrast with Theorem 3.6.

4.1. Optimal control problem in an interval: A closed graph property. In this section, we set $\Omega = (-1, 0)$ and $\Xi = [-1, 0] \times \mathbb{R}$. The optimal control problem consists of minimizing $J(\xi, \eta, \eta')$ given by (2.2) on the dynamics given by (2.1) and staying in Ξ.

The definition of Ξ^{ad} then becomes

$$\Xi^{\text{ad}} = \Xi \setminus \Big(\{0\} \times (0, +\infty) \cup \{-1\} \times (-\infty, 0)\Big).$$

We make the following assumptions.

ASSUMPTION 4.1. *The running cost* $\ell : \Xi \times [0, T] \to \mathbb{R}$ *is a continuous function, bounded from below. The terminal cost* $g : \Xi \to \mathbb{R}$ *is also assumed to be continuous and bounded from below. Set* $M = \|g_-\|_{L^\infty(\Xi)} + \|\ell_-\|_{L^\infty(\Xi\times[0,T])}$.

ASSUMPTION 4.2. *For all* $t \in [0, T]$ *and* $v > 0$,

$$\ell(0, v, t) \geq \ell(0, 0, t) \qquad \text{and} \qquad \ell(-1, -v, t) \geq \ell(-1, 0, t).$$

An interpretation of Assumption 4.2 is that the running cost ℓ penalizes (or at least does not favor) the trajectories that exit Ξ^{ad}. In that respect, Assumption 4.2 is rather natural.

For $(x, v) \in \Xi$, let Γ, $\Gamma[x, v]$, and $\Gamma^{\mathrm{opt}}[x, v]$ be defined as follows:

$$\Gamma = \left\{ (\xi, \eta) \in C^1([0, T]; \mathbb{R}) \times AC([0, T]; \mathbb{R}) \ : \ \left| \begin{array}{ll} \xi'(s) = \eta(s) & \forall s \in [0, T], \\ (\xi(s), \eta(s)) \in \Xi & \forall s \in [0, T] \end{array} \right. \right\},$$

$$\Gamma[x, v] = \{(\xi, \eta) \in \Gamma : \xi(0) = x,\ \eta(0) = v\},$$

$$\Gamma^{\mathrm{opt}}[x, v] = \mathrm{argmin}_{(\xi,\eta)\in\Gamma[x,v]} J(\xi, \eta, \eta').$$

THEOREM 4.1. *Under Assumptions* 4.1 *and* 4.2, *the graph of the multivalued map* $\Gamma^{\mathrm{opt}} : \Xi^{\mathrm{ad}} \rightrightarrows \Gamma$, $(x, v) \mapsto \Gamma^{\mathrm{opt}}[x, v]$, *is closed in the following sense: consider a sequence* $(x^i, v^i)_{i\in\mathbb{N}}$, $(x^i, v^i) \in \Xi^{\mathrm{ad}}$, *such that* $\lim_{i\to\infty}(x^i, v^i) = (x, v) \in \Xi^{\mathrm{ad}}$. *Consider a sequence* $(\xi^i, \eta^i)_{i\in\mathbb{N}}$ *such that for all* $i \in \mathbb{N}$, $(\xi^i, \eta^i) \in \Gamma^{\mathrm{opt}}[x^i, v^i]$. *If* (ξ^i, η^i) *tends to* (ξ, η) *uniformly, then* $(\xi, \eta) \in \Gamma^{\mathrm{opt}}[x, v]$.

Remark 4.1. Note that, by contrast with Proposition 2.2, Theorem 4.1 holds for Γ^{opt} and not only its restriction to a subset Θ of Ξ^{ad} satisfying suitable conditions. Hence, Theorem 4.1 is more general. On the other hand, it requires an additional assumption, namely Assumption 4.2.

Note also that the result stated in Theorem 4.1, namely the closed graph property of the multivalued map Γ^{opt}, is obtained despite the fact that the value function of the optimal control problem is not continuous and not locally bounded on Ξ^{ad}. This may seem surprising at first glance. Besides, the fact that the value function is singular at some points of Ξ^{ad} will be an important difficulty in the proofs.

The proof of Theorem 4.1 relies on several lemmas.

LEMMA 4.2. *Consider* $(x, v) \in \Xi^{\mathrm{ad}}$, $(\xi, \eta) \in \Gamma[x, v]$ *such that* $\eta \in W^{1,2}(0, T; \mathbb{R})$ *and a sequence* $(x^i, v^i)_{i\in\mathbb{N}}$ *such that for all* $i \in \mathbb{N}$, $(x^i, v^i) \in \Xi^{\mathrm{ad}}$ *and* $(x^i, v^i) \to (x, v)$ *as* $i \to \infty$.

If one of the following assumptions is satisfied,

1. $x \in \Omega$,
2. $x = 0$, $v \leq 0$, *and for all integer* i, $v^i \leq 0$,
3. $(x, v) = (0, 0)$, $v^i > 0$ *for all integer* i, *and* $\lim_{i\to\infty} \frac{(v^i)^3}{|x^i|} = 0$,
4. $x = -1$, $v \geq 0$, *and for all integer* i, $v^i \geq 0$,
5. $(x, v) = (-1, 0)$, $v^i < 0$ *for all integer* i, *and* $\lim_{i\to\infty} \frac{|v^i|^3}{|x^i+1|} = 0$,

then there exists a sequence $(\xi^i, \eta^i)_{i\in\mathbb{N}}$ *such that for all* $i \in \mathbb{N}$, $(\xi^i, \eta^i) \in \Gamma[x^i, v^i]$, $\eta^i \in W^{1,2}(0, T; \mathbb{R})$, *and* (ξ^i, η^i) *tends to* (ξ, η) *in* $W^{2,2}(0, T; \mathbb{R}) \times W^{1,2}(0, T; \mathbb{R})$, *and hence uniformly in* $[0, T]$.

Proof. Lemma 4.2 is the counterpart of Lemma 2.3. The proof is quite similar, so we skip it for brevity. ∎

COROLLARY 4.3. *Consider $(x, v) \in \Xi^{\mathrm{ad}}$ and a sequence $(x^i, v^i)_{i\in\mathbb{N}}$ such that for all $i \in \mathbb{N}$, $(x^i, v^i) \in \Xi^{\mathrm{ad}}$ and $(x^i, v^i) \to (x, v)$ as $i \to \infty$. Suppose that Assumption 4.1 and one of the five conditions in Lemma 4.2 are satisfied. Let a sequence $(\xi^i, \eta^i)_{i\in\mathbb{N}}$ be such that for all $i \in \mathbb{N}$, $(\xi^i, \eta^i) \in \Gamma^{\mathrm{opt}}[x^i, v^i]$. If (ξ^i, η^i) tends to (ξ, η) uniformly in $[0, T]$, then $\eta \in W^{1,2}(0, T; \mathbb{R})$ and $(\xi, \eta) \in \Gamma^{\mathrm{opt}}[x, v]$.*

Proof. Corollary 4.3 is the counterpart of Lemma 2.5. The proof is identical. ☐

Consider $(x, v) \in \Xi^{\mathrm{ad}}$ and a sequence $(x^i, v^i)_{i\in\mathbb{N}}$ such that for all $i \in \mathbb{N}$, $(x^i, v^i) \in \Xi^{\mathrm{ad}}$ and $(x^i, v^i) \to (x, v)$ as $i \to \infty$. Because it is always possible to extract subsequences, we can say that the only cases that have not yet been addressed in Lemma 4.2 are the following:

$$\begin{cases} (x, v) = (0, 0), \quad v^i > 0, \\ \text{and there exists a constant } C > 0 \text{ s.t. for all } i \in \mathbb{N},\ \frac{(v^i)^3}{|x^i|} \geq C, \end{cases} \tag{4.1}$$

or

$$\begin{cases} (x, v) = (-1, 0), \quad v^i < 0, \\ \text{and there exists a constant } C > 0 \text{ s.t. for all } i \in \mathbb{N},\ \frac{|v^i|^3}{|x^i+1|} \geq C. \end{cases} \tag{4.2}$$

Since the two cases are symmetrical, we may concentrate on (4.1).

It is clear that (4.1) implies that $|x^i|/v^i \to 0$ as $i \to +\infty$ because $v^i \to 0$. In the case when (4.1) is satisfied, we need two technical lemmas which provide a lower bound for the cost $\int_0^T |\frac{d\eta^i}{dt}(s)|^2 ds$ of the admissible trajectories starting at (x^i, v^i).

LEMMA 4.4. *Consider $(x, v) \in \Xi^{\mathrm{ad}}$ such that $x < 0$, $v > 0$, $3|x|/v < T$, and $\theta \in (0, T)$. Given a real number $w \in [0, |x|/T]$, set*

$$K_{\theta,w} = \left\{ \eta \in W^{1,2}(0, \theta; \mathbb{R}) : \left| \begin{array}{l} \eta(0) = v, \quad \eta(\theta) = w, \\ \eta(s) \geq w \ \forall s \in [0, \theta], \\ x + \displaystyle\int_0^\theta \eta(s) ds \leq 0 \end{array} \right. \right\}. \tag{4.3}$$

The quantity

$$I(\theta, w) = \inf_{\eta \in K_{\theta,w}} \frac{1}{2} \int_0^\theta \left| \frac{d\eta}{dt}(s) \right|^2 ds \tag{4.4}$$

is achieved by a function $\eta = \eta_{\theta,w}$ and is given by

$$I(\theta, w) = \begin{cases} \dfrac{1}{2} \dfrac{(w - v)^2}{\theta} & \text{if} \quad \theta \in \left[0, \frac{2|x|}{v+w}\right], \\ 6\dfrac{x^2}{\theta^3} + 6\dfrac{x(v + w)}{\theta^2} + 2\dfrac{v^2 + vw + w^2}{\theta} & \text{if} \quad \theta \in \left[\frac{2|x|}{v+w}, \frac{3|x|}{v+2w}\right], \\ \dfrac{2}{9} \dfrac{(v - w)^3}{|x| - w\theta} & \text{if} \quad \theta \in \left[\frac{3|x|}{v+2w}, T\right). \end{cases} \tag{4.5}$$

Remark 4.2. The partition of the interval $[0, T]$ in (4.5) is justified by the assumptions of Lemma 4.4. Indeed,

- $3|x|/v < T$ and $w \geq 0$ imply that $3|x|/(v + 2w) < T$;
- $2|x|/(v + w) < 3|x|/(v + 2w)$ because $0 \leq w \leq |x|/T < v/3$.

Note also that if $|x|/v \to 0$, then $3|x|/(v + 2w) \sim 3|x|/v \ll T$.

Proof. Problem (4.4) is the minimization of a strictly convex and continuous functional under linear and continuous constraints, and the set $K_{\theta,w}$ is nonempty, as we shall see below, convex, and closed. Hence there exists a unique minimizer, named η again. The Euler–Lagrange necessary conditions read as follows: there exists a real number $\mu \geq 0$ such that η is a weak solution of the linear complementarity problem (variational inequality)

$$\left\{\begin{array}{rcll} -\eta'' & \geq & -\mu & \text{in } (0,\theta), \\ \eta & \geq & w & \text{in } (0,\theta), \\ (-\eta''+\mu)(\eta-w) & = & 0 & \text{in } (0,\theta), \\ x+\displaystyle\int_0^\theta \eta(s)ds & \leq & 0, & \\ \mu & \geq & 0, & \\ \mu\left(x+\displaystyle\int_0^\theta \eta(s)ds\right) & = & 0, & \\ \eta(0) & = & v, & \\ \eta(\theta) & = & w. & \end{array}\right. \tag{4.6}$$

The solution of (4.6) can be written explicitly. Skipping the details, it has the following form:

1. If $\theta \geq 3|x|/(v+2w)$, then

$$\left\{\begin{array}{rcll} \eta(t) & = & v-\mu\tau t+\dfrac{\mu}{2}t^2, & 0\leq t\leq\tau, \\ \eta(t) & = & w, & \tau<t\leq\theta, \end{array}\right.$$

with

$$\tau = -3\frac{x+w\theta}{v-w} \quad \text{and} \quad \mu = \frac{2(v-w)^3}{9(x+w\theta)^2}.$$

Note that $-3\frac{x+w\theta}{v-w} \leq \theta$ because $\theta \geq \frac{3|x|}{v+2w}$. Note also that $x+\int_0^\theta \eta(s)ds = 0$. We see that $I(\theta,w) = \frac{\mu^2}{2}\int_0^\tau(-\tau+t)^2dt = \frac{\mu^2\tau^3}{6} = \frac{2}{9}\frac{(v-w)^3}{|x|-w\theta}$; we have obtained the third line in (4.5).

2. If $2|x|/(v+w) \leq \theta \leq 3|x|/(v+2w)$, then for all $t\in[0,\theta]$,

$$\eta(t) = v+kt+\frac{\mu}{2}t^2,$$

with

$$k = -\frac{6x+(4v+2w)\theta}{\theta^2}, \qquad \text{and} \qquad \mu = 6\frac{2x+(v+w)\theta}{\theta^3}.$$

Note that $x+\int_0^\theta \eta(s)ds = 0$. Easy algebra leads to $I(\theta,w) = 6\frac{x^2}{\theta^3}+6\frac{x(v+w)}{\theta^2}+2\frac{v^2+vw+w^2}{\theta}$; we have obtained the second line in (4.5).

3. If $\theta \leq 2|x|/(v+w)$, then for all $t\in[0,\theta]$,

$$\eta(t) = v-(v-w)\frac{t}{\theta}.$$

Then, $I(\theta,w) = \frac{1}{2}\frac{(w-v)^2}{\theta}$; we have obtained the first line in (4.5). Note that if $\theta < \frac{2|x|}{v+w}$, then $x+\int_0^\theta \eta(s)ds < 0$. ∎

LEMMA 4.5. *Consider a sequence $(x^i, v^i)_{i\in\mathbb{N}}$ such that $x^i < 0$, $v^i > 0$ for all $i \in \mathbb{N}$, and $v^i \to 0$, $|x^i|/v^i \to 0$ as $i \to +\infty$. Call $I^i(\theta, w)$ the quantity given by (4.4) for $v = v^i$, $x = x^i$, and $w \in \big[0, |x^i|/T\big]$. Then*

$$\inf\left\{I^i(\theta, w),\ \theta \in (0, T)\right\} = \frac{2}{9}\frac{(v^i)^3}{|x^i|} + o(1), \tag{4.7}$$

where $o(1)$ is a quantity that tends to 0 as i tends to infinity (which is in fact of the order of $(v^i)^2$ or smaller).

Proof. Recall that $I^i(\theta, w)$ is given by (4.5). It is easy to see that $\theta \mapsto I^i(\theta, w)$ is decreasing on $\big(0, 2|x^i|/(v^i + w)\big]$ and increasing on $\big[3|x^i|/(v^i + 2w), T\big]$.

In $\big[2|x^i|/(v^i + w), 3|x^i|/(v^i + 2w)\big]$, $I^i(\theta, w) = P(1/\theta)$, where P is the third order polynomial

$$P(z) = 6(x^i)^2 z^3 + 6x^i(v^i + w)z^2 + 2((v^i)^2 + v^i w + w^2)z.$$

The roots of the second order polynomial $P'(z) = 18(x^i)^2z^2 + 12x^i(v^i + w)z + 2((v^i)^2 + v^i w + w^2)$ are $\frac{v^i+w\pm\sqrt{v^iw}}{3|x^i|}$. Hence, $\theta \mapsto I^i(\theta, w)$ is decreasing in $[\frac{2|x^i|}{v^i+w}, \frac{3|x^i|}{v^i+w+\sqrt{v^iw}}]$ and increasing in $[\frac{3|x^i|}{v^i+w+\sqrt{v^iw}}, \frac{3|x^i|}{v^i+2w}]$. Therefore, the minimizer of $\theta \mapsto I^i(\theta, w)$ on $[0, T)$ is $\theta = \frac{3|x^i|}{v^i+w+\sqrt{v^iw}}$, and the minimal value is

$$\begin{aligned}
&P\left(\frac{v^i + w + \sqrt{v^i w}}{3|x^i|}\right)\\
&= \frac{2(v^i)^3}{9|x^i|}\left(1 + \sqrt{\frac{w}{v^i}} + \frac{w}{v^i}\right)^3 - \frac{2(v^i)^3}{3|x^i|}\left(1 + \sqrt{\frac{w}{v^i}} + \frac{w}{v^i}\right)^2 + \frac{2(v^i)^3}{3|x^i|}\left(1 + \sqrt{\frac{w}{v^i}} + \frac{w}{v^i}\right)\\
&\quad + O\left((v^i)^2\frac{w}{|x^i|}\right)\\
&= \frac{2(v^i)^3}{9|x^i|} + O\left((v^i)^2\frac{w}{|x^i|}\right).
\end{aligned}$$

∎

The next lemma is the counterpart of Lemma 4.2 when (4.1) holds. By contrast with the situations considered so far, Assumption 4.2 is used.

LEMMA 4.6. *Under Assumptions 4.1 and 4.2, consider a sequence $(x^i, v^i)_{i\in\mathbb{N}}$ tending to $(x, v) = (0, 0)$ as $i \to \infty$, and which satisfies (4.1). Let a sequence $(\xi^i, \eta^i)_{i\in\mathbb{N}}$ be such that for all $i \in \mathbb{N}$, $(\xi^i, \eta^i) \in \Gamma^{\rm opt}[x^i, v^i]$. If (ξ^i, η^i) tends to (ξ, η) uniformly in $[0, T]$, then $\eta \in W^{1,2}(0, T; \mathbb{R})$ and $(\xi, \eta) \in \Gamma^{\rm opt}[x, v]$.*

Proof. The proof is more difficult than that of Lemma 4.2 because we will see that in general, the sequence $u(x^i, v^i)$ does not converge to $u(0, 0)$ as $i \to \infty$, and that $\int_0^T (\frac{d\eta^i}{dt}(s))^2 ds$ may tend to $+\infty$.

Step 1. We start by building a particular competitor for the optimal control problem at (x^i, v^i). It will be used in Steps 2 and 3 below. Let us set $\widetilde{t}_i = 3|x^i|/v^i$ (observe that $\lim_{i\to\infty} \widetilde{t}_i = 0$ since $v^i \to 0$ and $(v^i)^3/|x_i| \geq C > 0$). As in the proof of Lemma 4.4 with $w = 0$, we construct a pair of continuous functions $(\widetilde{\xi}^i, \widetilde{\eta}^i)$ defined on $[0, \widetilde{t}_i]$ such that $-1 \leq \widetilde{\xi}^i \leq 0$ and $\frac{d\widetilde{\xi}^i}{dt} = \widetilde{\eta}^i$, and

1. $(\widetilde{\xi}^i(0), \widetilde{\eta}^i(0)) = (x^i, v^i)$;
2. $\widetilde{\eta}^i(\widetilde{t}_i) = 0$;

3. $\frac{1}{2}\int_0^{\widetilde{t}_i}\left(\frac{d\widetilde{\eta}^i}{dt}(s)\right)^2 ds \sim \frac{2}{9}\frac{(v^i)^3}{|x^i|}$

(we have also used Lemma 4.5 with $w = 0$ and Remark 4.2). Observe that $x^i \leq \widetilde{\xi}^i(\widetilde{t}_i) \leq 0$; hence $\lim_{i\to+\infty}\widetilde{\xi}^i(\widetilde{t}_i) = 0$. Then, with the same arguments as in Lemma 2.3, it is possible to extend continuously $(\widetilde{\xi}^i, \widetilde{\eta}^i)$ to $[0,T]$ in such a way that

1. $(\widetilde{\xi}^i, \widetilde{\eta}^i) \in \Gamma[x^i, v^i]$;
2. $\lim_{i\to\infty}\int_{\widetilde{t}_i}^T |\frac{d\widetilde{\eta}^i}{dt}(s) - \alpha(s-\widetilde{t}_i)|^2 ds = 0$, where α is an optimal control law for trajectories with initial values $(0,0)$.

Combining all the information above, we obtain that

$$J\left(\widetilde{\xi}^i, \widetilde{\eta}^i, \frac{d\widetilde{\eta}^i}{dt}\right) = \frac{2}{9}\frac{(v^i)^3}{|x^i|} + u(0,0) + o(1). \tag{4.8}$$

Step 2. Since $(\xi^i, \eta^i) \in \Gamma^{\mathrm{opt}}[x^i, v^i]$, we know that for all $t \in [0,T]$, $\xi^i(t) = x^i + \int_0^t \eta^i(s)ds \leq 0$. We claim that there exists $t_i \in (0,T]$ such that $\eta^i(t_i) \leq -x^i/T$. Indeed, if this were not the case, then $\xi^i(T)$ would be larger than $x^i - T(x^i/T) = 0$, which is not true. Since η^i is continuous, we may define θ_i as the minimal time t such that $\eta^i(t) \leq -x^i/T$, and we see that $\eta^i(\theta_i) = -x^i/T$.

Step 2 consists of proving that

$$\lim_{i\to\infty}\theta_i = 0. \tag{4.9}$$

Suppose by contradiction that there exists $\delta > 0$ such that $\theta_i \geq \delta$. We may apply Lemma 4.4 with $w = |x^i|/T$. Since $v^i \to 0$ and $(v^i)^3/|x^i| \geq C > 0$, we see that $|x^i|/v^i \to 0$, and then that $\lim_{i\to\infty}\frac{3|x^i|}{v^i+2|x^i|/T} = 0$. Hence, for i large enough, $\theta_i \geq \delta > \frac{3|x^i|}{v^i+2|x^i|/T}$, and the third line of (4.5) yields

$$\frac{1}{2}\int_0^T\left(\frac{d\eta^i}{dt}(s)\right)^2 ds \geq \frac{2T}{9(T-\delta)}\frac{(\frac{x^i}{T}+v^i)^3}{|x^i|} = \frac{2T}{9(T-\delta)}\frac{(v^i)^3}{|x^i|} + o(1), \tag{4.10}$$

where $o(1)$ is a quantity that tends to zero as $i \to \infty$ (in fact like $(v^i)^2$).

Note that $\eta^i \geq -x^i/T \geq 0$ in $[0,\theta_i]$ yields that $\xi^i \geq x^i$ in $[0,\theta_i]$. Therefore,

$$\lim_{i\to\infty}\|\xi^i\|_{L^\infty(0,\theta_i)} = 0. \tag{4.11}$$

Let us construct an admissible trajectory $(\widehat{\xi}^i, \widehat{\eta}^i)$ starting from $(x,v) = (0,0)$ as follows:

1. For $s \in [0,\theta_i]$, $\widehat{\xi}^i(s) = Q_{\theta_i,0,0,\xi^i(\theta_i),\eta^i(\theta_i)}(s)$ and $\widehat{\eta}^i(s) = Q'_{\theta_i,0,0,\xi^i(\theta_i),\eta^i(\theta_i)}(s)$; see Definition 2.4;
2. $(\widehat{\xi}^i(s), \widehat{\eta}^i(s)) = (\xi^i(s), \eta^i(s))$ for $s \in [\theta_i, T]$.

It is easy to check that, if $s \leq \theta_i$, then

$$\widehat{\xi}^i(s) = \left(\theta_i\eta^i(\theta_i) - 2\xi^i(\theta_i)\right)\frac{s^3}{\theta_i^3} - \left(\theta_i\eta^i(\theta_i) - 3\xi^i(\theta_i)\right)\frac{s^2}{\theta_i^2}, \tag{4.12}$$

$$\widehat{\eta}^i(s) = 3\left(\theta_i\eta^i(\theta_i) - 2\xi^i(\theta_i)\right)\frac{s^2}{\theta_i^3} - 2\left(\theta_i\eta^i(\theta_i) - 3\xi^i(\theta_i)\right)\frac{s}{\theta_i^2}, \tag{4.13}$$

$$\frac{d\widehat{\eta}^i}{dt}(s) = 6\left(\theta_i\eta^i(\theta_i) - 2\xi^i(\theta_i)\right)\frac{s}{\theta_i^3} - 2\left(\theta_i\eta^i(\theta_i) - 3\xi^i(\theta_i)\right)\frac{1}{\theta_i^2}. \tag{4.14}$$

Since $\eta^i(\theta_i) = -x^i/T > 0$ and $\xi^i(\theta_i) \le 0$, we see that $\big(\theta_i\eta^i(\theta_i) - 2\xi^i(\theta_i)\big) \ge 0$ and that $\big(\theta_i\eta^i(\theta_i) - 3\xi^i(\theta_i)\big) \ge 0$. Hence for $s \in [0,\theta_i]$,

$$\widehat{\xi^i}(s) = \big(\theta_i\eta^i(\theta_i) - 2\xi^i(\theta_i)\big)\left(\frac{s^3}{\theta_i^3} - \frac{s^2}{\theta_i^2}\right) + \xi^i(\theta_i)\frac{s^2}{\theta_i^2} \le 0$$

as the sum of two nonpositive terms. Therefore, $(\widehat{\xi^i}, \widehat{\eta}^i) \in \Gamma[0,0]$. On the other hand, using (4.11) and the fact that $\theta_i\eta^i(\theta_i) = \theta_i|x^i|/T$, (4.12), and (4.13), we see that

$$\lim_{i\to+\infty}\left(\|\widehat{\xi^i}\|_{L^\infty(0,\theta_i)} + \|\widehat{\eta}^i)\|_{L^\infty(0,\theta_i)}\right) = 0. \tag{4.15}$$

Moreover, since $\theta_i \ge \delta > 0$, it is easy to check that

$$\lim_{i\to+\infty}\int_0^{\theta_i}\left(\frac{d\widehat{\eta}^i}{dt}(s)\right)^2 ds = 0. \tag{4.16}$$

Since $(\widehat{\xi^i}, \widehat{\eta}^i) \in \Gamma[0,0]$,

$$\begin{aligned}
u(0,0) \le\ & J(\widehat{\xi^i}, \widehat{\eta}^i, 0)\\
=\ & \int_0^T \left(\ell(\widehat{\xi^i}(s), \widehat{\eta}^i(s), s) + \frac{1}{2}\left(\frac{d\widehat{\eta}^i}{dt}(s)\right)^2\right) ds + g(\widehat{\xi^i}(T), \widehat{\eta}^i(T))\\
=\ & \int_0^{\theta_i} \left(\ell(\widehat{\xi^i}(s), \widehat{\eta}^i(s), s) + \frac{1}{2}\left(\frac{d\widehat{\eta}^i}{dt}(s)\right)^2\right) ds\\
& + \int_{\theta_i}^T \left(\ell(\xi^i(s), \eta^i(s), s) + \frac{1}{2}\left(\frac{d\eta^i}{dt}(s)\right)^2\right) ds + g(\xi^i(T), \eta^i(T))\\
=\ & u(x^i, v^i) + \int_0^{\theta_i} \left(\ell(\widehat{\xi^i}(s), \widehat{\eta}^i(s), s) + \frac{1}{2}\left(\frac{d\widehat{\eta}^i}{dt}(s)\right)^2\right) ds\\
& - \int_0^{\theta_i} \left(\ell(\xi^i(s), \eta^i(s), s) + \frac{1}{2}\left(\frac{d\eta^i}{dt}(s)\right)^2\right) ds.
\end{aligned}$$

Therefore,

$$\begin{aligned}
u(x^i, v^i) \ge\ & u(0,0) + \frac{1}{2}\int_0^{\theta_i}\left(\frac{d\eta^i}{dt}(s)\right)^2 ds\\
& + \int_0^{\theta_i}\Big(\ell(\xi^i(s), \eta^i(s), s) - \ell(\widehat{\xi^i}(s), \widehat{\eta}^i(s), s)\Big) ds - \frac{1}{2}\int_0^{\theta_i}\left(\frac{d\widehat{\eta}^i}{dt}(s)\right)^2 ds.
\end{aligned} \tag{4.17}$$

Let us address the terms in the right-hand side of (4.17) separately.

Thanks to the continuity of ℓ, (4.11), (4.15), and Assumption 4.2, we see that

$$\begin{aligned}
& \liminf_{i\to\infty}\int_0^{\theta_i}\Big(\ell(\xi^i(s), \eta^i(s), s) - \ell(\widehat{\xi^i}(s), \widehat{\eta}^i(s), s)\Big) ds\\
=\ & \liminf_{i\to\infty}\int_0^{\theta_i}\big(\ell(0, \eta^i(s), s) - \ell(0,0,s)\big) ds \ge 0.
\end{aligned} \tag{4.18}$$

Combining (4.18), (4.16), and (4.10), we obtain that

$$u(x^i, v^i) \ge \frac{2T}{9(T-\delta)}\frac{(v^i)^3}{|x^i|} + u(0,0) + o(1), \tag{4.19}$$

where $o(1)$ is a quantity that tends to 0 as $i \to \infty$.

But for $(\widetilde{\xi}^i, \widetilde{\eta}^i)$ constructed in Step 1, $J(\widetilde{\xi}^i, \widetilde{\eta}^i, \frac{d\widetilde{\eta}^i}{dt}) \geq u(x^i, v^i)$. This fact and (4.8) lead to a contradiction with (4.19). We have proved (4.9).

Step 3. Since $\lim_{i\to\infty} \theta_i = 0$ and (ξ^i, η^i) converges uniformly to (ξ, η), we see that

$$\int_0^{\theta_i} \ell(\xi^i(s), \eta^i(s), s)ds = o(1).$$

Hence

(4.20)
$$\begin{aligned} u(x^i, v^i) =& \frac{1}{2}\int_0^{\theta_i} \left(\frac{d\eta^i}{dt}(s)\right)^2 ds \\ &+ \int_{\theta_i}^T \ell(\xi^i(s), \eta^i(s), s)ds + \frac{1}{2}\int_{\theta_i}^T \left(\frac{d\eta^i}{dt}(s)\right)^2 ds + g(\xi^i(T), \eta^i(T)) + o(1). \end{aligned}$$

On the other hand, we have seen above that (4.8) implies that

$$u(x^i, v^i) \leq \frac{2}{9}\frac{(v^i)^3}{|x^i|} + u(0,0) + o(1). \tag{4.21}$$

From Lemma 4.5, we know that

$$\frac{1}{2}\int_0^{\theta_i} \left(\frac{d\eta^i}{dt}(s)\right)^2 ds \geq \frac{2}{9}\frac{(v^i)^3}{|x^i|} - o(1). \tag{4.22}$$

Combining (4.20), (4.21), and (4.22) yields that

(4.23)
$$\int_{\theta_i}^T \ell(\xi^i(s), \eta^i(s), s)ds + \frac{1}{2}\int_{\theta_i}^T \left(\frac{d\eta^i}{dt}(s)\right)^2 ds + g(\xi^i(T), \eta^i(T)) \leq u(0,0) + o(1).$$

Since (ξ^i, η^i) converges uniformly to (ξ, η), (4.23) implies that $(\mathbb{1}_{(\theta_i,T)}\frac{d\eta^i}{dt})_{i\in\mathbb{N}}$ is a bounded sequence in $L^2(0,T)$. Hence there exists $\phi \in L^2(0,T)$ such that, after the extraction of subsequence, $\mathbb{1}_{(\theta_i,T)}\frac{d\eta^i}{dt} \rightharpoonup \phi$ in $L^2(0,T)$ weakly. By testing with compactly supported functions in $(0,T)$, it is clear that $\phi = \frac{d\eta}{dt}$. Hence, the whole sequence $(\mathbb{1}_{(\theta_i,T)}\frac{d\eta^i}{dt})_{i\in\mathbb{N}}$ converges in $L^2(0,T)$ weakly to $\frac{d\eta}{dt} \in L^2(0,T)$. Moreover, the weak convergence in $L^2(0,T)$ implies that

$$\int_0^T \left(\frac{d\eta}{dt}(s)\right)^2 ds \leq \liminf_{i\to+\infty} \int_0^T \left(\mathbb{1}_{(\theta_i,T)}\frac{d\eta^i}{dt}(s)\right)^2 ds.$$

This and (4.23) imply that

$$\int_0^T \ell(\xi(s), \eta(s), s)ds + \frac{1}{2}\int_0^T \left(\frac{d\eta}{dt}(s)\right)^2 ds + g(\xi(T), \eta(T)) \leq u(0,0).$$

Hence, $(\xi, \eta) \in \Gamma^{\mathrm{opt}}[0,0]$, and the above inequality is in fact an identity. The proof is achieved. ∎

Proof of Theorem 4.1. Consider $(x,v)\in\Xi^{\rm ad}$ and a sequence $(x^i,v^i)_{i\in\mathbb{N}}$ such that for all $i\in\mathbb{N}$, $(x^i,v^i)\in\Xi^{\rm ad}$ and $(x^i,v^i)\to(x,v)$ as $i\to\infty$. Consider a sequence $(\xi^i,\eta^i)_{i\in\mathbb{N}}$ such that for all $i\in\mathbb{N}$, $(\xi^i,\eta^i)\in\Gamma^{\rm opt}[x^i,v^i]$ and that (ξ^i,η^i) tends to (ξ,η) uniformly. Possibly after the extraction of a subsequence, we can always assume that either one of the five conditions in Lemma 4.2 or one of the two symmetrical conditions (4.1)–(4.2) holds. Then the conclusion follows from Corollary 4.3 in the former case or from Lemma 4.6 in the latter case. ∎

Remark 4.3. For costs of the form (3.6) with $1<p\neq 2$, it is not possible to reproduce the explicit calculations of Lemmas 4.4 and 4.5, which are crucial steps for Lemma 4.6 and finally for Theorem 4.1.

4.2. Bounds related to optimal trajectories.

PROPOSITION 4.7. *For positive numbers r and C, let us set*

$$\Theta_r=\left\{(x,v)\in\Xi: -r(x+1)\le v^3\le r|x|\right\},\tag{4.24}$$

$$K_C=\{(x,v)\in\Xi:\ |v|\le C\},\tag{4.25}$$

$$\Gamma_C=\left\{(\xi,\eta)\in\Gamma:\ \left|\ \begin{array}{l}(\xi(t),\eta(t))\in K_C\quad\forall t\in[0,T],\\ \left\|\frac{d\eta}{dt}\right\|_{L^2(0,T;\mathbb{R})}\le C\end{array}\right.\right\}.\tag{4.26}$$

Under Assumption 4.1*, for all $r>0$, there exists a positive number $C=C(r,M)$ (M is defined in Assumption* 4.1*) such that if $(x,v)\in\Theta_r$, then $\Gamma^{\rm opt}[x,v]\subset\Gamma_C$. Moreover, as $r\to+\infty$, $C(r,M)=O(\sqrt{r})$.*

Proof. A possible proof consists of building a suitable map j from Θ_r to Γ. We distinguish different cases:
Case 1: $0\le v\le -3x/T$. Let $j(x,v)=(\widetilde{\xi},\widetilde{\eta})\in\Gamma[x,v]$ be defined by

$$\begin{cases}\widetilde{\eta}(t)\ =\ v\left(1-\frac{3t}{2T}\right)\quad\text{and}\quad \widetilde{\xi}(t)=\ x+v\left(t-\frac{3t^2}{4T}\right)\quad \text{if } 0\le t\le\frac{2T}{3},\\ \widetilde{\eta}(t)\ =0\quad\text{and}\quad \widetilde{\xi}(t)=\ x+\frac{vT}{3}\quad\text{if } \frac{2T}{3}\le t\le T.\end{cases}$$

It is easy to check that there exists a constant $\widetilde{C}=\widetilde{C}(r,M)$ such that

$$\|\widetilde{\eta}\|_{L^\infty(0,T;\mathbb{R})}\le\widetilde{C};\qquad \left\|\frac{d\widetilde{\eta}}{dt}\right\|_{L^2(0,T;\mathbb{R})}\le\widetilde{C}.\tag{4.27}$$

Case 2: $-3x/T<v\le(r|x|)^{\frac13}$. In this case, we choose $j(x,v)=(\widetilde{\xi},\widetilde{\eta})\in\Gamma[x,v]$ where $\widetilde{\xi}'=\widetilde{\eta}$ and $\widetilde{\eta}$ is the solution of the linear complementarity problem (4.6) with $\theta=T$ and $w=0$. Here again $(\widetilde{\xi},\widetilde{\eta})$ satisfies (4.27) for some constant $\widetilde{C}=\widetilde{C}(r,M)$. From Lemma 4.4, we see that as $r\to+\infty$, $\widetilde{C}=O(\sqrt{r})$.
Case 3: $-3(1+x)/T\le v\le 0$. The situation is symmetric to Case 1, and $j(x,v)$ is given by the same formula.
Case 4: $-r^{\frac13}(x+1)^{\frac13}\le v<-3(1+x)/T$. The situation is symmetric to Case 2, and $j(x,v)$ is constructed in the symmetric way as in Case 2.
Then, using $j(x,v)$ as a competitor for the optimal control problem leads to the desired result with a constant C that depends only on r and M and that can always be taken larger than $\widetilde{C}$.

Note that j is piecewise continuous from Θ_r to Γ. Note also that the construction of j is independent of ℓ and g. ∎

Remark 4.4. Note that the sets Θ_r form an increasing family of compact subsets of Ξ^{ad} and that

$$\bigcup_{r\geq 0} \Theta_r = \Xi^{\mathrm{ad}}. \tag{4.28}$$

4.3. Mean field games with state constraints. In the example considered here, we take $\Xi = [-1,0]\times\mathbb{R}$. Let $\mathcal{P}(\Xi)$ be the set of probability measures on Ξ.

Let $F, G : \mathcal{P}(\Xi) \to C_b^0(\Xi;\mathbb{R})$ be bounded and continuous maps (the continuity is with respect to the narrow convergence in $\mathcal{P}(\Xi)$) and L be a continuous and bounded from below function defined on $\Xi \times [0,T]$. Set

$$M = \max\left(\sup_{(x,v,t)\in\Xi\times[0,T]} L_-(x,v,t) + \sup_{m\in\mathcal{P}(\Xi)} \|F[m]\|_{L^\infty(\Xi)}, \sup_{m\in\mathcal{P}(\Xi)} \|G[m]\|_{L^\infty(\Xi)} \right).$$

ASSUMPTION 4.3. *We assume that for all $t \in [0,T]$, $m \in \mathcal{P}(\Xi)$, and $v \geq 0$, $L(0,v,t) + F[m](0,v) \geq L(0,0,t) + F[m](0,0)$ and $L(-1,-v,t) + F[m](-1,-v) \geq L(-1,0,t) + F[m](-1,0)$.*

Using notation similar to that in section 3.1, we consider the cost given by (3.2). With M in (3.1), note that Proposition 4.7 can be applied to J^μ defined in (3.2) with constants $C(r,M)$ uniform in μ.

LEMMA 4.8. *Let r be a positive number. Under the assumptions made above on L, F, and G (including Assumption* 4.3*), let $C = C(r,M)$ be the constant appearing in Proposition* 4.7*. For any probability measure m_0 on Ξ supported in Θ_r defined in* (4.24)*, there exists a constrained MFG equilibrium associated with the initial distribution m_0, i.e., a probability measure $\mu \in \mathcal{P}_{m_0}(\Gamma_C)$ such that* (3.5) *holds.*

Proof. The proof is similar to that of Theorem 3.6. We skip it. ∎

Remark 4.5. In Lemma 4.8, we still suppose that the support of m_0 is contained in some compact set, namely Θ_r; this assumption is used to obtain an estimate on the cost and guarantees that the optimal trajectories with initial conditions in the support of m_0 belong to a compact subset of Γ, which allows us to use Kakutani's fixed point theorem. However, compared to Theorem 3.6, the restrictions made in Lemma 4.8 on the support of m_0 are weaker; yet, the latter requires the additional Assumption 4.3 on the running cost.

In Theorem 4.9 below, we get rid of the assumptions on the support of m_0 made in Lemma 4.8; we just require that $m_0(\Xi \setminus \Xi^{\mathrm{ad}}) = 0$. By contrast with Lemma 4.8, the strategy for proving Theorem 4.9 will not rely directly on Kakutani's theorem; it will consist of approximating the measure m_0 by a sequence of probability measures $(m_{0,n})_{n>0}$, such that $m_{0,n}$ is supported in Θ_n. Hence, it will be possible to apply Lemma 4.8 and obtain a sequence of relaxed mean field equilibrium μ_n related to $m_{0,n}$, which will be proved to be tight in $\mathcal{P}(\Gamma)$. The relaxed mean field equilibrium μ related to m_0 will then be obtained as a cluster point of the sequence μ_n for the narrow convergence in $\mathcal{P}(\Gamma)$. The assumption on m_0 will imply that $\mu \in \mathcal{P}_{m_0}(\Gamma)$. The fact that $\mathrm{supp}(\mu) \subset \bigcup_{(x,v)\in\mathrm{supp}(m_0)} \Gamma^{\mu,\mathrm{opt}}[x,v]$ will be obtained by applying a key argument which is a modification of Lemma 4.6.

THEOREM 4.9. *Let m_0 be a probability measure on Ξ such that*

$$m_0(\Xi\setminus\Xi^{\mathrm{ad}}) = 0. \tag{4.29}$$

Under the assumptions made above on L, F, and G (including Assumption 4.3), there exists a constrained MFG equilibrium associated with the initial distribution m_0, i.e., a probability measure $\mu \in \mathcal{P}_{m_0}(\Gamma)$ such that (3.5) holds.

Proof. From (4.28) and (4.29), there exists $n_0 > 0$ such that $m_0(\Theta_n) > 0$ for $n > n_0$. For $n > n_0$, we set $m_{0,n} = \frac{1}{m_0(\Theta_n)} m_0|_{\Theta_n}$. With a slight abuse of notation, let $m_{0,n}$ also denote the probability on Ξ obtained by extending $m_{0,n}$ by 0 outside Θ_n, i.e., $m_{0,n}(B) = \frac{1}{m_0(\Theta_n)} m_0(B \cap \Theta_n)$, for any measurable subset B of Ξ. It is clear that $m_{0,n}$ converges narrowly to m_0, from (4.28) and (4.29). Let $\mu_n \in \mathcal{P}_{m_{0,n}}\left(\Gamma_{C(n,M)}\right)$ be a constrained MFG equilibrium associated with the initial distribution $m_{0,n}$, the existence of which comes from Lemma 4.8. With a similar abuse of notation as above, let μ_n also denote the probability on Γ obtained by extending μ_n by 0 outside $\Gamma_{C(n,M)}$.

We claim that $\{\mu_n, n > n_0\}$ is tight in $\mathcal{P}(\Gamma)$, i.e., that for each $\epsilon > 0$, there exists a compact $K_\epsilon \subset \Gamma$ such that

$$\mu_n(\Gamma \setminus K_\epsilon) < \epsilon \qquad \text{for each } n > n_0. \tag{4.30}$$

From the increasing character of the sequence Θ_n, (4.28), and (4.29), we observe that for each $\epsilon > 0$, there exists $n_1 > 0$ such that $m_0(\Theta_{n_1}) > 1 - \epsilon$. Let us prove (4.30) with $K_\epsilon = \Gamma_{C(n_1,M)}$.

Since for all $n > n_0$, $\mu_n \in \mathcal{P}_{m_{0,n}}(\Gamma)$ is an MFG equilibrium, we see that for all measurable $B \subset \Xi$,

$$m_{0,n}(B) = \mu_n \left\{(\xi,\eta) \in \operatorname{supp}(\mu_n) \ : \ (\xi(0),\eta(0)) \in B\right\} \le \mu_n \left(\bigcup_{(x,v)\in B} \Gamma^{\mathrm{opt},\mu_n}[x,v] \right).$$

Taking $B = \Theta_{n_1}$ and using Proposition 4.7, we see that

$$m_{0,n}\left(\Theta_{n_1}\right) \le \mu_n \left(\bigcup_{(x,v)\in \Theta_{n_1}} \Gamma^{\mathrm{opt},\mu_n}[x,v] \right) \le \mu_n\left(\Gamma_{C(n_1,M)}\right)$$

(note that the constant $C(n_1, M)$ does not depend on μ_n).

On the other hand,

$$\begin{array}{rcll} m_{0,n}\left(\Theta_{n_1}\right) & \ge & m_0(\Theta_{n_1}) > 1-\epsilon & \text{if } n > n_1, \\ m_{0,n}\left(\Theta_{n_1}\right) & = & 1 & \text{if } n_0 < n \le n_1. \end{array}$$

In both cases, $\mu_n\left(\Gamma_{C(n_1,M)}\right) \ge 1 - \epsilon$ and therefore $\mu_n\left(\Gamma \setminus \Gamma_{C(n_1,M)}\right) \le \epsilon$, and the claim is proved.

Thanks to the Prokhorov theorem, possibly after the extraction of subsequence that we still name μ_n, we deduce that there exists $\mu \in \mathcal{P}(\Gamma)$ such that μ_n converges narrowly to μ.

We claim that μ is an MFG equilibrium related to m_0. We already know that $\mu \in \mathcal{P}(\Gamma)$. There remains to prove that

- $\mu \in \mathcal{P}_{m_0}(\Gamma)$, i.e., that $e_0 \sharp \mu = m_0$;
- μ satisfies (3.5).

The fact that $e_0 \sharp \mu = m_0$ stems from Lemma 3.1 and from the fact that $m_{0,n}$ narrowly converges to m_0.

In order to prove (3.5), we recall that from Kuratowski's theorem (see [6]),

$$\operatorname{supp}(\mu) \subset \liminf_{n\to\infty} \operatorname{supp}(\mu_n),$$

which means that for all $(\xi, \eta) \in \operatorname{supp}(\mu)$, there exists a sequence $(\xi_n, \eta_n) \in \operatorname{supp}(\mu_n)$, such that $(\xi_n, \eta_n) \to (\xi, \eta)$ uniformly. From Definition 3.5, $(\xi_n(0), \eta_n(0)) \in \operatorname{supp}(m_{0,n}) \subset \operatorname{supp}(m_0)$. Hence, $(\xi(0), \eta(0)) \in \operatorname{supp}(m_0)$. We also know that $(\xi(0), \eta(0)) \in \Xi^{\mathrm{ad}}$ because $(\xi, \eta) \in \Gamma$. Therefore, setting $(x_n, v_n) = (\xi_n(0), \eta_n(0))$ and $(x, v) = (\xi(0), \eta(0))$, we see that $(\xi_n, \eta_n) \in \Gamma^{\mathrm{opt},\mu_n}[x_n, v_n]$ and that $\lim_{n\to\infty}(x_n, v_n) = (x, v) \in \operatorname{supp}(m_0) \cap \Xi^{\mathrm{ad}}$. Applying Proposition 4.10 below, which is a generalization of Theorem 4.1, we may pass to the limit and conclude that $(\xi, \eta) \in \Gamma^{\mathrm{opt},\mu}[x, v]$; this achieves the proof. ∎

PROPOSITION 4.10. *Under the assumptions made above on L, F, and G (including Assumption 4.3), consider a sequence $(\mu^i)_{i\in\mathbb{N}}$, $\mu^i \in \mathcal{P}(\Gamma)$, such that μ^i converges narrowly to $\mu \in \mathcal{P}(\Gamma)$. Consider a sequence $(\xi^i, \eta^i)_{i\in\mathbb{N}}$, $(\xi^i, \eta^i) \in \Gamma$, such that*

1. *$(\xi^i, \eta^i) \in \Gamma^{\mathrm{opt},\mu^i}[x^i, v^i]$, where $(x^i, v^i) = (\xi^i(0), \eta^i(0))$;*
2. *(ξ^i, η^i) tends to $(\xi, \eta) \in \Gamma$ uniformly. Note that this implies that $(x, v) \in \Xi^{\mathrm{ad}}$ and $(\xi, \eta) \in \Gamma[x, v]$, where $(x, v) = \lim_{i\to\infty}(x^i, v^i)$.*

Then $(\xi, \eta) \in \Gamma^{\mathrm{opt},\mu}[x, v]$.

Proof. We skip the proof because it follows the same lines as that of Theorem 4.1 (see section 4.1). In particular, it includes an adaptation of Lemma 4.6. The necessary modifications are obvious. ∎

Remark 4.6. Finally, note that all the results contained in section 4 can be generalized in a straightforward manner to the case when Ω is a half-space (for example $\Omega = \{x \in \mathbb{R}^n : x_1 < 0\}$) or a strip (for example $\Omega = \{x \in \mathbb{R}^n : -1 < x_1 < 0\}$).

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