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RESEARCH ARTICLE

Global Strichartz estimates for the Dirac equation on symmetric spaces

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Abstract
In this paper, we study global-in-time, weighted Strichartz estimates for the Dirac equation on warped product spaces in dimension $n \geq 3$. In particular, we prove estimates for the dynamics restricted to eigenspaces of the Dirac operator on the compact spin manifolds defining the ambient manifold under some explicit sufficient condition on the metric and estimates with loss of angular derivatives for general initial data in the setting of spherically symmetric and asymptotically flat manifolds.

Contents

1. The radial Dirac equation on symmetric manifolds

In [10, 11], the second and third authors have started the study of the dynamics of the Dirac equation on curved spaces, the natural setting being a four-dimensional manifold $(\mathcal{M}, g)$ with signature $\{+, -, -, -\}$ that decouples space and time: namely, the metric g is assumed to take the form

$$g_{\mu\nu} = \begin{cases} 1 & \text{if } \mu = \nu = 0 \\ 0 & \text{if } \mu\nu = 0 \text{ and } \mu \neq \nu \\ -h_{\mu\nu}(\overrightarrow{x}) & \text{otherwise.} \end{cases} \tag{1.1}$$

We recall that the Cauchy problem for the Dirac equation in this setting can be written as

$$\begin{cases} i\partial_t u - \mathcal{D}u - m\beta u = 0, \\ u(0, x) = u_0(x), \end{cases} \tag{1.2}$$

where β is a square, complex matrix such that β^2 is the identity and $\mathcal{D}$ is the Dirac operator. By construction, the operator $\mathcal{D}$ satisfies the following property

$$\mathcal{D}^2 = -\Delta_h + \frac{1}{4}\mathcal{R}_h, \tag{1.3}$$

where Δ_h is the Laplace-Beltrami operator for Dirac bispinors: that is, $\Delta_h = D^j D_j$, where D_j is the covariant derivative for Dirac bispinors that we properly define later, and $\mathcal{R}_h$ is the scalar curvature associated to the spatial metric h.

In the case when $(\mathcal{M}, g)$ is the Minkoswki space, the literature of dispersive estimates and related problems for solutions to equation (1.2) is quite extensive. To the best of our knowledge, the first Strichartz estimates for the Dirac equation and its application to the well-posedness of some nonlinear models appeared in [25]. Some refinements of the results were later obtained in [30], which includes the extension to any space dimension, and in [29], in which the endpoint Strichartz estimate with angular regularity is proved. The study of the well-posedness of the cubic nonlinear Dirac equation, which is a delicate problem as it forces one to work at the level of the endpoint Strichartz estimates, has been only recently solved in [3] (see also [4] and [6]). Also, a lot of effort has been devoted to the study of the validity of dispersive estimates in the presence of potential perturbations: we mention at least the papers [18, 19, 20, 7, 9, 23, 24] for 'small' electric and magnetic potentials and [13, 8] for scaling critical perturbations (i.e., Dirac-Coulomb potential and Aharonov-Bohm magnetic field).

In [10], the authors exploited the classical Morawetz multiplier technique in order to obtain local smoothing (or weak dispersive) estimates for the solutions to equation (1.2) in the setting of asymptotically flat and (some) warped products manifolds. As it is often the case when dealing with equations on manifolds, it is not possible to rely on the classical Duhamel argument in order to obtain Strichartz estimates for the flow because, even in the asymptotically flat case, the perturbative term cannot be regarded as a zero-order perturbation of the flat dynamics.

In the subsequent paper [11], the authors considered three-dimensional spherically symmetric settings: that is, manifolds $(\mathcal{M}, g)$ defined by $\mathcal{M} = \mathbb{R}_t \times \Sigma$, where now $\Sigma = \mathbb{R}_r^+ \times \mathbb{S}^2_{\theta,\phi}$ is equipped with the Riemannian metric

$$dr^2 + \varphi(r)^2 d\omega^2_{\mathbb{S}^2}, \tag{1.4}$$

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where $d\omega^2_{\mathbb{S}^2} = (d\theta^2 + \sin^2\theta d\phi^2)$ is the Euclidean metric on the 2D sphere $\mathbb{S}^2$. Notice that taking $\varphi(r) = r$ reduces Σ to the standard 3D Euclidean space and therefore $\mathcal{M}$ to the standard Minkowski space. Within this setting, in [11], local-in-time, weighted Strichartz estimates for the Dirac dynamics were proved, under some quite general (and natural) assumptions on the function φ, which will be discussed in forthcoming Subsection 1.1: the main strategy consisted of exploiting the spherical symmetry of the space in order to separate variables and to reduce the problem to a 'sum' of much easier radial equations that could be regarded, after introducing weighted bispinors, as Dirac equations on the flat space perturbed with potentials, for which several results are available. Nevertheless, global-in-time Strichartz estimates turned out to be out of reach, the main problem being the lack of existence of dispersive estimates for the Dirac equation with scaling critical potentials in the Euclidean setting.

The purpose of this manuscript is to complement the results of [11], investigating the validity of weighted, global-in-time Strichartz estimates in the more general setting of *warped products* in any space dimension $n \geq 3$. We consider manifolds $(\mathcal{M}, g)$ defined by $\mathcal{M} = \mathbb{R}_t \times \Sigma$ with $\Sigma = \mathbb{R}^+_r \times \mathbb{K}^{n-1}$, where $\mathbb{K}^{n-1}$ is now a generic $n-1$-dimensional compact and Riemannian spin manifold, and Σ is a Riemannian manifold equipped with the Riemannian metric

$$dr^2 + \varphi(r)^2 d\omega^2_{\mathbb{K}^{n-1}}. \tag{1.5}$$

Here, φ is a map from $\mathbb{R}_+$ to itself, and $d\omega^2_{\mathbb{K}^{n-1}}$ is the Riemannian metric on $\mathbb{K}^{n-1}$. Of course, this case includes the spherically symmetric one when choosing $\mathbb{K}^{n-1} = \mathbb{S}^{n-1}$, and thus this paper can be regarded in fact as an extension of [11]. On the other hand, as we will see, the assumptions on the admissible functions φ will be much stronger: this is because, as mentioned, we cannot directly rely on the theory of the flat Dirac equation with potentials, but we need to square the equation at the radial level in order to reduce to a system of Klein-Gordon equations and then, via Kato smoothing arguments, rely on the existing theory for this dynamics. Let us give some more details on the strategy. Recall that Dirac bispinors in dimension $n+1$ are maps from $\mathcal{M}$ to $\mathbb{C}^M$, with M an integer bigger than $2^{\lfloor \frac{n+1}{2} \rfloor}$ (in Section 2, we will review the construction of the Dirac operator on curved spaces). Due to equation (1.3), it is often useful to exploit the identity

$$(i\partial_t u - \mathcal{D}u - m\beta u)(i\partial_t u + \mathcal{D}u + m\beta u) = \big(-\partial_t^2 + \Delta_h - \frac{1}{4}\mathcal{R}_h u - m^2\big)\mathbb{I}_M u \tag{1.6}$$

where $\mathbb{I}_M$ denotes the M-dimensional identity matrix, so that if u solves equation (1.2) then u also solves system

$$\begin{cases} -\partial_t^2 u + \Delta_h u - \frac{1}{4}\mathcal{R}_h u - m^2 u = 0. \\ u(0,x) = u_0(x), \\ \partial_t u(0,x) = (\mathcal{D} + m)u_0(x) \end{cases} \tag{1.7}$$

which shows the close relationship between the Dirac and wave/Klein-Gordon flows. This is sometimes referred to as the 'squaring trick' and turns out to be extremely useful, at least in the flat case, to reduce the study of the algebraically rich dynamics of the Dirac equation to the much easier one of the Klein-Gordon one. Let us stress that in this non-flat setting, the operator Δ_h is the bispinorial Laplacian, and not the scalar one; consequently, it is not straightforward to adapt the results known for the wave/Klein-Gordon equation on manifolds to deal with the Dirac one. Nevertheless, by using separation of variables, in some symmetric cases it is possible to bring this strategy at a 'radial' level: we intend to walk this path here. However, this plan is not going to work in the 'general' setting of assumptions **(A0)** (the assumptions taken on the metric in [11]; see equation (1.11) below), and it will force us to impose stronger ones.

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Before stating our main results, let us recall some basic (and classical) facts about the decomposition of the Dirac operator. On three-dimensional spherically symmetric manifolds – that is, if the metric enjoys the structure in equation (1.4) – the Dirac equation can be written in the convenient form

$$i\partial_t\psi = H_\varphi\psi,$$

where

$$H_\varphi = \begin{pmatrix} m & -i\sigma_3\left(\partial_r + \frac{\varphi'}{\varphi}\right) + \frac{1}{\varphi}\mathcal{D}_{\mathbb{S}^2} \\ i\sigma_3\left(\partial_r + \frac{\varphi'}{\varphi}\right) + \frac{1}{\varphi}\mathcal{D}_{\mathbb{S}^2} & -m \end{pmatrix}.$$

Here, σ_3 is one of the Pauli matrices

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and $\mathcal{D}_{\mathbb{S}^2}$ is the Dirac operator on the sphere $\mathbb{S}^2$ (see [36] Section 4.6 and [22]). It is well-known that the operator $\mathcal{D}_{\mathbb{S}^2}$ can be diagonalized (see [14]); as a consequence, one has the following natural decomposition

$$L^2(\mathbb{R}^3)^4 \cong \bigoplus_{\mu,j_\mu} L^2((0,+\infty),\varphi^2(r)dr)\otimes\mathcal{H}_{\mu,j_\mu} \tag{1.8}$$

where the indexes are $\mu\in\mathbb{Z}^* = \mathbb{Z}\backslash\{0\}$, $j_\mu = -|\mu|+1, -|\mu|, \dots, |\mu|$ (the μ in j_μ is a standard notation in the Physics literature and aims at recalling that the range for j_μ depends on μ), and the two-dimensional Hilbert spaces $\mathcal{H}_{\mu,j_\mu}$ are generated by two orthogonal functions $\{\Phi^+_{\mu,j_\mu}, \Phi^-_{\mu,j_\mu}\}$ that essentially are normalized spherical harmonics. The action of H_φ on the spaces $H^1(\varphi(r)^2dr)\otimes\mathrm{Vect}(\Phi^+_{\mu,j_\mu}, \Phi^-_{\mu,j_\mu})$ is given by

$$h_\mu = \begin{pmatrix} m & -\left(\partial_r + \frac{\varphi'}{\varphi}\right) + \frac{\mu}{\varphi} \\ \left(\partial_r + \frac{\varphi'}{\varphi}\right) + \frac{\mu}{\varphi} & -m \end{pmatrix}, \tag{1.9}$$

where the $\mu\in\mathbb{Z}^*$ are the eigenvalues of the angular operator $\mathcal{D}_{\mathbb{S}^2}$ (notice that we are using a slightly different but equivalent decomposition with respect to [36] and [11] that allows a much easier generalization). More in general, this decomposition holds in the setting of warped product metrics (see equation (1.5)) in dimension $n\geq 3$. Indeed, there exists a decomposition of $L^2(\mathbb{K}^{n-1})$

$$L^2(\mathbb{K}^{n-1}) = \bigoplus_{\mu,j_\mu}\mathcal{H}_{\mu,j_\mu},$$

where μ is taken over the spectrum of $\mathcal{D}_{\mathbb{K}^{n-1}}$ (which is purely discrete) and where $j_\mu\in[1,r_\mu]\cap\mathbb{N}$, where r_μ is the multiplicity of μ. On $\mathcal{H}_{\mu,j_\mu}$, the action of $\mathcal{D}_\Sigma$ can be represented by h_μ. Subsection 2.1 will be devoted to present an overview of the topic.

1.1. Admissible manifolds: discussion

Let us now briefly discuss and compare the different assumptions we will have to make about the function φ in the definition of the metric in equation (1.5) in order to prove our estimates.

Assumptions (A0). Let $(\mathcal{M}, g)$ be a Lorentzian manifold of dimension $n+1\geq 4$ defined by $\mathcal{M} = \mathbb{R}_t\times\Sigma$, with (Σ, h) a warped product: that is, a Riemannian manifold in the form $\Sigma = \mathbb{R}^+_r\times\mathbb{K}^{n-1}$, where $\mathbb{K}^{n-1}$ is an $(n-1)$-dimensional compact spin manifold and Σ is equipped with the Riemannian metric

$$dr^2 + \varphi(r)^2 d\omega^2_{\mathbb{K}^{n-1}}, \tag{1.10}$$

where $d\omega^2_{\mathbb{K}^{n-1}}$ the Riemannian metric on $\mathbb{K}^{n-1}$ and where the function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is $C^\infty(\mathbb{R}^+)$, is strictly positive on $(0, +\infty)$ and is such that

$$\varphi(0) = \varphi^{(2n)}(0) = 0, \qquad \varphi'(0) = 1, \qquad \frac{\varphi'(r)}{\varphi(r)} \in L^\infty. \tag{1.11}$$

Notice that Assumptions **(A0)**, in the case $n = 3$ and $\mathbb{K} = \mathbb{S}$, are essentially the ones we retained in [11] in order to prove local-in-time Strichartz estimates.

In order to prove global-in-time Strichartz estimates for the dynamics restricted to an eigenspace of $\mathcal{D}_{\mathbb{K}^{n-1}}$, we need to complement **(A0)** with the following:

Assumptions (A1). Let $(\mathcal{M}, g)$ satisfy assumptions **(A0)**. Assume that the operator $\mathcal{D}_{\mathbb{K}^{n-1}}$ has no eigenvalue μ with $|\mu| < \frac{1}{2}$, and let μ be in the spectrum of $\mathcal{D}_{\mathbb{K}^{n-1}}$. Let $V_\mu = \frac{\mu(\mu+\varphi')}{\varphi^2}$ and

$$\delta_\varphi(\mu) = \min(1, \inf(4r^2 V_\mu + 1), \inf(-4r^2 V_\mu - 4r^3 V'_\mu + 1)). \tag{1.12}$$

Assume that the function φ in the metric in equation (1.10) satisfies

$$\delta_\varphi(-\mu), \delta_\varphi(\mu) > 0, \quad r^2 V_\mu \in L^\infty. \tag{1.13}$$

In order to prove global-in-time Strichartz estimates for the complete flow, we need to strengthen our assumptions some more, having in mind as a main example *asymptotically flat manifolds*. We thus set the following:

Assumptions (A2). Let $(\mathcal{M}, g)$ be defined by $\mathcal{M} = \mathbb{R}_t \times \Sigma$, with (Σ, h) a spherically symmetric manifold of dimension $n \geq 3$ with the metric given by equation (1.5) with $\mathbb{K} = \mathbb{S}$. Let $\varphi \in C^\infty(\mathbb{R}^+)$ be such that $\varphi(0) = 0$, $\varphi'(0) = 1$, and for all $k \in \mathbb{N}$, $\varphi^{(2k)}(0) = 0$. We assume that there exists $\varphi_1 \in \mathcal{C}^\infty(\mathbb{R}^+)$ such that

$$\varphi : r \mapsto r(1 + \varphi_1(r))$$

with the following assumptions on φ_1:

- φ_1 is non-negative;
- $\sup_{r\geq 0}(|\varphi_1(r)| + |r\varphi_1(r)'| + |r^2\varphi_1''(r)|) \ll 1$.

Remark 1.1. The map

$$\varphi_1 = \varepsilon \frac{r^\alpha}{\langle r\rangle^\beta}$$

with $\beta \geq \alpha > 0$, with $\alpha, \beta \in \mathbb{N}$, α even, satisfies these assumptions.

Remark 1.2. Our aim is to apply this type of result to well-known Lorentzian manifolds such as black holes. It is known that there are spherically symmetric back holes, such as the Schwarzschild or Reissner-Nordström one; in these cases, the metric (outside the black hole) writes

$$ds^2 = F(r)dt^2 - F^{-1}(r)dr^2 - r^2 d\omega_{\mathbb{S}^2},$$

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where $d\omega_{\mathbb{S}^2}$ is the metric on the sphere $\mathbb{S}^2$ and F is defined as

$$F(r) = 1 - \frac{A}{r} + \frac{B}{r^2},$$

with $A, B \geq 0$ and $B = 0$ in the case of the Schwarzschild metric.

The metric couples time and space, but the Dirac equation does not, and in fact it is written (we refer to [31, Eq. (12)]) as follows

$$\left[\gamma^0 \partial_t + F\gamma^1\left(\partial_r + \frac{1}{r} + \frac{F'}{4F}\right) + \frac{F^{1/2}}{r}\mathcal{D}_{\mathbb{S}^2} + iF^{1/2}m\right]u = 0,$$

where $\mathcal{D}_{\mathbb{S}^2}$ is an operator acting only on the angular variable and can be diagonalized in the same way as we did in this paper.

By changing variables, we get an equation of the type

$$\left[\gamma^0 \partial_t + \gamma^1\left(\partial_r + \psi_1(r)\right) + \psi_2(r)\mathcal{D}_{\mathbb{S}^2} + i\psi_3(r)m\right]u = 0,$$

where ψ_j, $j = 1, 2, 3$ are functions of the radial variable such that

$$\psi_1(r), \psi_2(r) = \frac{1}{r} + \mathcal{O}(1/r^2), \quad \psi_3(r) = 1 + \mathcal{O}(1/r).$$

The behavior as $r \to \infty$ is the same as in our current case, especially if $m = 0$. The difficulty arises when one looks at the region close to the black hole, as there, indeed, the functions ψ_2 and ψ_3 are not differentiable.

Therefore, this paper has to be thought of as a first step toward the study of the dispersion of the Dirac operator in a spherically symmetric black hole metric, as we here tackle a similar case for the behavior at ∞, but we do not tackle the difficult task of looking at what happens close to the black hole.

1.2. Main results

We are now ready to state the main results. For a definition of functional spaces, we refer to Subsection 1.3.

Definition 1.3. We say that the triple (p, q, m) is admissible, either if $m = 0$

$$\frac{2}{p} + \frac{n-1}{q} = \frac{n-1}{2}, \quad p > 2, \quad q \in [2, \infty),$$

or if $m \neq 0$

$$\frac{2}{p} + \frac{n}{q} = \frac{n}{2}, \quad p > 2, \quad q \in [2, \infty).$$

The first result we prove is a global-in-time Strichartz estimate for the Dirac flow restricted to eigenspaces of the operator $\mathcal{D}_{\mathbb{K}^{n-1}}$.

Theorem 1.4. *Let $(\mathcal{M}, g)$ satisfy assumptions **(A0)** and **(A1)**. Then, for any admissible triple (p, q, m) in Definition 1.3 and any $\varepsilon > 0$, there exists a constant C depending only on $m, p, q, \varphi, \varepsilon$ (but not on μ) such that for all $v_0 \in H^{1/2}_\varphi$,*

$$\begin{aligned} &\left\|\left(\frac{\varphi(r)}{r}\right)^{\frac{(n-1)}{2}\left(1-\frac{2}{q}\right)} e^{-ith_\mu} v_0\right\|_{L^p(\mathbb{R}, W_\varphi^{1/q-1/p,q})} \\ &\qquad \leq C|\mu|^{5/p+\varepsilon}(\delta_\varphi(\mu)^{1/p+\varepsilon} + \delta_\varphi(-\mu)^{1/p+\varepsilon})\|v_0\|_{H_\varphi^{1/2}}, \end{aligned} \tag{1.14}$$

where $W^{1/q-1/p,q}_{\varphi}$ and $H^{1/2}_{\varphi}$ are Sobolev spaces on the manifold Σ for radial functions defined in Subsection 1.3.

Remark 1.5. The need for an $\varepsilon > 0$ in the estimate in equation (1.14) is connected to the non-admissibility of the endpoint triple (p, q, m), as we will briefly discuss in the proof of Corollary 3.10.

Remark 1.6. Note that assuming that the compact manifold $\mathbb{K}^{n-1}$ satisfies that the (discrete) spectrum of the Dirac operator on $\mathbb{K}^{n-1}$ is included in $(-\infty, -\frac{1}{2}] \cup [\frac{1}{2}, \infty)$ ensures that the Dirac operator on Σ with $\varphi = r$ is self-adjoint (see [15, Theorem 3.2] and references therein). The operator h_μ being isomorphic to an L^∞ perturbation of

$$\tilde{h}_\mu = \begin{pmatrix} m & -\left(\partial_r + \frac{1}{r}\right) + \frac{\mu}{r} \\ \left(\partial_r + \frac{1}{r}\right) + \frac{\mu}{r} & -m \end{pmatrix},$$

we get that h_μ is self-adjoint. This will be further commented upon in Remark 2.2.

Remark 1.7. When $\mathbb{K}^{n-1}$ is the $(n-1)$-dimensional sphere, then the manifold Σ is smooth and in fact geometrically complete, which ensures the self-adjointness of the Dirac operator. Also, in this case, by relying on the endpoint Strichartz estimate proved in [29] and on mixed Strichartz-local smoothing estimates ([9]), it is possible to recover the endpoint as well but with extra (global) derivatives, namely the estimates

$$\left\| u \left(\frac{\varphi(r)}{r} \right)^{\frac{(n-1)}{2}} \right\|_{L^2_t(I, L^\infty(\mathbb{R}_+, L^p(\mathbb{S}^{n-1})))} \leq \sqrt{p} |\mu|^{5/2} (\delta_\varphi(\mu)^{1/2} + \delta_\varphi(-\mu)^{1/2}) \|u_0\|_{H^2(\Sigma)} \tag{1.15}$$

when $m = 0, n = 3$ and

$$\begin{aligned} &\left\| \left(\frac{\varphi(r)}{r} \right)^{\frac{(n-1)}{2}\left(1-\frac{2}{q}\right)} e^{-ith_\mu} v_0 \right\|_{L^2(\mathbb{R}, W^{1/q-1/2,q}_{\varphi})} \\ &\qquad \leq C |\mu|^{5/2} (\delta_\varphi(\mu)^{1/2} + \delta_\varphi(-\mu)^{1/2}) \|v_0\|_{H^{3/2}_{\varphi}}, \end{aligned}$$

where $q = \frac{2(n-1)}{n-3}$ if $m = 0$ and $q = \frac{2n}{n-2}$ otherwise.

Remark 1.8. The dependence on the angular parameter μ in our Strichartz estimates (which can be ultimately intended as a loss of angular derivatives and is most likely not sharp) is due to the method of our proof: the action of the 'radial Dirac operator' equation (1.9) depends on the 'angular' eigenvalue μ, and as a consequence, the Strichartz estimates for the flow e^{ith_μ} will necessarily depend on μ. The additional ε-loss in the massless case is due to the lack of the endpoint Strichartz estimates in this case, as indeed, these estimates will be obtained by interpolation. We refer to [11], Section 5 for all the details.

Remark 1.9. With slight additional care, the result above could be generalized in order to include spaces with conical singularities; the study of the Dirac operator in this context, mostly from the spectral point of view, has been developed in detail in [15]. The analysis of dispersive flows on conical spaces (and on spaces with conical singularities) has seen increasing interest in recent years; we don't intend to provide a precise picture of the literature here. We mention that the present work in fact originally motivated the paper [5], in which we have analyzed the dispersive dynamics of the Klein-Gordon equation on spaces with conical singularities. In any case, we need to stress once more that it is not possible to directly adapt those results to the context of the Dirac flow, as the Laplacian operators are of a different nature (spinorial vs. scalar).

The fact that the constant on the right-hand side of the estimate in equation (1.14) is a function of μ suggests that it might be possible to prove Strichartz estimates with loss of angular derivatives: such estimates are quite classical in the context of dispersive PDEs, and the local-in-time case (in dimension

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3) has been already discussed in the predecessor of this paper ([11]). For the next theorem, we shall indeed restrict to the case $\mathbb{K}^{n-1} = \mathbb{S}^{n-1}$ in order to be able to resort to the well-established Littlewood-Paley theory on the sphere. It is in fact possible to 'sum' the Strichartz estimates in equation (1.14) in order to obtain Strichartz estimates for general initial data by requiring additional regularity in the angular variable (we postpone to Subsection 1.3 the precise definitions of the spaces $H^{a,b}(\Sigma)$).

The result is the following:

Theorem 1.10. *Let $(\mathcal{M}, g)$ satisfy assumptions **(A2)**. Let $p, q \in [2, \infty]$ and $a, b \geq 0$. Assume that (p, q, m) is admissible and $\frac{5}{pb} + \frac{1}{2a} < 1$. Then the solutions u to equation (1.2) with initial data $u_0 \in H^{a,b}(\Sigma)$ satisfy the estimates*

$$\left\| \left(\frac{\varphi(r)}{r} \right)^{\frac{(n-1)}{2}\left(1-\frac{2}{q}\right)} u \right\|_{L^p_t(\mathbb{R}, W^{1/q-1/p,q}(\Sigma))} \leq C\|u_0\|_{H^{a,b}(\Sigma)}. \tag{1.16}$$

Remark 1.11. The analogue of Theorem 1.10 could be proved in the more general case of warped products under the assumption in equation (1.13), provided one has a suitable Littlewood-Paley theory on the manifold $\mathbb{K}^{n-1}$. This might be the object of forthcoming works.

As a matter of fact, the starting point in the proof of Theorem 1.10 is showing that within the assumptions **(A2)**, the crucial condition given by **(A1)** is satisfied. In other words, Assumptions **(A2)** (asymptotically flat, spherically symmetric manifolds) provide an explicit example of 'admissible manifolds' for the validity of Theorem 1.4. In Subsection 4.2, we will thus prove the following:

Proposition 1.12. *Let $(\mathcal{M}, g)$ be defined by $\mathcal{M} = \mathbb{R}_t \times \Sigma$, with (Σ, h) a warped product with the metric given by equation (1.5). Let μ_0 be the infimum of the positive part of the spectrum of the Dirac operator on $\mathbb{K}^{n-1}$, and assume that $\mu_0 > 1/2$. If φ satisfies the assumptions in **(A2)**, where the required smallness of C is determined by μ_0, then the assumptions in equation (1.13) are fulfilled.*

Remark 1.13. We will provide more precise assumptions on φ_1 and in particular on the size of the constant C, with explicit dependence on the space dimension, at the beginning of Section 4.

Remark 1.14. It is a natural question to ask whether there exist other possible choices of the function φ that satisfy the conditions in equation (1.13). We will devote the appendix to a small discussion.

The plan of the paper is the following. In Section 2, we review the separation of variables procedure for the Dirac equation in the warped products setting and show how to reduce to the Klein-Gordon dynamics. In Section 3, we discuss the classical Kato argument to obtain the Strichartz estimates for the Klein-Gordon dynamics with potentials of critical decay. Finally, in Section 4, we show that asymptotically flat manifolds are admissible, and we prove Strichartz estimates for general initial data in the spherically symmetric setting.

1.3. Notations

We will use the standard notation L^p, $\dot{H}^s$, H^s, $W^{p,q}$ to denote, respectively, the Lebesgue and homogeneous/non homogeneous Sobolev spaces of functions from $\mathbb{R}^n$ to $\mathbb{C}^M$. We will use the same notation to denote these functional spaces on the (spatial) manifold (Σ, h), which is in our structure in equation (1.1), that is, with time and space already decoupled, by adding the dependence $L^p(\Sigma)$, $\dot{H}^s(\Sigma)$, $H^s(\Sigma)$, $W^{p,q}(\Sigma)$: for example, the norm $L^p(\Sigma)$ will be given by

$$\|f\|^p_{L^p(\Sigma)} := \int |f(x)|^p \sqrt{\det(h(x))}dx$$

and so on.

The space $L^2(\Sigma)$ is thus endowed with the usual Hilbertian structure.

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The space $\dot{H}^1(\Sigma)$, is induced by the norm

$$\|f\|^2_{\dot{H}^1(\Sigma)} := \|\sqrt{h^{ij}\langle D_i f, D_j f\rangle_{\mathbb{C}^M}}\|_{L^2(\Sigma)},$$

where the D_j are covariant derivatives for Dirac bispinors.

The space $W^{1,p}$, $p \in [1,\infty]$ is induced by

$$\|f\|_{W^{1,p}(\Sigma)} = \|\sqrt{h^{ij}\langle D_i f, D_j f\rangle_{\mathbb{C}^M}}\|_{L^p(\Sigma)} + \|f\|_{L^p(\Sigma)}.$$

The spaces $\dot{H}^s(\Sigma)$ and $W^{s,p}(\Sigma)$ with $s \in [-1,1]$ are defined by interpolation and duality.

Due to the warped product structure of the metric in equation (1.5), for a radial function $f_{rad}(|x|)$ we define

$$\|f_{rad}\|^p_{L^p_\varphi} := \int_0^{+\infty} |f_{rad}(r)|^p \varphi(r)^{n-1} dr \sim \|f_{rad}\|^p_{L^p(\Sigma)}.$$

For the Sobolev spaces, we use the compatible notations

$$\|f_{rad}\|_{\dot{H}^1_\varphi} := \|\partial_r f_{rad}\|_{L^2(\Sigma)}$$

and

$$\|f_{rad}\|_{W^{1,p}_\varphi} := \|\partial_r f_{rad}\|_{L^p(\Sigma)} + \|f_{rad}\|_{L^p(\Sigma)}.$$

We define $\dot{H}^s_\varphi$ and $W^{s,p}_\varphi$, $s \in (0,1)$, by interpolation, and $\dot{H}^s_\varphi, W^{s,p}_\varphi$, $s \in [-1,1]$, $p \in (1,\infty)$, by duality.

Note that since we are dealing with vectors in $\mathbb{C}^M$, $|f(x)|$ should be understood as

$$|f(x)| = \sqrt{\langle f(x), f(x)\rangle_{\mathbb{C}^M}}.$$

The norms in time will be denoted by L^p_t. The mixed Strichartz spaces will be standardly denoted by $L^p_t L^q(\Sigma) = L^p(I; L^q(\Sigma, \mathbb{C}^M))$.

We finally introduce the spaces $H^{a,b}$ for $a \in [-1,1]$, $b \in \mathbb{R}$ by defining the norms

$$\|f\|_{H^{a,b}(\Sigma)} = \left(\|f\|^2_{H^a(\Sigma)} + \|(-\Delta_{\mathbb{S}^{n-1}})^{b/2} f\|_{L^2(\Sigma)}\right)^{1/2}.$$

2. The setup: separation of variables and reduction to Klein-Gordon

The construction of the Dirac operator on a 4D manifold is a rather delicate task in general and requires the introduction of the so-called *vierbein*, which essentially defines some proper frames that connect the metric of the manifold $(\mathcal{M}, g)$ to the Lorentzian one η; details can be found in the predecessor of this paper, [11], and in [32]. In order to properly define those frames, also known as Cartan's formalism, one needs the hypothesis that the manifold admits a spin structure: we will take this as an assumption. The fact that admitting a spin structure is a homological property has been proved and commented upon in [26]. In fact, Σ (and $\mathcal{M}$) inherit a spin structure from the spin structure of $\mathbb{K}^{n-1}$; we will explain this in the next subsection.

In this section, we show how to exploit the separation of variables and the classical spectral theory for the Dirac equation on compact manifolds to reduce the study of the dynamics of the Dirac equation on warped products to the one of a system of radial Klein-Gordon equations. We refer the interested reader to [15, 1] for further details on various aspects we will discuss. We mention the fact that most of the geometric objects that will appear in the next pages will only have the role of allowing the definition of the Dirac operator in a curved setting, and therefore we will be quite sketchy on some of them, as it would be impossible to make the presentation self-contained. Nevertheless, we will try our best to indicate precise references to help the interested reader's comprehension. Also, we include a short (and informal) appendix at the end of the paper in which we introduce and briefly discuss the necessary tools needed for the computations developed in this section.

For a complete derivation of the Dirac equation in curved space-time, we refer to Section 5.6 in [32].

2.1. *The separation of variables*

We start by recalling that the Dirac operator on a Lorentzian manifold $(\mathcal{M}, g)$ of dimension $n+1$ admitting a spin structure[1] and with decoupled space and time writes

$$\mathcal{D} = m\gamma^0 - i\gamma^0 \underline{\gamma}^j D_j,$$

where the implicit summation on j is taken from 1 to n. In the above, $m \in \mathbb{R}$ is the mass of the electron, γ^0 is a self-adjoint matrix of size $M \times M$ with $M = 2^{\lfloor (n+1)/2 \rfloor}$ with values in $\mathbb{C}$ whose square is the identity, $\underline{\gamma}^j$ are anti-hermitian matrix bundles that satisfy

$$\forall j,k = 1,\dots,n \quad \{\underline{\gamma}^j, \underline{\gamma}^k\} = 2g^{jk}, \quad \forall j = 1,\dots,n \quad \{\gamma^0, \underline{\gamma}^j\} = 0$$

and D_j are covariant derivatives for spinor bundles.

Writing $(\mathcal{M}, g)$ as $\mathcal{M} = \mathbb{R}_t \times \Sigma$ and

$$g = \begin{pmatrix} 1 & (0) \\ (0) & -h \end{pmatrix},$$

where h is the (Riemannian) metric of Σ, we endow the spinorial Riemannian manifold (i.e., a Riemannian manifold with a spin structure) (Σ, h) with a vierbein $e^j_{\ a}$ (chosen such that for all j, k, $e^j_{\ a}\delta^{ab}e^k_{\ b} = h^{jk}$), and we fix

$$\underline{\gamma}^j = e^j_{\ a}\gamma^a.$$

The implicit summation for a is taken from 1 to n. The family $(\gamma^a)_{0\le a\le n}$ satisfies the anticommutation relations

$$\{\gamma^a, \gamma^b\} = 2\eta^{ab}$$

where

$$\eta = \begin{pmatrix} 1 & & & \\ & -1 & (0) & \\ & (0) & \ddots & \\ & & & -1 \end{pmatrix}$$

is the Minkowski metric in $\mathbb{R}^{1+n}$. Writing $\alpha^0 = \gamma^0$ and $\alpha^a = \gamma^0\gamma^a$, we have that the family $(\alpha^a)_{0\le a\le n}$ satisfy the *canonical anticommutation relations*

$$\{\alpha^a, \alpha^b\} = 2\delta^{ab}$$

and are self-adjoint matrices. What is more, the Dirac operator now writes

$$\mathcal{D} = m\alpha^0 - ie^j_{\ a}\alpha^a D_j.$$

[1]For the definition and properties of spin manifold, spin structure and spinora in this 'geometric' setting, we refer readers to the Section 1 'preliminaries' in [15, Page 6].

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Details on this construction (as well as the definition and the main properties of a vierbein) can be found in [32, Section 3.9 page, 144]. The minimal dimension for such a family of matrices is $2^{\lfloor (n+1)/2 \rfloor}$. An easy way to see that this dimension is big enough is to consider for $n = 2$, the Pauli matrices

$$\alpha^0 = \sigma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \alpha^1 = \sigma_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad \alpha^2 = \sigma_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and for $n = 2k + 2$ even, given a family $(\tilde{\alpha}^a)_{0 \le a \le 2k}$ of self-adjoint matrices of size $K \times K$ satisfying canonical anticommutation relations, the matrices written by block

$$\alpha^0 = \begin{pmatrix} \mathrm{Id}_K & (0) \\ (0) & -\mathrm{Id}_K \end{pmatrix},$$
$$\forall a = 0, \ldots, 2k, \alpha^{a+1} = \begin{pmatrix} (0) & \tilde{\alpha}^a \\ \tilde{\alpha}^a & (0) \end{pmatrix}, \quad \alpha^n = \begin{pmatrix} (0) & i\mathrm{Id}_K \\ -i\mathrm{Id}_K & (0) \end{pmatrix}. \tag{2.1}$$

Therefore, a natural way to pass from dimension $n = 2k$ even to $n + 1 = 2k + 1$ odd is to pass from the family of matrices $(\tilde{\alpha}^a)_{0 \le a \le n}$ to

$$\alpha^0 = \begin{pmatrix} \mathrm{Id}_K & (0) \\ (0) & -\mathrm{Id}_K \end{pmatrix}, \quad \forall a = 0, \ldots, 2k, \alpha^{a+1} = \begin{pmatrix} (0) & \tilde{\alpha}^a \\ \tilde{\alpha}^a & (0) \end{pmatrix};$$

and to pass from dimension $n + 1 = 2k + 1$ odd to dimension $n + 2$ even is simply to add the matrix

$$\alpha^n = \begin{pmatrix} (0) & i\mathrm{Id}_K \\ -i\mathrm{Id}_K & (0) \end{pmatrix}.$$

However, it is also natural to pass from an odd to an even dimension in the same way as to pass from an even to an odd. The reason is that, because of the theory of Clifford algebras, the algebra generated by the family $(\alpha^a)_{0 \le a \le n+2}$ defined as in equation (2.1) is canonically isomorphic to the one generated by

$$\begin{pmatrix} \mathrm{Id}_{2K} & (0) \\ (0) & -\mathrm{Id}_{2K} \end{pmatrix}, \quad \forall a = 0, \ldots, n+1, \begin{pmatrix} (0) & \alpha^a \\ \alpha^a & (0) \end{pmatrix}$$

(see [32] Section 5.6.2, page 229, for further details). We now consider the following setting: (Σ, σ) is a warped product – that is, a Riemannian manifold in the form $\Sigma = \mathbb{R}^+_r \times \mathbb{K}^{n-1}_\phi$ – where $\mathbb{K}^{n-1}$ is a $(n-1)$-dimensional compact spin manifold, and Σ is equipped with the Riemannian metric

$$dh^2 = dr^2 + \varphi(r)^2 d\phi^2,$$

where $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ and $d\phi^2$ is the Riemannian metric over $\mathbb{K}^{n-1}$. In other words,

$$h = \begin{pmatrix} 1 & (0) \\ (0) & \varphi^2 \kappa \end{pmatrix},$$

where κ is the Riemannian metric of $\mathbb{K}^{n-1}$.

In the case that interests us, we assume that a vierbein $\tilde{e} = (\tilde{e}_j^{\ a})$ has been set for $\mathbb{K}^{n-1}$, which we assume admits a spin structure. As the equation is covariant, we may choose any convenient vierbein for Σ: we use as a vierbein for Σ

$$e = \begin{pmatrix} 1 & (0) \\ (0) & \varphi(r)\tilde{e} \end{pmatrix}.$$

We set $(\tilde{\alpha}^a)_{0\le a\le n-1}$ a family of matrices satisfying canonical anticommutation relations and

$$\alpha^0 = \begin{pmatrix} \mathrm{Id} & (0) \\ (0) & -\mathrm{Id} \end{pmatrix}, \quad \forall a = 0, \dots, n-1, \ \alpha^{a+1} = \begin{pmatrix} (0) & \tilde{\alpha}^a \\ \tilde{\alpha}^a & (0) \end{pmatrix}.$$

We set also

$$\gamma^0 = \alpha^0, \forall a = 1, \dots, n, \ \gamma^a = \alpha^0\alpha^a,$$

and finally

$$\tilde{\gamma}^0 = \tilde{\alpha}^0, \quad \forall a = 1, \dots, n, \ \tilde{\gamma}^a = \tilde{\alpha}^0\tilde{\alpha}^a.$$

We recall that the covariant derivatives for Dirac spinors are given by

$$D_\mu = \partial_\mu + i\omega_\mu^{ab}\Sigma_{a,b},$$

where ω is the spin connection and

$$\Sigma_{a,b} = -\frac{i}{8}[\gamma_a, \gamma_b].$$

We have for all $a, b = 1, \dots, n$

$$[\gamma^a, \gamma^b] = [\alpha^0\alpha^a, \alpha^0\alpha^b] = [\alpha^b, \alpha^a] = \begin{pmatrix} [\tilde{\alpha}^{b-1}, \tilde{\alpha}^{a-1}] & (0) \\ (0) & [\tilde{\alpha}^{b-1}, \tilde{\alpha}^{a-1}] \end{pmatrix} = \begin{pmatrix} [\tilde{\gamma}^{a-1}, \tilde{\gamma}^{b-1}] & (0) \\ (0) & [\tilde{\gamma}^{a-1}, \tilde{\gamma}^{b-1}] \end{pmatrix}.$$

Therefore, we have

$$\Sigma_{a,b} = \begin{pmatrix} \tilde{\Sigma}_{a-1,b-1} & (0) \\ (0) & \tilde{\Sigma}_{a-1,b-1} \end{pmatrix},$$

where $\tilde{\Sigma}_{a,b} = -\frac{i}{8}[\tilde{\gamma}^a, \tilde{\gamma}^b]$.

We also have

$$de^a + \omega^a_{\ b} \wedge e^b = 0.$$

Since $e^1 = dr$, we have $de^1 = 0$ and thus

$$\omega^1_{\ b} \wedge e^b = 0.$$

Therefore, we get $\omega^1_{\ b} \sim e_{\ b}$ for all b and then $\omega_1^{\ 1b} = 0$ for all $b \ge 1$.

Since for all $a > 1$, we have $e^a = \varphi(r)\tilde{e}^{a-1}$, we get

$$de^a = \varphi' e^1 \wedge \tilde{e}^{a-1} + \varphi d\tilde{e}^{a-1} = \varphi' e^1 \wedge \tilde{e}^{a-1} - \varphi\tilde{\omega}^{a-1}_{\ b} \wedge \tilde{e}^b$$

and thus

$$\omega^a_{\ b} \wedge e^b = \varphi'\tilde{e}^{a-1} \wedge e^1 + \varphi\tilde{\omega}^{a-1}_{\ b} \wedge \tilde{e}^b = \varphi'\tilde{e}^{a-1} \wedge e^1 + \tilde{\omega}^{a-1}_{\ b} \wedge e^{b+1}.$$

Therefore,

$$\omega^a_{\ 1} = \varphi'\tilde{e}^{a-1} \sim e^a \Rightarrow \omega_1^{\ a1} = 0 \text{ and for all } j > 1, \ \omega_j^{\ a1} = \varphi'\tilde{e}^{a-1}_{j-1},$$

and for all $b > 1$,

$$\omega^a_{\ b} = \tilde{\omega}^{a-1}_{\ (b-1)} \Rightarrow \omega_1^{\ ab} = 0 \text{ and for all } j > 1,\ \omega_j^{\ ab} = \tilde{\omega}_{j-1}^{(a-1)(b-1)}.$$

Summing up, we get $D_1 = \partial_r$ and for all $j > 1$,

$$D_j = \partial_j + 2i\varphi' \tilde{e}^{\ a}_{j-1}\Sigma_{1(a+1)} + i\tilde{\omega}_{j-1}^{(a-1)(b-1)}\Sigma_{a,b}.$$

Since

$$\Sigma_{1,(a+1)} = \begin{pmatrix} \tilde{\Sigma}_{0,a} & (0) \\ (0) & \tilde{\Sigma}_{0,a} \end{pmatrix} \text{ and } \Sigma_{a,b} = \begin{pmatrix} \tilde{\Sigma}_{a-1,b-1} & (0) \\ (0) & \tilde{\Sigma}_{a-1,b-1} \end{pmatrix},$$

we have

$$D_j = \begin{pmatrix} \tilde{D}_{j-1} + 2i\varphi' \tilde{e}^{\ a}_{j-1}\tilde{\Sigma}_{0,a} & (0) \\ (0) & \tilde{D}_{j-1} + 2i\varphi' \tilde{e}^{\ a}_{j-1}\tilde{\Sigma}_{0,a} \end{pmatrix},$$

where $\tilde{D}_j$ are covariant derivatives for spinor bundles over $\mathbb{K}^2$.

We deduce

$$\begin{aligned} ie^j_{\ b}\alpha^b D_j &= i\alpha_1\partial_r + i\frac{1}{\varphi}\tilde{e}^{j-1}_{\ b}\begin{pmatrix} (0) & \tilde{\alpha}^b \\ \tilde{\alpha}^b & (0) \end{pmatrix}\begin{pmatrix} \tilde{D}_{j-1} + 2i\varphi' \tilde{e}^{\ a}_{j-1}\tilde{\Sigma}_{0,a} & (0) \\ (0) & \tilde{D}_{j-1} + 2i\varphi' \tilde{e}^{\ a}_{j-1}\tilde{\Sigma}_{0,a} \end{pmatrix} \\ &= i\alpha_1\partial_r + \frac{1}{r}\begin{pmatrix} (0) & i\tilde{e}^j_{\ b}\tilde{\alpha}^b(\tilde{D}_j + i\tilde{e}_j^{\ a}\tilde{\Sigma}_{0,a}) \\ i\tilde{e}^j_{\ b}\tilde{\alpha}^b(\tilde{D}_j + i\tilde{e}_j^{\ a}\tilde{\Sigma}_{0,a}) & (0) \end{pmatrix}. \end{aligned}$$

Using that $\tilde{e}^j_{\ b}\tilde{e}_j^{\ a} = \delta^a_{\ b}$ and that $\tilde{\alpha}^a\tilde{\Sigma}_{0,a} = -i\frac{n-1}{4}\tilde{\alpha}^0$, we deduce

$$\mathcal{D} = m\alpha^0 + i\alpha_1\partial_r + \begin{pmatrix} (0) & \frac{1}{\varphi(r)}\mathcal{D}_{\mathbb{K}^{n-1}} + i\frac{n-1}{2}\frac{\varphi'(r)}{\varphi(r)}\tilde{\alpha}^0 \\ \frac{1}{\varphi(r)}\mathcal{D}_{\mathbb{K}^{n-1}} + i\frac{n-1}{2}\frac{\varphi'(r)}{\varphi(r)}\tilde{\alpha}^0 & (0) \end{pmatrix},$$

where $\mathcal{D}_{\mathbb{K}^{n-1}}$ is the Dirac operator on $\mathbb{K}^{n-1}$. We thus get

$$\mathcal{D} = \begin{pmatrix} m & i\tilde{\alpha}^0\left(\partial_r + \frac{n-1}{2}\frac{\varphi'(r)}{\varphi(r)}\right) + \frac{1}{\varphi(r)}\mathcal{D}_{\mathbb{K}^{n-1}} \\ i\tilde{\alpha}^0\left(\partial_r + \frac{n-1}{2}\frac{\varphi'(r)}{\varphi(r)}\right) + \frac{1}{\varphi(r)}\mathcal{D}_{\mathbb{K}^{n-1}} & -m \end{pmatrix}. \tag{2.2}$$

Now the key step (for us) consists of ensuring that the operator $\mathcal{D}_{\mathbb{K}^{n-1}}$ can in fact be diagonalized. In the case $\mathbb{K}^{n-1}$ being the two-dimensional unit sphere, this fact is classical and well-known; the eigenvalues and eigenfunctions are explicit (see, e.g., [36] or [14]). In the general case, we can nevertheless evoke the following result, which can be found, for example, in [35, Theorem 5.27].

Proposition 2.1. *Let H be the Dirac operator on a smooth compact manifold $\mathbb{K}^{n-1}$. Then there is a direct sum decomposition of H into a sum of countably many orthogonal spaces H_μ, each of which is a finite-dimensional space of smooth sections and an eigenspace for H with eigenvalue μ. The eigenvalues μ form a discrete subset of $\mathbb{R}$.*

Let $\mu > 0$ be in the spectrum of $\mathcal{D}_{\mathbb{K}^{n-1}}$; we fix $(\psi_{\mu,j})_j$ an orthogonal basis of the eigenspace of $\mathcal{D}_{\mathbb{K}^{n-1}}$ with eigenvalue μ. We set $\psi_{-\mu,j} = i\tilde{\alpha}^0\psi_{\mu,j}$. Since $\tilde{\alpha}^0$ anticommutes with $\mathcal{D}_{\mathbb{K}^{n-1}}$, we get that $\psi_{-\mu,j}$ is an eigenfunction of $\mathcal{D}_{\mathbb{K}^{n-1}}$ with eigenvalue $-\mu$. Note that for all μ in the spectrum of $\mathcal{D}_{\mathbb{K}^{n-1}}$,

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since $(i\alpha^0)^2=-1$, we have for $\mu>0$, $(1+i\tilde{\alpha}^0)\psi_{\mu,j}=\psi_{\mu,j}+\psi_{-\mu,j}$ and $(1-i\tilde{\alpha}^0)\psi_{-\mu,j}=\psi_{-\mu,j}-\psi_{\mu,j}$. Similarly,

$$(1-i\tilde{\alpha}^0)\psi_{\mu,j}=\psi_{\mu,j}-\psi_{-\mu,j} \quad \text{and} \quad (1-i\tilde{\alpha}^0)\psi_{-\mu,j}=\psi_{-\mu,j}+\psi_{\mu,j}.$$

Therefore, the family

$$\mathcal{B}=\left(\begin{pmatrix}\frac{1+i\tilde{\alpha}^0}{\sqrt{2}}\psi_{\mu,j}\\ 0\end{pmatrix},\begin{pmatrix}0\\ -\frac{1-i\tilde{\alpha}^0}{\sqrt{2}}\psi_{\mu,j}\end{pmatrix}\right)_{\mu\in Sp(\mathcal{D}_{\mathbb{K}^{n-1}}),j}$$

forms an orthonormal basis of $L^2(\mathbb{K}^{n-1},\mathbb{C}^M)$.

We deduce that we have the decomposition

$$L^2(\Sigma,\mathbb{C}^M)=\bigoplus_{\mu,j}\mathcal{H}_{\mu,j},$$

where $\mathcal{H}_{\mu,j}$ is the tensor product of L^2_φ (the L^2 maps of $\mathbb{R}_+$ with measure φ^2dr) and with values in $\mathbb{C}$ and the vector space generated by

$$\left(\begin{pmatrix}\frac{1+i\tilde{\alpha}^0}{\sqrt{2}}\psi_{\mu,j}\\ 0\end{pmatrix},\begin{pmatrix}0\\ -\frac{1-i\tilde{\alpha}^0}{\sqrt{2}}\psi_{\mu,j}\end{pmatrix}\right).$$

In other words, any map $u\in L^2(\Sigma,\mathbb{C}^M)$ may be written as

$$u(r,\omega)=\sum_{\mu,j}u^+_{\mu,j}(r)\begin{pmatrix}\frac{1+i\tilde{\alpha}^0}{\sqrt{2}}\psi_{\mu,j}(\omega)\\ 0\end{pmatrix}+u^-_{\mu,j}(r)\begin{pmatrix}0\\ -\frac{1-i\tilde{\alpha}^0}{\sqrt{2}}\psi_{\mu,j}(\omega)\end{pmatrix},$$

where $\omega\in\mathbb{K}^{n-1}$, and $u^\pm_{\mu,j}\in L^2_\varphi$ are such that

$$\sum_{\mu,j}\|u^+_{\mu,j}(r)\|^2_{L^2_\varphi}+\|u^-_{\mu,j}(r)\|^2_{L^2_\varphi}<\infty.$$

For any $\mu\in Sp(\mathcal{D}_{\mathbb{K}^{n-1}})$ and any j, and $f(r)$ a radial test function, we have

$$\mathcal{D}\left[f\begin{pmatrix}(1+i\tilde{\alpha}^0)\psi_{\mu,j}\\ 0\end{pmatrix}\right]= \\ mf\begin{pmatrix}(1+i\tilde{\alpha}^0)\psi_{\mu,j}\\ 0\end{pmatrix}+\left(\left[-\left(\partial_r+\frac{n-1}{2}\frac{\varphi'(r)}{\varphi(r)}\right)+\frac{\mu}{\varphi(r)}\right]f\right)\begin{pmatrix}0\\ (1-i\tilde{\alpha}^0)\psi_{\mu,j}\end{pmatrix}$$

and

$$\mathcal{D}\left[f\begin{pmatrix}0\\ (1-i\tilde{\alpha}^0)\psi_{\mu,j}\end{pmatrix}\right]= \\ -mf\begin{pmatrix}0\\ (1-i\tilde{\alpha}^0)\psi_{\mu,j}\end{pmatrix}+\left(\left[\left(\partial_r+\frac{n-1}{2}\frac{\varphi'(r)}{\varphi(r)}\right)+\frac{\mu}{\varphi(r)}\right]f\right)\begin{pmatrix}(1+i\tilde{\alpha}^0)\psi_{\mu,j}\\ 0\end{pmatrix}.$$

We are thus left with studying the dispersion of the equation

$$i\partial_tF+h_\mu F=0, \tag{2.3}$$

with

$$h_\mu = \begin{pmatrix} m & -\left(\partial_r + \frac{n-1}{2}\frac{\varphi'(r)}{\varphi(r)}\right) + \frac{\mu}{\varphi(r)} \\ \left(\partial_r + \frac{n-1}{2}\frac{\varphi'(r)}{\varphi(r)}\right) + \frac{\mu}{\varphi(r)} & -m \end{pmatrix} \tag{2.4}$$

for any $\mu \in Sp(\mathcal{D}_{\mathbb{K}^{n-1}})$.

Remark 2.2. If we take ψ_μ an eigenfunction of $\mathcal{D}_{\mathbb{K}^{n-1}}$ on $\mathbb{K}^{n-1}$ with eigenvalue $\mu \neq 0$ and we suppose that $\theta = f(r)\begin{pmatrix}(1+i\tilde{\alpha}^0)\psi_\mu \\ 0\end{pmatrix}$ is an eigenspinor of $\mathcal{D}^2$ with eigenvalue $\rho^2 \neq 0$, then we have that f satisfies the following ODE

$$f'' + \frac{n-1}{r} + \left[\rho^2 - \left(\mu^2 - \mu - \frac{n^2-4n+3}{4}\right)\frac{1}{r^2}\right] f = 0,$$

which has solutions $f(r) = \gamma^c J_{\pm\nu^+}(\rho r)$, where $c = (2-n)/2$, $\nu^+ = |2\mu - 1|/2$ and J_{ν^+} is the standard Bessel function of order ν^+. Analogously, assuming that now $\theta = f(r)\begin{pmatrix}0 \\ (1-i\tilde{\alpha}^0)\psi_\mu\end{pmatrix}$ is an eigenspinor of $\mathcal{D}^2$ with eigenvalue $\rho^2 \neq 0$, we see that f satisfies the ODE

$$f'' + \frac{n-1}{r} + \left[\rho^2 - \left(\mu^2 + \mu - \frac{n^2-4n+3}{4}\right)\frac{1}{r^2}\right] f = 0,$$

which has solutions $f(r) = r^c J_{\pm\nu^+}(\rho r)$, with c as before and $\nu^+ = |2\mu + 1|/2$. These equations recall the connection with the Klein-Gordon equation, which has now been brought to the 'radial' level. In particular, in [5], these equations are the starting point in order to prove the crucial local smoothing estimates for the Klein-Gordon equation; nevertheless, we stress once again the fact that the argument of deducing dispersive estimates for the Dirac flow from the corresponding Klein-Gordon ones does not work for free, as indeed the Laplace operator that comes into play when squaring the Dirac operator is the spinorial one (and not the standard scalar one that we dealt with in [5]).

We can explicitly write down the positive and negative eigenspinors of the operator $\mathcal{D}_{\mathbb{K}^{n-1}}$, when we are calling 'positive' (respectively, 'negative') the ones corresponding to Bessel functions of positive (respectively, negative) order. It can then be shown that both positive and negative ones fall in the domain of $\mathcal{D}_{\mathbb{K}^{n-1}}$. The negative ones, though, correspond to eigenvalues μ of $\mathcal{D}_{\mathbb{K}^{n-1}}$ such that $|\mu| \leq 1/2$ (this can be seen by studying the asymptotic behaviours of the Bessel functions). Finally, negative solutions in the domain of $\mathcal{D}_{\mathbb{K}^{n-1}}$ prevent the operator $\mathcal{D}_{\mathbb{K}^{n-1}}$ from being self-adjoint. This is why we need the assumption $|\mu| > 1/2$.

2.2. The squaring trick and weighted spinors

We now introduce *weighted spinors*, the main goal being transforming the system equation (2.3) into a system of wave equations on $\mathbb{R}^n$ perturbed by a radial, electric potential in order to exploit the existing theory to obtain dispersive estimates. This strategy has been already employed in [33, 2, 21] in different contexts (the Schrödinger equation on Damek-Ricci spaces and on spherically symmetric manifolds and equivariant wave maps, respectively) and in the predecessor of this paper, [11], to deal with the local-in-time case.

Take $\sigma : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ such that for all $r > 0$,

$$\sigma(r) = \frac{r}{\varphi(r)},$$

where $\varphi(r)$ satisfies the assumptions of Theorem 1.4 and write, for $n \geq 3$,

$$\sigma_n = \sigma^{(n-1)/2}.$$

Lemma 2.3. *The map σ prolonged by continuity at 0 is $\mathcal{C}^1$ and the map*

$$\frac{\sigma'}{\sigma}$$

is bounded on $(0, \infty)$.

Proof. Indeed, for $r \geq 0$,

$$\sigma'(r) = \Big(\frac{1}{r} - \frac{\varphi'(r)}{\varphi(r)}\Big)\sigma.$$

The map σ at 0 converges to 1 and we have, as $r \downarrow 0$, writing $a = \frac{\varphi''(0)}{2}$,

$$\frac{\sigma(r)-1}{r} = \frac{1}{r}\Big(\frac{r}{r + ar^2 + o(r^2)} - 1\Big) \to -a,$$

hence $\sigma'(0) = -a$. What is more, as $r \to 0$,

$$\sigma'(r) = \frac{1}{\varphi(r)} - \frac{r\varphi'(r)}{\varphi^2} = \frac{1}{r}(-ar + o(r)) \to -a.$$

Finally, since $\sigma \to 1$ at 0 and $\sigma > 0$, we deduce that $\frac{\sigma'}{\sigma}$ is continuous on $[0, \infty)$.

Finally,

$$\frac{\sigma'(r)}{\sigma(r)} = \frac{1}{r} - \frac{\varphi'}{\varphi},$$

which ensures its boundedness. □

Lemma 2.4. *The multiplication by σ_n is an isometry from L^2_r to L^2_φ. What is more, the multiplication by σ_n is an isomorphism from H^1_r to H^1_φ that satisfies*

$$\Big[1 + c_\varphi \frac{n-1}{2}\Big]^{-1} \|f\|_{H^1_r} \leq \|\sigma_n f\|_{H^1_\varphi} \leq \Big[1 + c_\varphi \frac{n-1}{2}\Big] \|f\|_{H^1_r}$$

with $c_\varphi = \|\frac{\sigma'}{\sigma}\|_{L^\infty((0,\infty))}$. In particular, by interpolation, we get, for all $s \in [0,1]$,

$$\Big[1 + c_\varphi \frac{n-1}{2}\Big]^{-s} \|f\|_{H^s_r} \leq \|\sigma_n f\|_{H^s_\varphi} \leq \Big[1 + c_\varphi \frac{n-1}{2}\Big]^{s} \|f\|_{H^s_r}.$$

Proof. The fact that σ_n is an isometry at the L^2-level follows by the definition of the norms, as indeed

$$\|\sigma_n f\|^2_{L^2_\varphi} = \int \sigma_n^2 f^2 \varphi^{n-1} dr = \int r^{n-1} f^2 dr = \|f\|^2_{L^2_r}.$$

We now estimate $\|\sigma_n f\|_{H^1_\varphi}$. We have, by the isometry in L^2,

$$\|\sigma_n f\|_{H^1_\varphi} \leq \|f\|_{H^1_r} + \|\sigma_n' \sigma_n^{-1} f\|_{L^2_r}.$$

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A direct computation yields

$$\sigma_n' \sigma_n^{-1} = \frac{n-1}{2} \frac{\sigma'}{\sigma}.$$

By Hölder's inequality, we get

$$\|\sigma_n' \sigma_n^{-1} f\|_{L^2_r} \leq \frac{n-1}{2} c_\varphi \|f\|_{L^2_r}.$$

We now estimate $\|\sigma_n^{-1} g\|_{H^1_r}$. We have, by isometry in L^2,

$$\|\sigma_n^{-1} g\|_{H^1_r} \leq \|g\|_{H^1_\varphi} + \|(\sigma_n^{-1})' \sigma_n g\|_{L^2_\varphi}.$$

We have

$$(\sigma_n^{-1})' \sigma_n = -\frac{\sigma_n'}{\sigma_n} = -\frac{n-1}{2} \frac{\sigma'}{\sigma}.$$

We use Hölder's inequality to get

$$\|(\sigma_n^{-1})' \sigma_n g\|_{H^1_\varphi} \leq \frac{n-1}{2} c_\varphi \|g\|_{L^2_\varphi}.$$

This concludes the proof. □

Lemma 2.5. *We have*

$$h_{\mu,n} := \sigma_n^{-1} h_\mu \sigma_n = \begin{pmatrix} m & -\partial_r - \frac{n-1}{2r} + \frac{\mu}{\varphi} \\ \partial_r + \frac{n-1}{2r} + \frac{\mu}{\varphi} & -m \end{pmatrix}. \tag{2.5}$$

Proof. Straightforward computation. □

Proposition 2.6. *Let $s \in [-1, 1]$ and $p, q \geq 1$. If $e^{-ith_{\mu,n}}$ is a continuous operator from $H^{1/2}_r$ to $L^p(\mathbb{R}, W^{s,q}_r)$, then e^{-ith_μ} is a continuous operator from $H^{1/2}_\varphi$ to $\sigma_n^{1-2/q} L^p(\mathbb{R}, W^{s,q}_\varphi)$ and*

$$\left\|e^{-ith_\mu}\right\|_{H^{1/2}_\varphi \to \sigma_n^{1-2/q} L^p(\mathbb{R}, W^{s,q}_\varphi)} \leq C_\varphi \left\|e^{-ith_{\mu,n}}\right\|_{H^{1/2}_r \to L^p(\mathbb{R}, W^{s,q}_r)}$$

with a constant C_φ that does not depend on μ.

This is a consequence of the fact that

$$e^{-ith_\mu} = \sigma_n e^{-ith_{\mu,n}} \sigma_n^{-1},$$

of Lemma 2.4 and of the following lemma.

Lemma 2.7. *The multiplication by σ_n is a continuous operator from*

$$L^p(\mathbb{R}, W^{s,q}_r)$$

to

$$\sigma_n^{1-2/q} L^p(\mathbb{R}, W^{s,q}_\varphi)$$

for any $s \in [-1, 1]$ and any $q \in (1, \infty)$.

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Proof. First of all, the norm in the t variable is not relevant in the proof; hence we only prove that the multiplication by σ_n is continuous from $W_r^{s,q}$ to $\sigma_n^{1-2/q} W_\varphi^{s,q}$ for $q \in (1,\infty)$ and $s \in [-1,1]$. This is equivalent to proving that the multiplication by $\sigma_n^{2/q}$ is continuous from $W_r^{s,q}$ to $W_\varphi^{s,q}$.

For non-negative s, by interpolation, we can reduce the proof to the cases $s = 0, 1$.

For negative s, by duality, the continuity of the multiplication by $\sigma_n^{2/q}$ from $W_r^{s,q}$ to $W_\varphi^{s,q}$ is implied by the continuity of the multiplication by $\sigma_n^{-2/q'}$ from $W_\varphi^{-s,q'}$ to $W_r^{-s,q'}$, where q' is the conjugated exponent of q.

Therefore, it sufficient to prove the following, for all $q \in (1,\infty)$:

1. the multiplication by $\sigma_n^{2/q}$ is an isometry from L_r^q to L_φ^q,
2. the multiplication by $\sigma_n^{2/q}$ is continuous from $W_r^{1,q}$ to $W_\varphi^{1,q}$,
3. the multiplication by $\sigma_n^{-2/q}$ is continuous from $W_\varphi^{1,q}$ to $W_r^{1,q}$.

(1) Let $f \in L_r^q$; we have by definition

$$\|\sigma_n^{2/q} f\|_{L_\varphi^q}^q = \int_0^\infty \sigma_n^2 \varphi^{n-1} |f|^q,$$

and using the definition of σ_n,

$$\|\sigma_n^{2/q} f\|_{L_\varphi^q}^q = \int_0^\infty r^{n-1} |f(r)|^q \, dr = \|f\|_{L_r^q}^q.$$

(2) From (1), it is sufficient to prove that for all $f \in W_r^{1,q}$, we have

$$\|\partial_r(\sigma_n^{2/q} f)\|_{L_\varphi^q} \lesssim \|f\|_{W_r^{1,q}}.$$

By the Leibniz rule, we have

$$\|\partial_r(\sigma_n^{2/q} f)\|_{L_\varphi^q} = \|\sigma_n^{2/q}\Big(\frac{n-1}{q}\frac{\sigma'}{\sigma} f + \partial_r f\Big)\|_{L_\varphi^q},$$

and from (1), we get

$$\|\partial_r(\sigma_n^{2/q} f)\|_{L_\varphi^q} = \|\frac{n-1}{q}\frac{\sigma'}{\sigma} f + \partial_r f\|_{L_r^q}.$$

We conclude by using the fact that $\frac{\sigma'}{\sigma}$ is bounded.

(3) From (1), it is sufficient to prove that

$$\|\partial_r(\sigma_n^{-2/q} f)\|_{L_r^q} \lesssim \|f\|_{W_\varphi^{1,q}}.$$

By the Leibniz rule, we have

$$\|\partial_r(\sigma_n^{-2/q} f)\|_{L_r^q} = \|\sigma_n^{-2/q}\Big(-\frac{n-1}{q}\frac{\sigma'}{\sigma} f + \partial_r f\Big)\|_{L_r^q},$$

and from (1), we get

$$\|\partial_r(\sigma_n^{-2/q} f)\|_{L_r^q} = \| -\frac{n-1}{q}\frac{\sigma'}{\sigma} f + \partial_r f\|_{L_\varphi^q}.$$

We conclude by using the fact that $\frac{\sigma'}{\sigma}$ is bounded. □

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As recalled in the introduction, the (massless) Dirac operator has been constructed as some square root of the Laplacian; in other words, every solution to the free Dirac equation on $\mathbb{R}^n$ satisfies a system of decoupled free wave/Klein-Gordon equations. This point of view can be carried at the 'radial' level:

Lemma 2.8. *Let $V \in C^2(0,\infty)$, and let*

$$h_{V,n} = \begin{pmatrix} m & -\left(\partial_r + \frac{n-1}{2r}\right) + V \\ \left(\partial_r + \frac{n-1}{2r}\right) + V & -m \end{pmatrix}.$$

Then we have

$$h^2_{V,n} = \begin{pmatrix} m^2 + H_{c_-} & 0 \\ 0 & m^2 + H_{c_+} \end{pmatrix}$$

with

$$H_{c_\pm} = -\left(\partial_r^2 + \frac{n-1}{r}\partial_r\right) + c_\pm \tag{2.6}$$

and

$$c_\pm = -\frac{(n-1)(n-3)}{4r^2} + V^2 \pm \partial_r V.$$

Remark 2.9. In other words, if $v = \begin{pmatrix} v_+ \\ v_- \end{pmatrix}$ solves the equation

$$i\partial_t v = \tilde{h}_\mu v$$

with initial datum $v_0 = \begin{pmatrix} v_{0,+} \\ v_{0,-} \end{pmatrix}$, then v_+ and v_- solve, respectively,

$$\partial_t^2 v_+ = -m^2 v_+ - H_{c_-} v_+ \quad \text{and} \quad \partial_t^2 v_- = -m^2 v_- - H_{c_+} v_-$$

with initial data

$$\begin{pmatrix} v_+(t=0) \\ \partial_t v_+(t=0) \end{pmatrix} = \begin{pmatrix} v_{0,+} \\ -imv_{0,+} + i\left(\partial_r + \frac{1}{r} - V\right)v_{0,-} \end{pmatrix}$$

and

$$\begin{pmatrix} v_-(t=0) \\ \partial_t v_-(t=0) \end{pmatrix} = \begin{pmatrix} v_{0,-} \\ imv_{0,-} - i\left(\partial_r + \frac{1}{r} + V\right)v_{0,+} \end{pmatrix}.$$

Proof. Straightforward computation. □

3. The wave and Klein-Gordon equation with potentials of critical decay

In this section, we review the well-known theory on dispersive estimates for critical perturbations of the wave and Klein-Gordon flows, discussing in particular how the available results can be adapted to deal with our problem. As the strategy and the results below are classical, we will only sketch most of them, providing references to fill in the details.

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3.1. General Kato-smoothing

First of all, let us recall the following definition (see [17, Definition 2.1]):

Definition 3.1. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces and H a self-adjoint operator on $\mathcal{H}_1$. Let R be the resolvent operator of H. A closed operator A from $\mathcal{H}_1$ to $\mathcal{H}_2$ with dense domain $D(A)$ is called

1. H-smooth with constant a if there exists ε_0 such that for every $\varepsilon, \lambda \in \mathbb{R}$ with $0 < |\varepsilon| < \varepsilon_0$, the following uniform bound holds:

$$|(\Im R(\lambda + i\varepsilon)A^* v, A^* v)_{\mathcal{H}_1}| \le a\|v\|^2_{\mathcal{H}_2}, \qquad v \in D(A^*).$$

2. H-supersmooth with constant a if there exists ε_0 such that for every $\varepsilon, \lambda \in \mathbb{R}$ with $0 < |\varepsilon| < \varepsilon_0$, the following uniform bound holds:

$$|(R(\lambda + i\varepsilon)A^* v, A^* v)_{\mathcal{H}_1}| \le a\|v\|^2_{\mathcal{H}_2}, \qquad v \in D(A^*).$$

We prove the following proposition:

Proposition 3.2. *Let $n \ge 3$ be the dimension, and let $c \in C^1((0,\infty))$ and $r = |x|$. Assume that $r^2 c \in L^\infty$ and*

$$\delta_c := \min\left[\frac{1}{4}, \inf\left(cr^2 + \frac{(n-2)^2}{4}\right), \inf\left(-r^3 c' - r^2 c + \frac{(n-2)^2}{4}\right)\right] > 0. \tag{3.1}$$

Then the operator H_c defined in equation (2.6) is positive on the Hilbert space $L^2(\mathbb{R}^n)$ and the operator $|x|^{-1}$ (from $L^2(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$) is H_c super-smooth with constant δ_c^{-1}.

Proof. Because we have

$$\inf\left(cr^2 + \frac{(n-2)^2}{4}\right) \ge \delta_c,$$

we get, for any $v \in C^\infty((0,\infty))$ with compact support

$$\langle v, H_c v\rangle_{L^2} \ge \langle v, \left(-\partial_r^2 - \frac{n-1}{r}\partial_r\right)v\rangle_{L^2} - \langle v, \frac{(n-2)^2}{4r^2}v\rangle_{L^2} + \delta_c\langle v, r^{-2}v\rangle_{L^2}$$

and by Hardy's inequality, as we are in dimension $n \ge 3$,

$$\langle v, H_c v\rangle_{L^2} \ge \delta_c\langle v, r^{-2}v\rangle_{L^2}.$$

Therefore, H_c is positive.

The fact that $|x|^{-1}$ is H_c super-smooth is a consequence of [21, Theorem 3.3] with $a = \frac{n-1}{r}$. Indeed, for v in the domain of $|x|^{-1}$, write $f = |x|^{-1}v$; writing $R(\lambda + i\varepsilon)$ the resolvent of H_c with $\varepsilon \ne 0$, we have that $R(\lambda + i\varepsilon)|x|^{-1}v$ is the solution to

$$u'' + \frac{n-1}{r}u' + (\lambda + i\varepsilon)u - cu = -f$$

from which we deduce

$$\| |x|^{-1}u\|_{L^2(\mathbb{R}^n)} \le \delta_c^{-1}\| |x| f\|_{L^2(\mathbb{R}^n)} = \delta_c^{-1}\|v\|_{L^2(\mathbb{R}^n)}.$$

We get from this

$$\langle R(\lambda + i\varepsilon)|x|^{-1}v, |x|^{-1}v\rangle_{L^2} = \langle |x|^{-1}u, v\rangle_{L^2} \le \| |x|^{-1}u\|_{L^2(\mathbb{R}^n)}\|v\|_{L^2(\mathbb{R}^n)},$$

which yields

$$\langle R(\lambda+i\varepsilon)|x|^{-1}v,|x|^{-1}v\rangle_{L^2}\le \delta_c^{-1}\|v\|^2_{L^2(\mathbb{R}^n)}$$

and concludes the proof. □

Proposition 3.3. *Under the same assumptions as Proposition 3.2, we have that* $|x|^{-1}(H_c+\nu)^{-1/4}$ *is* $\sqrt{H_c+\nu}$ *super-smooth for any* $\nu\in\mathbb{R}^+$ *with constant* $C_c^2=(3+\pi)\delta_c^{-1}$ *and, in particular, for all v in the domain of* $(H_c+\nu)^{1/4}$*, we have that*

$$\big\|\,|x|^{-1}e^{-it\sqrt{\nu+H_c}}v\big\|_{L^2(\mathbb{R}\times\mathbb{R}^n)}\le C_c\|(H_c+\nu)^{1/4}v\|_{L^2(\mathbb{R}^n)}.$$

Proof. This is a direct consequence of [17, Theorem 2.4]. □

Proposition 3.4. *Under the assumptions of Proposition 3.2 and assuming* ν *nonnegative, set* $U_c(t)$ *to be the flow of the equation*

$$\begin{cases}\partial_t^2 v+H_c v+\nu v=0,\\ \quad v(t=0)=v_0,\qquad \partial_t v(t=0)=v_1.\end{cases}\tag{3.2}$$

Set also $\mathcal{X}_{c,\nu}$ *and* $\mathcal{H}^{1/2}_{c,\nu}$ *to be spaces respectively induced by the norms*

$$\|(f,g)\|^2_{\mathcal{X}_{c,\nu}}=\|\,|x|^{-1}f\|^2_{L^2(\mathbb{R}\times\mathbb{R}^n)}+\|\,|x|^{-1}(H_c+\nu)^{-1/2}g\|^2_{L^2(\mathbb{R}\times\mathbb{R}^n)}$$

and

$$\|(v_0,v_1)\|^2_{\mathcal{H}^{1/2}_{c,\nu}}=\|(H_c+\nu)^{1/4}v_0\|^2_{L^2(\mathbb{R}^n)}+\|(H_c+\nu)^{-1/4}v_1\|^2_{L^2(\mathbb{R}^n)}.$$

Then we have for all $(v_0,v_1)\in\mathcal{H}^{1/2}_{c,\nu}$*,*

$$\|U_c(t)(v_0,v_1)\|_{\mathcal{X}_{c,\nu}}\le C_c\|(v_0,v_1)\|_{\mathcal{H}^{1/2}_{c,\nu}}.$$

Proof. The proof follows the usual lines, assuming, without loss of generality, that ν is real and using the transform

$$U=v+i(H_c+\nu)^{-1/2}\partial_t v.$$ □

3.2. Application to the Dirac equation with critical potentials

Proposition 3.5. *Let* $V\in C^2((0,\infty))$*. Write* $c_\pm=-\frac{(n-1)(n-3)}{4r^2}+V^2\pm V'$*. Set* $S_{V,n}(t)$ *to be the flow of equation* $i\partial_t u=h_{V,n}u$ *with* $h_{V,n}$ *as in Lemma 2.8. Assume that*

$$\delta_V^\pm:=\min\left[\frac14,\inf\left(\frac14+r^2(V^2\pm V')\right),\inf\left(\frac14-r^3(2VV'\pm V'')-r^2(V^2\pm V')\right)\right]>0.\tag{3.3}$$

Then we have that for all $u_0\in\mathcal{H}^{1/2}_{c_+,c_-,m}$ *the solution* $S_{V,n}(t)u_0$ *satisfies*

$$\||x|^{-1}S_{V,n}(t)u_0\|_{L^2(\mathbb{R}\times\mathbb{R}^n)}\le 3(C_{c_+}+C_{c_-})\|u_0\|_{\mathcal{H}^{1/2}_{c_+,c_-,m}},$$

where $\mathcal{H}^{1/2}_{c_+,c_-,m}$ *is the space induced by the norm*

$$\left\|\begin{pmatrix}f\\ g\end{pmatrix}\right\|_{\mathcal{H}^{1/2}_{c_+,c_-,m}}=\|(m^2+H_{c_-})^{1/4}f\|_{L^2(\mathbb{R}^n)}+\|(m^2+H_{c_+})^{1/4}g\|_{L^2(\mathbb{R}^n)}.$$

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Finally, we have $\delta_{c_\pm} = \delta_V^\pm$.

Proof. Set

$$u_0 = \begin{pmatrix} f_0 \\ g_0 \end{pmatrix} \quad \text{and} \quad S_{V,n}(t)(u_0) = \begin{pmatrix} f \\ g \end{pmatrix}.$$

Write $V_\pm = V \pm \left(\partial_r + \frac{n-1}{2r}\right)$. By a straightforward computation we get that $V_- V_+ = H_{c_-}$ and $V_+ V_- = H_{c_+}$. Therefore, we recall that

$$h_{V,n}^2 = \begin{pmatrix} m^2 + H_{c_-} & 0 \\ 0 & m^2 + H_{c_+} \end{pmatrix} \tag{3.4}$$

and that f is the solution to $\partial_t^2 f + H_{c_-} f + m^2 f = 0$ with initial datum

$$f(t=0) = f_0 \quad \text{and} \quad \partial_t f(t=0) = f_1 := -imf_0 - iV_- g_0.$$

A direct computation yields $\delta_V^\pm = \delta_{c_\pm}$ and thus

$$\| |x|^{-1} f\|_{L^2_{t,x}} \le C_{c_-}\left(\|(m^2 + H_{c_-})^{1/4} f_0\|_{L^2_x} + \|(m^2 + H_{c_-})^{-1/4} f_1\|_{L^2_x}\right)$$

We have that

$$\|(m^2 + H_{c_-})^{-1/4} m f_0\|_{L^2} \le \sqrt{|m|}\|f_0\|_{L^2}$$

since $m^2 + H_{c_-} \ge m^2$.

What is more,

$$\|(m^2 + H_{c_-})^{-1/4} V_- g_0\|_{L^2}^2 = \langle (m^2 + H_{c_-})^{-1/4} V_- g_0, (m^2 + H_{c_-})^{-1/4} V_- g_0\rangle_{L^2}.$$

Since $m^2 + H_{c_-} \ge H_{c_-}$, we have

$$\|(m^2 + H_{c_-})^{-1/4} V_- g_0\|_{L^2}^2 \le \langle H_{c_-}^{-1/4} V_- g_0, H_{c_-}^{-1/4} V_- g_0\rangle_{L^2}.$$

By taking adjoints, we get

$$\|(m^2 + H_{c_-})^{-1/4} V_- g_0\|_{L^2}^2 \le \langle V_+ H_{c_-}^{-1/2} V_- g_0, g_0\rangle_{L^2}.$$

Let $A = V_+ H_{c_-}^{-1/2} V_-$. The operator A is positive and

$$A^2 = V_+ H_{c_-}^{-1/2} V_- V_+ H_{c_-}^{-1/2} V_-.$$

Since $V_- V_+ = H_{c_-}$ we get

$$A^2 = V_+ H_{c_-}^{-1/2} H_{c_-} H_{c_-}^{-1/2} V_- = V_+ V_- = H_{c_+}.$$

Finally,

$$\| |x|^{-1} f\|_{L^2} \le C_{c_-}\left(2\|(m^2 + H_{c_-})^{1/4} f_0\|_{L^2} + \|(m^2 + H_{c_+})^{1/4} g_0\|_{L^2}\right).$$

With a similar computation, we get

$$\| |x|^{-1} g\|_{L^2} \le C_{c_+}\left(2\|(m^2 + H_{c_+})^{1/4} g_0\|_{L^2} + \|(m^2 + H_{c_-})^{1/4} f_0\|_{L^2}\right).$$

□

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We can now exploit the powerful Rodnianski-Schlag argument (see [34]) to deduce Strichartz estimates from Proposition 3.5.

Proposition 3.6. *Assume that (p,q,m) is admissible, as in Definition 1.3. Then there exists a constant $C = C(p,q,m)$ such that for all $u_0 \in H^{1/2}_r \cap \mathcal{H}^{1/2}_{c_+,c_-,m}$, we have*

$$\|S_{V,n}(t)u_0\|_{L^p,W_r^{1/q-1/p,q}} \le C\Big((1+\|rV\|_{L^\infty((0,\infty))})\|u_0\|_{H^{1/2}} + (\|r^2c_+\|_{L^\infty((0,\infty))} + \|r^2c_-\|_{L^\infty((0,\infty))})((\delta_V^+)^{-1} + (\delta_V^-)^{-1})\|u_0\|_{\mathcal{H}^{1/2}_{c_+,c_-,m}}\Big). \tag{3.5}$$

Proof. Let

$$S_{V,n}(t)u_0 = \begin{pmatrix} f \\ g \end{pmatrix} \quad \text{and} \quad u_0 = \begin{pmatrix} f_0 \\ g_0 \end{pmatrix},$$

where, we recall, $S_{V,n}(t)$ is the flow of equation $i\partial_t u = h_{V,n}u$ with $h_{V,n}$ as in Lemma 2.8. We then have

$$f = V_{c_-}(t)(f_0,f_1) = V_0(t)(f_0,f_1) - \int_0^t \frac{\sin((m^2+H_0)^{1/2}(t-\tau))}{(m^2+H_0)^{1/2}}(c_-f(\tau))d\tau$$

with $f_1 = -imf_0 - i(V - (\partial_r + \frac{n-1}{2r}))g_0$, and where $V_{c_-}(t)(f_0,f_1)$ is the solution to

$$\begin{cases} \partial_t^2 f + m^2 f + H_{c_-}f = 0 \\ f(t=0) = f_0, \partial_t f(t=0) = f_1. \end{cases}$$

From standard arguments, we get the following estimate on f:

$$\|f\|_{L^p,W_r^{1/q-1/p,q}} \le C\Big(\|f_0\|_{H_r^{1/2}} + \|f_1\|_{H_r^{-1/2}} + \|rc_-f\|_{L^2(\mathbb{R}\times\mathbb{R}^n)}\Big).$$

This estimate can be obtained by combining standard Strichartz estimates for Klein-Gordon (see, e.g., [19]), the Christ-Kiselev lemma ($p>2$) and local smoothing on $e^{i\sqrt{m^2+H_0}t}$ (by, for example, taking the dual form of estimate (3.7) in [17]). Indeed, we have

$$\|V_0(t)(f_0,f_1)\|_{L^p,W_r^{1/q-1/p,q}} \lesssim \|f_0\|_{H_r^{1/2}} + \|f_1\|_{H_r^{-1/2}}$$

due to the Strichartz inequality for the Klein-Gordon equation. We also have that

$$\Big\|\int_0^\infty \frac{\sin((m^2+H_0)^{1/2}(t-\tau))}{(m^2+H_0)^{1/2}}(c_-f(\tau))d\tau\Big\|_{L^p,W_r^{1/q-1/p,q}} \lesssim \Big\|\int_0^\infty \frac{\sin((m^2+H_0)^{1/2}(t-\tau))}{(m^2+H_0)^{1/2}}(c_-f(\tau))d\tau\Big\|_{H^{1/2}(\mathbb{R}^n)}.$$

Since $H_0 = \sqrt{-\Delta}$ and the Laplacian commute, we get

$$\Big\|\int_0^\infty \frac{\sin((m^2+H_0)^{1/2}(t-\tau))}{(m^2+H_0)^{1/2}}(c_-f(\tau))d\tau\Big\|_{H^{1/2}(\mathbb{R}^n)} = \Big\|\int_0^\infty \sin((m^2+H_0)^{1/2}(t-\tau))(c_-f(\tau))d\tau\Big\|_{H^{-1/2}(\mathbb{R}^n)}.$$

We then use the dual form of local smoothing to get

$$\Big\| \int_0^\infty \sin((m^2+H_0)^{1/2}(t-\tau))(c_- f(\tau))d\tau \Big\|_{H^{-1/2}(\mathbb{R}^n)} \lesssim \| |x| c_- f\|_{L^2(\mathbb{R}\times\mathbb{R}^n)}.$$

Now, exploiting the Christ-Kiselev lemma, we get that since

$$F \mapsto \int_0^\infty \frac{\sin((m^2+H_0)^{1/2}(t-\tau))}{(m^2+H_0)^{1/2}} (|x|^{-1}F(\tau))d\tau$$

is continuous from $L^2(\mathbb{R}\times\mathbb{R}^n)$ to $L^p, W_r^{1/q-1/p,q}$ and since $p>2$, so is

$$F \mapsto \int_0^t \frac{\sin((m^2+H_0)^{1/2}(t-\tau))}{(m^2+H_0)^{1/2}} (|x|^{-1}F(\tau))d\tau.$$

We get that

$$\Big\| \int_0^t \frac{\sin((m^2+H_0)^{1/2}(t-\tau))}{(m^2+H_0)^{1/2}} (c_- f(\tau))d\tau \Big\|_{L^p, W_r^{1/q-1/p,q}} \lesssim \|r c_- f\|_{L^2(\mathbb{R}\times\mathbb{R}^n)}.$$

Because of the Hardy inequality, we have

$$\|f_1\|_{H_r^{-1/2}} \lesssim \|f_0\|_{L^2} + (1+\|rV\|_{L^\infty})\|g_0\|_{H_r^{1/2}}.$$

Besides,

$$\|r c_- f\|_{L^2(\mathbb{R}\times\mathbb{R}^n)} \leq \|r^2 c_-\|_{L^\infty} \|r^{-1} f\|_{L^2(\mathbb{R}\times\mathbb{R}^n)}.$$

Because of local smoothing on $S_{V,n}(t)$, we get

$$\|r^{-1} f\|_{L^2(\mathbb{R}\times\mathbb{R}^n)} \lesssim (C_{c_+} + C_{c_-})\|u_0\|_{\mathcal{H}^{1/2}_{c_+,c_-,m}}.$$

A similar computation on g yields the result. □

3.3. *Application to the Dirac equation in curved manifolds*

In this section, we set $V = V_\mu = \frac{\mu}{\varphi}$, $\delta_\pm(\mu) = \delta^\pm_{V_\mu}$ and

$$c_\pm(\mu) = -\frac{(n-1)(n-3)}{4r^2} + V_\mu^2 \pm V_\mu' = -\frac{(n-1)(n-3)}{4r^2} + \mu\frac{\mu \mp \varphi'}{\varphi^2}.$$

Finally, we set $H_\pm(\mu) = H_{c_\pm}(\mu)$. Here, we are assuming that φ satisfies assumptions **(A0)-(A1)**.

Lemma 3.7. *For any $|s| \leq 1$, we have the bound*

$$\|(m^2 + H_\pm(\mu))^{s/2} v\|_{L^2(\mathbb{R}^n)} \lesssim_{\varphi,m} (1+\mu^2)^{s/2}\|v\|_{H^s(\mathbb{R}^n)}.$$

In particular, we have

$$\|u_0\|_{\mathcal{H}^{1/2}_{c_+(\mu),c_-(\mu),m}} \lesssim_{\varphi,m} \sqrt{|\mu|}\|u_0\|_{H^{1/2}_{r,n}}.$$

Remark 3.8. Differently from [21], we need to keep track of the dependence on μ of the inequalities above: indeed, if on the one hand, for the purpose of Theorem 1.4, such a dependence is irrelevant (as it is in [21]), in view of Theorem 1.10 it will play an important role, as powers of μ will be traded with angular derivatives on the initial data.

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Proof. We have that, for any μ,

$$c_{\pm}(\mu)(r) \leq \frac{C_\varphi \mu^2}{r^2}, \quad |c_{\pm}(\mu)(r)| \leq \frac{(n-1)(n-3)}{4r^2} + \frac{C_\varphi \mu^2}{r^2} \tag{3.6}$$

due to the assumption in equation (1.13) with

$$C_\varphi = \max\left(\|\frac{r^2}{\varphi^2}\|_{L^\infty((0,\infty))}, 2\|\frac{r^2\varphi'}{\varphi^2}\|_{L^\infty((0,\infty))}\right).$$

As done in [21, Section 2], the result follows from the application of Hardy inequality and interpolation in a standard way. We omit the details. □

We deduce from Proposition 3.6 the following result.

Proposition 3.9. *Assume that* (p, q, m) *is admissible, as in Definition 1.3. Then there exists a constant* $C = C(p, q, m, \varphi)$ *such that for all* $u_0 \in H_r^{1/2}$*, we have*

$$\|S_{V_\mu,n}(t)u_0\|_{L^p, W_r^{1/q-1/p,q}} \leq C|\mu|^{5/2}((\delta_{V_\mu}^+)^{-1/2} + (\delta_{V_\mu}^-)^{-1/2})\|u_0\|_{H_r^{1/2}}. \tag{3.7}$$

Proof. The estimate in equation (3.7) is a direct consequence of Proposition 3.6, Lemma 3.7 and the bounds

$$\|rV_\mu\|_{L^\infty} \lesssim_\varphi |\mu|, \quad \|r^2 c_{\pm}\|_{L^\infty} \lesssim_\varphi \mu^2.$$

The bound on c is due to equation (3.6). For the bound on V_μ, we recall that

$$V_\mu = \frac{\mu}{\varphi}$$

and thus

$$\|rV_\mu\|_{L^\infty} \leq |\mu| \; \|\frac{r}{\varphi}\|_{L^\infty}.$$

□

By interpolation, we get the following:

Corollary 3.10. *Assume that* (p, q, m) *is admissible, as in Definition 1.3, and let* $\varepsilon > 0$*. There exists* $C = C(p, q, m, \varphi, \varepsilon)$ *such that for all* $u_0 \in H_r^{1/2}$*, we have*

$$\|S_{V_\mu,n}(t)u_0\|_{L^p, W_r^{1/q-1/p,q}} \leq C|\mu|^{5/p+\varepsilon}((\delta_V^+)^{-1/2} + (\delta_V^-)^{-1/2})^{2/p+\varepsilon}\|u_0\|_{H_r^{1/2}}. \tag{3.8}$$

Proof. If $\theta := \frac{2}{p} + \varepsilon > 1$, then the result is a consequence of the estimate in equation (3.7). Otherwise, we obtain equation (3.10) by interpolating equation (3.7), taking p close enough to 2 with the standard $L^\infty H^s$ estimate. Notice that the assumption $\varepsilon > 0$ is needed because the endpoint couple is not admissible: we refer to the proofs of Lemmas 5.1 and 5.5 in [11]. □

Exploiting Proposition 2.6, we eventually get Theorem 1.4. In Proposition 2.6, we used the notation $e^{-ith_{\mu,n}}$ for $S_{V_\mu,n}(t)$ and $\sigma_n(r) = \left(\frac{r}{\varphi(r)}\right)^{(n-1)/2}$. Hence, this proposition allows to pass from Strichartz estimates for $S_{V_\mu,n}(t)$ to Strichartz estimates for the operator h_μ.

4. Strichartz estimates in the asymptotically flat case

In this section, we specialize to the 'asymptotically flat' case. First of all, we provide a slightly more precise version of Assumptions **(A2)** and in particular of the constant C. As a consequence, we are able

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to give some explicit conditions in order for the hypothesis in equation (1.13) to be satisfied. Then, after further restricting to the case $\mathbb{K}^{n-1} = \mathbb{S}^{n-1}$, we prove Theorem 1.10.

4.1. Assumptions

Let us assume that the infimum of the positive part of the spectrum of the Dirac operator on $\mathbb{K}^{n-1}$, denoted by μ_0, is strictly bigger than $\frac{1}{2}$, and that φ is asymptotically flat: in other words, that

$$\varphi = r(1+\varphi_1)$$

with the following assumption on φ_1:

- φ_1 is non-negative and bounded,
- $A_\varphi = \|\varphi_1 + r\varphi_1'\|_\infty$ and

$$B_\varphi = \|r\varphi_1' + (1+\varphi_1)(\varphi_1 + r\varphi_1')\|_\infty + \|2r^2(\varphi_1')^2 + (1+\varphi_1)r^2\varphi_1''\|_\infty$$

  are well-defined,
-
$$\max(A_\varphi, B_\varphi)\begin{cases} \leq 1 & \text{if } \mu_0 \geq 2 \\ < \min(\frac{1}{4} + \mu_0^2 - \mu_0, \frac{1}{8}) & \text{otherwise} \end{cases}.$$

4.2. Asymptotically flat manifolds are admissible

In this subsection, we prove Proposition 1.12: if $\varphi(r)$ satisfies the assumptions above, the condition in equation (1.13) is satisfied, and therefore the Strichartz estimates proved in Theorem 1.4 hold. The only thing we need to prove is the following:

Lemma 4.1. *Under the above assumptions on φ_1, we have for all $\mu \geq \mu_0$,*

$$\delta_\pm(\mu) \geq \begin{cases} \frac{1}{4} & \text{if } \mu_0 \geq 2, \\ \min(\frac{1}{4} + \mu_0^2 - \mu_0, \frac{1}{8}) - \max(A_\varphi, B_\varphi) & \text{otherwise.} \end{cases}.$$

Proof. We have

$$I(r) := \frac{1}{4} + r^2(V_\mu \pm V_\mu') = \frac{1}{4} + \frac{\mu^2 \mp \mu}{(1+\varphi_1)^2} \mp \mu\frac{\varphi_1 + r\varphi_1'}{(1+\varphi_1)^2}.$$

Therefore,

$$I(r) \geq \frac{1}{4} + \frac{\mu^2 - \mu}{(1+\varphi_1)^2} - \mu\frac{A_\varphi}{(1+\varphi_1)^2}.$$

Case 1: $\mu \geq 2$, we have since $\mu^2 - \mu \geq \mu$,

$$I(r) \geq \frac{1}{4} + \frac{\mu}{(1+\varphi_1)^2}(1 - A_\mu)$$

and since $A_\mu \leq 1$, we have $I(r) \geq \frac{1}{4}$.

Case 2: $\mu \in [1,2)$, we have since $\varphi_1 \geq 0$ and $\mu^2 - \mu \geq 0$,

$$I(r) \geq \frac{1}{4} - 2A_\varphi \geq \frac{1}{8} - A_\varphi,$$

which is positive since $A_\varphi < \frac{1}{8}$.

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Case 3: $\mu \in [\mu_0, 1)$, we have, since $\varphi_1 > 0$ and $\mu^2 - \mu \geq \mu_0^2 - \mu_0$,

$$I(r) \geq \frac{1}{4} + \mu_0^2 - \mu_0 - A_\varphi,$$

which is positive.

Set

$$Q_\pm(r) = \frac{1}{4} - r^3(2V_\mu V_\mu' \pm V_\mu'') - r^2(V_\mu^2 \pm V_\mu').$$

We have

$$Q_\pm(r) = \frac{1}{4} + \frac{\mu^2 \mp \mu}{(1+\varphi_1)^2} + \frac{\mu^2 \mp \mu}{(1+\varphi_1)^3} f(r) \mp \frac{\mu}{(1+\varphi_1)^3} g(r)$$

with

$$f(r) = r\varphi_1' + (1+\varphi_1)(\varphi_1 + r\varphi_1') \quad \text{and} \quad g(r) = 2r^2(\varphi_1')^2 + (1+\varphi_1)r^2\varphi_1''.$$

Case 1: We consider $Q_+(r)$. We have

$$Q_+(r) = \frac{1}{4} + \frac{\mu^2 + \mu}{(1+\varphi_1)^2} + \frac{\mu^2 + \mu}{(1+\varphi_1)^3} f(r) + \frac{\mu}{(1+\varphi_1)^3} g(r),$$

hence

$$Q_+(r) \geq \frac{1}{4} + \frac{\mu^2 + \mu}{(1+\varphi_1)^2}(1 - B_\varphi),$$

and since $B_\varphi \leq 1$, we have $Q_+(r) \geq \frac{1}{4}$.

Case 2: We consider $Q_-(r)$. We have

$$Q_-(r) = \frac{1}{4} + \frac{\mu^2 - \mu}{(1+\varphi_1)^2} + \frac{\mu^2 - \mu}{(1+\varphi_1)^3} f(r) - \frac{\mu}{(1+\varphi_1)^3} g(r).$$

Case 2.1: $\mu \geq 2$, we have $\mu^2 - \mu \geq \mu$, hence

$$Q_-(r) \geq \frac{1}{4} + \frac{\mu^2 - \mu}{(1+\varphi_1)^2}(1 - B_\varphi),$$

and since $B_\varphi \leq 1$, we have $Q_-(r) \geq \frac{1}{4}$.

Case 2.2: $\mu \in [1, 2)$. We have $\mu^2 - \mu \leq 2$ and $\mu \leq 2$, hence

$$Q_-(r) \geq \frac{1}{4} - 2B_\varphi.$$

Finally, case 2.3: $\mu \in [\mu_0, 1)$, we have $0 > \mu^2 - \mu \leq \mu_0^2 - \mu_0$ and $|\mu^2 - \mu| \leq 1$ hence

$$Q_-(r) \geq \frac{1}{4} + \mu_0^2 - \mu_0 - B_\varphi,$$

which concludes the proof. ◻

4.3. Local smoothing in the asymptotically flat case

From this subsection, we assume that $\mathbb{K}^{n-1}$ is $\mathbb{S}^{n-1}$. We have that the positive spectrum of the Dirac operator on the sphere is $\frac{n-1}{2}+\mathbb{N}$. Hence, we see that in dimensions higher than 5, we have that $\delta_\pm(\mu) \geq \frac{1}{4}$ for all μ in the spectrum. In any case, for a fixed φ satisfying Assumptions in (**A2**), we have that δ is uniformly bounded in μ by the below.

For μ in the spectrum of the Dirac operator on the sphere, we write $\mathcal{H}_\mu$ the space generated by

$$\left\{\begin{pmatrix}(1+i\tilde{\alpha}^0)\psi_\mu\\ 0\end{pmatrix}, \begin{pmatrix}0\\ (1-i\tilde{\alpha}^0)\psi_\mu\end{pmatrix}, \quad \mathcal{D}_{\mathbb{S}^{n-1}}\psi_\mu = \mu\psi_\mu\right\}.$$

For $0 \leq a < b$, we set

$$\mathcal{H}_{a,b} = \bigoplus_{|\mu|\in[a,b]} \mathcal{H}_\mu$$

and $p_{a,b}$ the orthogonal projection onto $L^2_r \otimes \mathcal{H}_{a,b}$.

We recall that $\sigma_n^{-1}\mathcal{D}_\Sigma\sigma_n$ is entirely described by the $h_{\mu,n}$ and thus commute with $p_{a,b}$. We write $S_n(t)$ the flow of

$$i\partial_t - \sigma_n^{-1}\mathcal{D}_\Sigma\sigma_n = 0.$$

We deduce the following proposition.

Proposition 4.2. *Let $u_0 \in H^{1/2}(\mathbb{R}^n)$. We have*

$$\|r^{-1}p_{a,b}S_n(t)u_0\|_{L^2(\mathbb{R}\times\mathbb{R}^n)} \lesssim_{m,\varphi,n} b^{1/2}\|p_{a,b}u_0\|_{H^{1/2}(\mathbb{R}^n)}.$$

Proof. Let $u_{0,\mu}$ be the orthogonal projection of u_0 over $L^2_r \otimes \mathcal{H}_\mu$ and $u_\mu = S_n(t)u_{0,\mu}$. Because the orthogonal projection over $L^2_r \otimes \mathcal{H}_\mu$ and $S_n(t)$ commute, we get

$$\|r^{-1}p_{a,b}S_n(t)u_0\|^2_{L^2(\mathbb{R}\times\mathbb{R}^n)} = \sum_{|\mu|\in[a,b]} \|r^{-1}u_\mu\|^2_{L^2(\mathbb{R}\times\mathbb{R}^n)}.$$

From Proposition 3.5, we have

$$\|r^{-1}u_\mu\|_{L^2(\mathbb{R}\times\mathbb{R}^n)} \leq 3(C_{c_+(\mu)} + C_{c_-(\mu)})\|u_{0,\mu}\|_{\mathcal{H}_{c_+(\mu),c_-(\mu),m}},$$

where, by abuse of notation, we identified $u_{0,\mu}$ with

$$\sum_j \frac{f_j}{\sqrt{2}}\begin{pmatrix}(1+i\tilde{\alpha}^0)\psi_{\mu,j}\\ 0\end{pmatrix} + \frac{g_j}{\sqrt{2}}\begin{pmatrix}0\\ (1-i\tilde{\alpha}^0)\psi_{\mu,j}\end{pmatrix},$$

where the (finite) family $(\psi_{\mu,j})_j$ is an orthonormal basis of the eigenspace of $\mathcal{D}_{\mathbb{S}^{n-1}}$ associated to μ, and we identified $\|u_{0,\mu}\|^2_{\mathcal{H}_{c_+(\mu),c_-(\mu),m}}$ with

$$\sum_j \left\|\begin{pmatrix}f_j\\ g_j\end{pmatrix}\right\|^2_{\mathcal{H}_{c_+(\mu),c_-(\mu),m}}.$$

From Lemma 3.7, we have for all j

$$\left\|\begin{pmatrix}f_j\\ g_j\end{pmatrix}\right\|_{\mathcal{H}_{c_+(\mu),c_-(\mu),m}} \lesssim_{m,\varphi} \sqrt{|\mu|}\left\|\begin{pmatrix}f_j\\ g_j\end{pmatrix}\right\|_{H^{1/2}(\mathbb{R}^n)},$$

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from which we deduce

$$\|u_{0,\mu}\|_{\mathcal{H}_{c+(\mu),c-(\mu),m}} \lesssim_{m,\varphi} \sqrt{|\mu|}\|u_{0,\mu}\|_{H^{1/2}(\mathbb{R}^n)} \leq \sqrt{b}\|u_{0,\mu}\|_{H^{1/2}(\mathbb{R}^n)}.$$

We conclude by using the fact that $C_{c_+(\mu)}$ and $C_{c_-(\mu)}$ are uniformly bounded in μ. □

4.4. Restricted Strichartz estimates in the asymptotically flat case

In this subsection, we prove the following proposition.

Proposition 4.3. *Let $0 \leq a < b$, and let m, p, q be admissible. We have, for all $u_0 \in H^{1/2}(\mathbb{R}^n)$ and all $\varepsilon > 0$,*

$$\|p_{a,b}S_n(t)u_0\|_{L^p(\mathbb{R},W^{s,q}(\mathbb{R}^n))}\begin{cases} \lesssim_{m,\varphi,n,\varepsilon,p,q} b^{5/p+\varepsilon}\|p_{a,b}u_0\|_{H^{1/2}} & \text{if } n=3, m=0 \\ \lesssim_{m,\varphi,n,p,q} b^{5/p}\|p_{a,b}u_0\|_{H^{1/2}} & \text{otherwise}, \end{cases}$$

with $s = \frac{1}{q} - \frac{1}{p}$.

Proof. We prove that

$$\|p_{a,b}S_n(t)u_0\|_{L^p(\mathbb{R},W^{s,q}(\mathbb{R}^n))} \lesssim_{m,\varphi,n,} b^{5/2}\|p_{a,b}u_0\|_{H^{1/2}}$$

for all admissible triplets (m, p, q) and conclude by interpolation.

First, we have

$$\sigma_n^{-1}\mathcal{D}_\Sigma\sigma_n = \mathcal{D}_{\mathbb{R}^n} + \mathcal{V}$$

with $\mathcal{V}$ the operator

$$\mathcal{V} = \left(\frac{1}{\varphi} - \frac{1}{r}\right)\begin{pmatrix} 0 & \mathcal{D}_{\mathbb{S}^{n-1}} \\ \mathcal{D}_{\mathbb{S}^{n-1}} & 0 \end{pmatrix}.$$

Writing $u = S_n(t)u_0$, we get that u satisfies

$$\partial_t^2 u + (\mathcal{D}_{\mathbb{R}^n} + \mathcal{V})^2 u = 0$$

with initial data $u(t=0) = u_0$ and $\partial_t u(t=0) =: u_1 = -i(\mathcal{D}_{\mathbb{R}^n} + \mathcal{V})u_0$.

We have

$$(\mathcal{D}_{\mathbb{R}^n} + \mathcal{V})^2 = \mathcal{D}_{\mathbb{R}^n}^2 + \mathcal{W} = m^2 - \Delta_{\mathbb{R}^n} + \mathcal{W}$$

with

$$\mathcal{W} = \{\mathcal{V}, \mathcal{D}_{\mathbb{R}^n}\} + \mathcal{V}^2.$$

By the Rodnianski-Schlag argument that we previously used, we get

$$\|p_{a,b}u\|_{L^p,W^{s,q}(\mathbb{R}^n)} \lesssim_{n,p,q} \|p_{a,b}u_0\|_{H^{1/2}(\mathbb{R}^n)} + \|p_{a,b}u_1\|_{H^{-1/2}(\mathbb{R}^n)} + \|r\mathcal{W}p_{a,b}u\|_{L^2(\mathbb{R}^{n+1})}.$$

By the commutativity of $p_{a,b}$ and $\mathcal{D}_{\mathbb{R}^n} + \mathcal{V}$, we get

$$\|p_{a,b}u_1\|_{H^{-1/2}} = \|(\mathcal{D}_{\mathbb{R}^n} + \mathcal{V})p_{a,b}u_0\|_{H^{-1/2}}$$

and since $r(\frac{1}{\varphi} - \frac{1}{r})$ is bounded, by Hardy's inequality, we get

$$\|p_{a,b}u_1\|_{H^{-1/2}} \lesssim_{n,\varphi} \|p_{a,b}u_0\|_{H^{1/2}}.$$

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For the other part, we use that

$$\|r\mathcal{W}p_{a,b}u\|_{L^2(\mathbb{R}^{n+1})} \le \|p_{a,b}r\mathcal{W}rp_{a,b}\|_{L^2\to L^2}\|r^{-1}p_{a,b}u\|_{L^2(\mathbb{R}^{n+1})}.$$

It remains to use Proposition 4.2 and prove that $p_{a,b}r\mathcal{W}rp_{a,b}$ is a bounded operator from $L^2(\mathbb{R}^{n+1})$ to itself and compute the dependence of its norm in a, b to conclude.

Because multiplication by a radial function and the Dirac operator on the sphere commute, we get that

$$p_{a,b}r\mathcal{V}^2rp_{a,b} = \left(\frac{r}{\varphi}-1\right)^2\begin{pmatrix} p_{a,b}\mathcal{D}^2_{\mathbb{S}^{n-1}}p_{a,b} & 0 \\ 0 & p_{a,b}\mathcal{D}^2_{\mathbb{S}^{n-1}}p_{a,b}\end{pmatrix}$$

and we deduce

$$\|p_{a,b}r\mathcal{V}^2rp_{a,b}\|_{L^2\to L^2} \le \left\|\left(\frac{r}{\varphi}-1\right)^2\right\|_\infty b^2,$$

which is finite because of the assumptions on φ.

What is more, we have

$$\mathcal{D}_{\mathbb{R}^n} = \begin{pmatrix} m & i\tilde{\alpha}^0\left(\partial_r + \frac{n-1}{2r}\right) + \frac{1}{r}\mathcal{D}_{\mathbb{S}^{n-1}} \\ i\tilde{\alpha}^0\left(\partial_r + \frac{n-1}{2r}\right) + \frac{1}{r}\mathcal{D}_{\mathbb{S}^{n-1}} & -m \end{pmatrix}.$$

We deduce

$$\{\mathcal{D}_{\mathbb{R}^n}, \mathcal{V}\} = \begin{pmatrix} \mathcal{L} & 0 \\ 0 & \mathcal{L}\end{pmatrix}$$

with

$$\mathcal{L} = \{i\tilde{\alpha}^0\left(\partial_r + \frac{n-1}{2r}\right) + \frac{1}{r}\mathcal{D}_{\mathbb{S}^{n-1}}, \left(\frac{1}{\varphi}-\frac{1}{r}\right)\mathcal{D}_{\mathbb{S}^{n-1}}\}.$$

We have that $i\tilde{\alpha}^0$ and $\mathcal{D}_{\mathbb{S}^{n-1}}$ anticommute, that ∂_r and $\mathcal{D}_{\mathbb{S}^{n-1}}$ commute and that the multiplication by a radial function commutes with $\mathcal{D}_{\mathbb{S}^{n-1}}$. Hence we get

$$\mathcal{L} = i\tilde{\alpha}^0\mathcal{D}_{\mathbb{S}^{n-1}}\left[\partial_r + \frac{n-1}{2r}, \frac{1}{\varphi}-\frac{1}{r}\right] + 2\left(\frac{1}{\varphi}-\frac{1}{r}\right)\frac{1}{r}\mathcal{D}^2_{\mathbb{S}^{n-1}}.$$

We deduce

$$p_{a,b}r\{\mathcal{V}, \mathcal{D}_{\mathbb{R}^n}\}rp_{a,b} = \begin{pmatrix} \mathcal{L}_{a,b} & 0 \\ 0 & \mathcal{L}_{a,b}\end{pmatrix}$$

with

$$\mathcal{L}_{a,b} = ip_{a,b}\tilde{\alpha}^0\mathcal{D}_{\mathbb{S}^{n-1}}p_{a,b}r^2\partial_r\left(\frac{1}{\varphi}-\frac{1}{r}\right) + 2r\left(\frac{1}{\varphi}-\frac{1}{r}\right)p_{a,b}\mathcal{D}^2_{\mathbb{S}^{n-1}}p_{a,b}.$$

Because

$$r^2\partial_r\left(\frac{1}{\varphi}-\frac{1}{r}\right) = \frac{\varphi_1}{1+\varphi_1} - \frac{r\varphi_1'}{(1+\varphi_1)^2}$$

belongs to L^∞, and so does $r\left(\frac{1}{\varphi} - \frac{1}{r}\right) = \frac{1}{1+\varphi_1} - 1$, we get

$$\|p_{a,b} r\{\mathcal{V}, \mathcal{D}_{\mathbb{R}^n}\} r p_{a,b}\|_{L^2 \to L^2} \lesssim_\varphi b^2.$$

This concludes the proof. □

4.5. Setup for the Littlewood-Paley argument

In this subsection, we draw a link between the spherical harmonics and the eigenfunctions of the Dirac operator on the sphere.

Proposition 4.4. *Let π_j be the orthogonal projection on $\mathcal{S}_j \otimes L^2_r \otimes \mathbb{C}^M$, where $\mathcal{S}_j$ are the spherical harmonics of degree in $[2^j, 2^{j+1})$, and let $u \in L^2(\mathbb{R}^n, \mathbb{C}^M)$. We have*

$$\pi_j u = \pi_j p_{a_j, b_j} u$$

with $a_j = \frac{n-1}{2} + 2^j - 1$ and $b_j = \frac{n-1}{2} + 2^{j+1}$.

Before proving this proposition, we prove the following short lemma.

Lemma 4.5. *We have for all μ in the spectrum of the Dirac operator on the sphere $\mathbb{S}^{n-1}$,*

$$\mathcal{H}_\mu \subseteq (\mathcal{S}_{|\mu| - \frac{n-1}{2}} \oplus \mathcal{S}_{|\mu| - \frac{n-1}{2} + 1}) \otimes \mathbb{C}^M.$$

Proof. Let ψ_μ be an eigenfunction of $\mathcal{D}_{\mathbb{S}^{n-1}}$ with eigenvalue μ, and write

$$\Psi^+_\mu = \begin{pmatrix} (1 + i\tilde{\alpha}^0)\psi_\mu \\ 0 \end{pmatrix}, \quad \Psi^-_\mu = \begin{pmatrix} 0 \\ (1 - i\tilde{\alpha}^0)\psi_\mu \end{pmatrix}.$$

We have

$$\mathcal{D}_{\mathbb{R}^n} \Psi^+_\mu = \left(\mu - \frac{n-1}{2}\right) \frac{1}{r} \Psi^-_\mu$$

and thus

$$-\Delta_{\mathbb{R}^n} \Psi^+_\mu = \mathcal{D}^2_{\mathbb{R}^n} \Psi^+_\mu = \mathcal{D}_{\mathbb{R}^n} \left(\mu - \frac{n-1}{2}\right) \frac{1}{r} \Psi^-_\mu = \left(\mu - \frac{n-1}{2}\right)\left(\mu + \frac{n-1}{2} - 1\right) \frac{1}{r^2} \Psi^+_\mu.$$

Because Ψ^+_μ does not depend on r, we deduce that it is a spherical harmonics of degree $\mu - \frac{n-1}{2}$ if $\mu > 0$ and $-\mu - \frac{n-1}{2} + 1$ otherwise.

The same type of computation yields

$$-\Delta_{\mathbb{R}^n} \Psi^-_\mu = \left(\mu + \frac{n-1}{2}\right)\left(\mu - \frac{n-1}{2} + 1\right) \frac{1}{r^2} \Psi^-_\mu,$$

hence Ψ^-_μ is a spherical harmonics of degree $\mu - \frac{n-1}{2} + 1$ if $\mu > 0$ and $-\mu - \frac{n-1}{2}$ otherwise.

In other words,

$$\mathcal{H}_\mu \subseteq (\mathcal{S}_{|\mu| - \frac{n-1}{2}} \oplus \mathcal{S}_{|\mu| - \frac{n-1}{2} + 1}) \otimes \mathbb{C}^M.$$ □

Proof of Proposition 4.4. We have

$$\pi_j u = \sum_\mu \pi_j u_\mu,$$

where u_μ is the orthogonal projection of u over $\mathcal{H}_\mu \otimes L^2_r$. If $|\mu| > b_j$, then

$$|\mu| - \frac{n-1}{2} > 2^{j+1},$$

hence u_μ is a combination of spherical harmonics of degree higher than 2^{j+1}, hence $\pi_j u_\mu = 0$.

If $|\mu| < a_j$, then

$$|\mu| - \frac{n-1}{2} + 1 < 2^j,$$

hence u_μ is a combination of spherical harmonics of degree lesser than 2^j; we have $\pi_j u_\mu = 0$, and therefore

$$\pi_j u = \sum_{|\mu| \in [a_j, b_j]} \pi_j u_\mu = \pi_j p_{a_j,b_j} u.$$

□

4.6. Proof of Theorem 1.10.

As done in [11], by relying on Littlewood-Paley theory on the sphere, we are able to prove Strichartz estimates for the Dirac equation with general initial conditions in the setting of spherically symmetric manifolds. As the proof is very similar, we omit some details.

Proposition 4.6. *Let m, p, q be admissible. Let $a, b > 0$ be such that*

$$\frac{1}{2a} + \frac{5}{pb} < 1.$$

We have for all $u_0 \in H^{a,b}(\mathbb{R}^n)$,

$$\|S_n(t)u_0\|_{L^p(\mathbb{R}, W^{s,q}(\mathbb{R}^n))} \lesssim_{n,\varphi,m,p,q,a,b} \|u_0\|_{H^{a,b}}.$$

Proof. We have by the Littlewood-Paley theory ($q \in [2, \infty)$)

$$\|S_n(t)u_0\|^2_{L^p(\mathbb{R}, W^{s,q}(\mathbb{R}^n))} \lesssim \sum_j \|\pi_j S_n(t)u_0\|^2_{L^p(\mathbb{R}, W^{s,q}(\mathbb{R}^n))}.$$

By Proposition 4.4, we have

$$\|\pi_j S_n(t)u_0\|^2_{L^p(\mathbb{R}, W^{s,q}(\mathbb{R}^n))} = \|\pi_j p_{a_j,b_j} S_n(t)u_0\|^2_{L^p(\mathbb{R}, W^{s,q}(\mathbb{R}^n))}.$$

Again by Littlewood-Paley theory, we have

$$\|\pi_j p_{a_j,b_j} S_n(t)u_0\|^2_{L^p(\mathbb{R}, W^{s,q}(\mathbb{R}^n))} \lesssim \|p_{a_j,b_j} S_n(t)u_0\|^2_{L^p(\mathbb{R}, W^{s,q}(\mathbb{R}^n))}.$$

We apply Proposition 4.3, and we get

$$\|\pi_j p_{a_j,b_j} S_n(t)u_0\|^2_{L^p(\mathbb{R}, W^{s,q}(\mathbb{R}^n))} \lesssim b_j^{10/p+\varepsilon} \|p_{a_j,b_j} u_0\|^2_{H^{1/2}}$$

with $\varepsilon > 0$ if $m = 0$ and $n = 3$ (and 0 otherwise). From the inequality

$$xy \le x^c + y^d$$

for any $x, y \in [1, \infty)$ and $\frac{1}{c} + \frac{1}{d} \le 1$, we deduce

$$\|\pi_j p_{a_j,b_j} S_n(t)u_0\|^2_{L^p(\mathbb{R},W^{s,q}(\mathbb{R}^n))} \lesssim b_j^{(10/p+\varepsilon)d}\|p_{a_j,b_j}u_0\|^2_{L^2} + \|p_{a_j,b_j}u_0\|^2_{H^{c/2}}.$$

Because $[a_j, b_j]$ is localized around 2^j, we get

$$\|S_n(t)u_0\|_{L^p(\mathbb{R},W^{s,q}(\mathbb{R}^n))} \lesssim \|u_0\|_{H^{c/2,(5/p+\varepsilon/2)d}}.$$

Setting $a = c/2$ and $b = (5/p + \varepsilon/2)d$, the condition on c and d becomes

$$\frac{1}{2a} + \frac{5/p + \varepsilon/2}{b} \le 1,$$

which is equivalent to the hypothesis of Proposition 4.6 by discussing the possible values of ε. □

We now extend Lemmas 2.4 and 2.7 to include the angular dependence.

Lemma 4.7. *The multiplication by σ_n is an isometry from $L^2(\mathbb{R}^n)$ to $L^2(\Sigma)$. The multiplication by σ_n is an isomorphism from $H^1(\mathbb{R}^n)$ to $H^1(\Sigma)$, the immediate consequence of which is that for all $a \in [0, 1]$, $b \in \mathbb{R}$, the multiplication by σ_n is an isomorphism from $H^{a,b}(\mathbb{R}^n)$ to $H^{a,b}(\Sigma)$.*

Proof. The fact that the multiplication by σ_n is an isometry from $L^2(\mathbb{R}^n)$ to $L^2(\Sigma)$ is already present in Lemma 2.4.

We have for all $F \in \mathcal{C}^\infty(\Sigma, \mathbb{C}^M)$, writing

$$F = \begin{pmatrix} f \\ g \end{pmatrix}$$

with f and g in $\mathcal{C}^\infty(\Sigma, \mathbb{C}^{M/2})$

$$h^{ij}\langle D_i F, D_j F\rangle_{\mathbb{C}^M} = \langle \partial_r F, \partial_r F\rangle_{\mathbb{C}^M} + \frac{1}{\varphi^2}\Big(\tilde{h}^{ij}\langle \mathbb{D}_i^\varphi f, \mathbb{D}_j^\varphi f\rangle_{\mathbb{C}^{M/2}} + \tilde{h}^{ij}\langle \mathbb{D}_i^\varphi g, \mathbb{D}_j^\varphi g\rangle_{\mathbb{C}^{M/2}}\Big),$$

where $\mathbb{D}_j^\varphi = \tilde{D}_j + 2i\varphi'\tilde{e}^a_{\ j}\tilde{\Sigma}_{0,a}$ with $\tilde{D}$ the covariant derivatives for spinors on the sphere and $\tilde{h}$ is the metric of the sphere.

The fact that

$$\|\langle \partial_r(\sigma_n F), \partial_r(\sigma_n F)\rangle_{\mathbb{C}^M}\|_{L^1(\Sigma)} \sim \|\partial_r F\|^2_{L^2(\mathbb{R}^n)}$$

is due to Lemma 2.4.

We have

$$\mathbb{D}_j^\varphi = \mathbb{D}_j^r + 2i(\varphi' - 1)\tilde{e}^a_{\ j}\tilde{\Sigma}_{0,a}.$$

Thanks to the Cauchy-Schwarz inequality applied to the scalar product $x, y \mapsto h^{ij}x_i y_j$, we get

$$\sqrt{\frac{1}{\varphi^2}\tilde{h}^{ij}\langle \mathbb{D}_i^\varphi, \mathbb{D}_j^\varphi f\rangle_{\mathbb{C}^{M/2}}} \lesssim \frac{r}{\varphi}\sqrt{\frac{1}{r^2}\tilde{h}^{ij}\langle \mathbb{D}_i^r, \mathbb{D}_j^r f\rangle_{\mathbb{C}^{M/2}}} + \frac{|\varphi' - 1|}{\varphi}\sqrt{\langle f, f\rangle_{\mathbb{C}^{M/2}}} \tag{4.1}$$

and conversely

$$\sqrt{\frac{1}{r^2}\tilde{h}^{ij}\langle \mathbb{D}_i^r, \mathbb{D}_j^r f\rangle_{\mathbb{C}^{M/2}}} \lesssim \frac{\varphi}{r}\sqrt{\frac{1}{\varphi^2}\tilde{h}^{ij}\langle \mathbb{D}_i^\varphi, \mathbb{D}_j^\varphi f\rangle_{\mathbb{C}^{M/2}}} + \frac{|\varphi' - 1|}{r}\sqrt{\langle f, f\rangle_{\mathbb{C}^{M/2}}}.$$

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Because $\mathbb{D}$ and σ_n commute, we get

$$\mathbb{D}^{\varphi}_j(\sigma_n f) = \sigma_n \mathbb{D}^r_j f + \sigma_n 2i(\varphi' - 1)\tilde{e}^a_j \tilde{\Sigma}_{0,a}.$$

To ensure that

$$\Big\| \frac{1}{\varphi^2} \tilde{h}^{ij} \langle \mathbb{D}^{\varphi}_i \sigma_n f, \mathbb{D}^{\varphi}_j \sigma_n f \rangle_{\mathbb{C}^{M/2}} \Big\|_{L^1(\Sigma)}$$

it is thus sufficient to prove that $\frac{\varphi}{r}$, $\frac{r}{\varphi}$ and $\frac{\varphi'-1}{\varphi}$ are bounded. But $\varphi = r(1+\varphi_1)$ with φ_1 non-negative, bounded, $O(r)$ in 0 and thus that φ_1' is bounded, hence

$$\frac{\varphi}{r} = 1 + \varphi_1, \qquad \frac{r}{\varphi} = \frac{1}{1+\varphi_1}, \qquad \frac{\varphi' - 1}{\varphi} = \frac{\varphi_1}{r(1+\varphi_1)} + \frac{\varphi_1'}{1+\varphi_1}$$

are bounded. □

Lemma 4.8. *The multiplication by σ_n is a continuous operator from $L^p(\mathbb{R}, W^{s,q}(\mathbb{R}^n))$ to $\sigma_n^{1-2/q} L^p(\mathbb{R}, W^{s,q}(\Sigma))$ for any $p \in [1,\infty]$, $q \in (1,\infty)$, $s \in [-1,1]$.*

Proof. As in Lemma 2.7, we reduce our proof to the proof of, for all $q \in (1,\infty)$,

1. the multiplication by $\sigma_n^{2/q}$ is an isometry from $L^q(\mathbb{R}^n)$ to $L^q(\Sigma)$,
2. the multiplication by $\sigma_n^{2/q}$ is continuous from $W^{1,q}(\mathbb{R}^n)$ to $W^{1,q}(\Sigma)$,
3. the multiplication by $\sigma_n^{-2/q}$ is continuous from $W^{1,q}(\Sigma)$ to $W^{1,q}(\mathbb{R}^n)$.

(1) The proof of (1) is similar to what we have already done in the proof of Lemma 2.7.

(2) With the same notations as in the proof of Lemma 4.7, and keeping in mind (2) in Lemma 2.7, it remains to prove (with a slight abuse of notation) that for all $f \in W^{1,q}(\mathbb{R}^n)$,

$$\Big\| \sqrt{\frac{1}{\varphi^2} \tilde{h}^{ij} \langle \mathbb{D}^{\varphi}_i (\sigma_n^{2/q} f), \mathbb{D}^{\varphi}_j (\sigma_n^{2/q} f) \rangle_{\mathbb{C}^{M/2}}} \Big\|_{L^q(\Sigma)} \lesssim \|f\|_{W^{1,q}(\mathbb{R}^n)}.$$

But because of (1) and the fact that $\sigma_n^{2/q}$ and $\mathbb{D}^{\varphi}$ commute, it sufficient to prove that

$$\Big\| \sqrt{\frac{1}{\varphi^2} \tilde{h}^{ij} \langle \mathbb{D}^{\varphi}_i f, \mathbb{D}^{\varphi}_j f \rangle_{\mathbb{C}^{M/2}}} \Big\|_{L^q(\mathbb{R}^n)} \lesssim \|f\|_{W^{1,q}(\mathbb{R}^n)}.$$

We now use the inequality in equation (4.1) and the fact that $\frac{r}{\varphi}$ and $\frac{\varphi'-1}{\varphi}$ are bounded to get the result.

(3) Similar to (2). □

Therefore, combining Proposition 4.6 with Lemma 4.8 eventually yields the Proof of Theorem 1.10.

A. Comments on admissible manifolds

It is natural to ask whether the conditions in equation (1.13) are fulfilled by other natural choices of the function $\varphi(r)$, as for example $\varphi(r) = \sinh(r)$ (which corresponds to *hyperbolic spaces*), or $\varphi(r) = r + r^2 + \cdots + r^p$ with $p > 2$ (manifolds with polynomial growth). It turns out that with both these choices, the conditions in equation (1.13) are only satisfied for large r; more precisely, the following result holds:

Proposition A.1. *Let $(\mathcal{M}, g)$ defined by $\mathcal{M} = \mathbb{R}_t \times \Sigma$, with (Σ, σ) a warped product manifold with the metric given by equation (1.5), and let $\varphi(r) = \sinh(r)$ or $\varphi(r) = r + r^2 + \cdots + r^p$ with $p \in \mathbb{N}$ and $p > 2$. Then the condition in equation (1.13) is not satisfied.*

Proof. It is quite immediate to see that condition

$$4r^2 V_\mu + 1 > 0 \Leftrightarrow 4r^2\mu^2 \pm 4r^2\mu\cosh(r) + \sinh(r)^2 > 0$$

is true for any μ if and only if

$$\left(\frac{\varphi'(r)}{\varphi(r)}\right)^2 < \frac{1}{r^2},$$

and this last condition is not satisfied by the choices $\varphi(r) = \sinh(r)$ or $\varphi(r) = r + r^2 + \cdots + r^p$. We omit the details. □

Remark A.2. As a matter of fact, it might be possible to prove that with the choices of $\varphi(r)$ of Proposition A.1, the condition in equation (1.13) is actually satisfied for r larger than a sufficiently large $R = R(\mu)$; as a consequence, it would be tempting to consider manifolds that are flat inside some balls, and then present different asymptotic behaviors (like, for instance, asymptotically hyperbolic manifolds). These cases would correspond to choosing a function $\varphi(r) \in C^\infty(\mathbb{R}^+)$ that takes the form

$$\varphi(r) = \begin{cases} r & \text{if } r \leq R, \\ \psi(r) & \text{if } R \leq r \leq 2R, \\ \sinh(r) & \text{if } r > 2R \end{cases} \tag{A.1}$$

(and analogous in the case of manifolds with polynomial growth). The existence of such a function is quite standard; on the other hand, we are not able to show that the condition in equation (1.13) is satisfied everywhere. In any case, the fact that the quantity R will depend on μ makes the analysis in these cases not as relevant from a geometrical point of view, and therefore we prefer to leave the study of these other geometries to future investigations.

B. Comments on the construction of the Dirac operators

Before stating anything, let us be precise that our aim here is to provide tools to do the computations of Section 2. We do not pretend to provide precise geometrical definitions. In particular, we assume that we have chosen a set of coordinates and present different notions within this choice of coordinates. We sometimes simplify definitions in a way that fits our context.

We recall that a Lorentzian manifold of dimension $n + 1$ is a differentiable manifold $\mathcal{M}$ equipped with a metric tensor g of signature $(1, n)$. In the rest of this discussion, we set $\mathcal{M}, g$ to be a Lorentzian manifold that admits an orientation and a causality: that is, 'a time arrow'. This manifold is said to have a spin structure if there exists a matrix bundle $e_\mu^{\ a}$ (μ and a belong to $\mathbb{N} \cap [0, n]$) such that

$$e_\mu^{\ a}\eta_{ab}e_\nu^{\ b} = g_{\mu\nu}$$

and that is coherent with the orientation and causality. This matrix bundle fixes a frame bundle for the tangent space of $\mathcal{M}$. It relates this tangent space to the Minkowski tangent space. It is called a vierbein (as it was originally used in dimension 1 + 3) or sometimes vielbein. Notice that this bundle is not uniquely defined: indeed, if L_a^b is a Lorentz transform – that is, a linear map belonging to $SO_0(1, n)$ (the connex component of the identity of $SO(1, n)$) – then

$$f_\mu^{\ a} = L_b^a e_\mu^{\ b}$$

satisfies the same equation as $e_\mu^{\ a}$, and the coherence with orientation and causality is due to the fact that Lorentz transforms preserve orientation and causality.

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The spin connection is given by the formula

$$\omega_\mu^{\ ab} = e_\nu^{\ a}\Gamma^\nu_{\ \sigma\mu}e^{\sigma b} + e_\nu^{\ a}\partial_\mu e^{\nu b},$$

where

$$\Gamma^\nu_{\sigma\mu} = \frac{1}{2}g^{\rho\nu}\left[\partial_\sigma g_{\rho\mu} + \partial_\mu g_{\rho\sigma} - \partial_\rho g_{\sigma\mu}\right]$$

are the Christoffel symbols (or affine connection). Notice that $\Gamma^\nu_{\ \mu\sigma} = \Gamma^\nu_{\ \sigma\mu}$, and also $\omega_\mu^{\ ab} = -\omega_\mu^{\ ba}$ (long but straightforward computations).

One important property of the spin connection is that it satisfies the Leibniz rule

$$de^a + \omega^a_{\ b} \wedge e^b = 0,$$

where d is the exterior derivative, $\wedge$ is the exterior product and

$$\omega^a_{\ b} = \omega_{\mu\ b}^{\ a}dx^\mu, \quad e^a = e_\mu^{\ a}dx^\mu.$$

In terms of coordinates, this means

$$H_{\mu\nu}^{\ \ a} - H_{\nu\mu}^{\ \ a} = 0$$

with

$$H_{\mu\nu}^{\ \ a} = \partial_\mu e_\nu^{\ a} + \omega_\nu^{\ ab}e_{\mu c}.$$

This is called the Leibniz rule because ω is used to define (covariant) derivatives, and this property ensures that these derivatives satisfy the Leibniz rule. One can compare with the requirement that the Christoffel symbols satisfy

$$\partial_\mu g_{\nu\sigma} - g_{\rho\sigma}\Gamma^\rho_{\ \nu\mu} - g_{\rho\nu}\Gamma^\rho_{\ \sigma\mu}.$$

In the same way, one can compute ω using that it is skew-symmetric in a and b and satisfies the Leibniz rule.

To define the Dirac operator, let us recall that it models the free evolution of a pair electron-positron. The behavior of this pair of particles is dictated by a (specific) linear representation U of the Lorentz group $SO_0(1, n)$. Roughly speaking, U dictates how the representation of a pair of electron-positron changes when the referential is changed under the action of a Lorentz transform. More pragmatically, this means the Dirac operator should commute with U. The Dirac operator is given, in analogy with the flat case, by

$$\mathcal{D} = i\underline{\gamma}^\mu D_\mu,$$

where $\underline{\gamma}^\mu = e^\mu_{\ a}\gamma^a$, with γ^a the standard gamma matrices described in Section 2, and D_μ is the 'covariant' derivative for Dirac spinors. When letting a Lorentz transform L act on $\mathcal{D}$, we change the vierbein e into $f_\mu^{\ a} = L^a_{\ b}e_\mu^{\ b}$. This changes $\mathcal{D}$ into $\mathcal{D}'$. For the Dirac equation to be invariant under the action of Lorentz transforms, we thus require

$$\mathcal{D}'U(L) = U(L)\mathcal{D}.$$

This condition can be absorbed into the covariant derivative and becomes

$$D'_\mu U(L) = U(L)D_\mu.$$

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This is where the spin connection intervenes. For D_μ to be a derivative satisfying the above condition, one sets

$$D_\mu = \partial_\mu + \omega_\mu^{ab}\Sigma_{ab},$$

where the Σ_{ab} are the generators of U. In our case, we have

$$\Sigma_{ab} = -\frac{1}{8}[\gamma_a, \gamma_b].$$

This is a generic way to define covariant derivatives for spin particles, and this concludes our heuristic about the Dirac operator.

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