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**A BOUND ON THE EXPECTED NUMBER
OF RANDOM ELEMENTS TO GENERATE A FINITE GROUP
ALL OF WHOSE SYLOW SUBGROUPS ARE d -GENERATED.**

ANDREA LUCCHINI

ABSTRACT. Assume that all the Sylow subgroups of a finite group G can be generated by d elements. Then the expected number of elements of G which have to be drawn at random, with replacement, before a set of generators is found, is at most $d + \eta$ with $\eta \sim 2.875065$.

1. INTRODUCTION

In 1989, R. Guralnick [5] and the author [11] independently proved that if all the Sylow subgroups of a finite group G can be generated by d elements, then the group G itself can be generated by $d + 1$ elements. The aim of this paper is to obtain a probabilistic version of this result.

Let G be a nontrivial finite group and let $x = (x_n)_{n \in \mathbb{N}}$ be a sequence of independent, uniformly distributed G -valued random variables. We may define a random variable τ_G by $\tau_G = \min\{n \geq 1 \mid \langle x_1, \dots, x_n \rangle = G\}$. We denote by $e(G)$ the expectation $E(\tau_G)$ of this random variable. In other word $e(G)$ is the expected number of elements of G which have to be drawn at random, with replacement, before a set of generators is found. Some estimations of the value $e(G)$ have been obtained in [12]. The main result of this paper is:

Theorem 1. *Let G be a finite group. If all the Sylow subgroups of G can be generated by d elements, then*

$$e(G) \leq d + \eta \quad \text{with} \quad \eta = \frac{5}{2} + \sum_{p \geq 3} \frac{1}{(p-1)^2} < 3.$$

From an accurate estimation of $\sum_p (p-1)^{-2}$ given in [1], it follows $\eta \sim 2.875065\dots$. This result is near to be best possible. For any prime p , let $A_{p,d}$ be the elementary abelian p -group of rank d and for any positive integer n consider $A_{n,d} = \prod_{p \leq n} A_{p,d}$. C. Pomerance [13] proved that $\lim_{n \rightarrow \infty} e(A_{n,d}) = d + \sigma$, where $\sigma \sim 2.11846\dots$ (the exact value of σ can be explicitly described in terms of the Riemann zeta-function).

If G is a p -subgroup of $\text{Sym}(n)$, then G can be generated by $\lfloor n/p \rfloor$ elements (see [8]), so Theorem 2 has the following consequence:

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Corollary 2. *If G is a permutation group of degree n , then $e(G) \leq \lfloor n/2 \rfloor + \eta$.*

A profinite group G , being a compact topological group, can be seen as a probability space. If we denote with μ the normalized Haar measure on G , so that $\mu(G) = 1$, the probability that k random elements generate (topologically) G is defined as

$$P_G(k) = \mu(\{(x_1, \dots, x_k) \in G^k \mid \langle x_1, \dots, x_k \rangle = G\}),$$

where μ denotes also the product measure on G^k . The definition of $e(G)$ can be extended to finitely generated profinite groups. In particular $e(G) = \sup_{N \in \mathcal{N}} e(G/N)$, being \mathcal{N} the set of the open normal subgroups of G (see for example [12, Section 6]), hence Theorem 2 remains true for profinite groups: *if all the Sylow subgroups of a profinite group G are (topologically) d -generated, then G is (topologically) $(d+1)$ -generated and $e(G) \leq d + \eta$.* A profinite group G is said to be positively finitely generated, PFG for short, if $P_G(k)$ is positive for some natural number k , and the least such natural number is denoted by $d_P(G)$. Not all finitely generated profinite groups are PFG (for example if \hat{F}_d is the free profinite group of rank $d \geq 2$ then $P_{\hat{F}_d}(t) = 0$ for every $t \geq d$, see [7]): if G is not PFG we set $d_P(G) = \infty$. It can be easily seen that $e(G) = \sum_{n \geq 0} 1 - P_G(n)$ (see (2.1) in Section 2). Since $P_G(n) = 0$ whenever $n \leq d_P(G)$, we immediately deduce that $e(G) > d_P(G)$. In particular, if all the Sylow subgroups of G are d -generated, then $d_P(G) < e(G) < d + 3$, and therefore we obtain the following result:

Theorem 3. *If all the Sylow subgroups of a profinite group G are (topologically) d -generated, then $d_P(G) \leq d + 2$.*

The previous result is best possible. For a given $d \in \mathbb{N}$, let $A_{p,d}$ be an elementary abelian p -group of rank d , $A_d = \prod_{p \neq 2} A_{p,d}$ and consider the semidirect product $G_d = A_d \rtimes B$, where $B = \langle b \rangle$ is cyclic of order 2 and $a^b = a^{-1}$ for every $a \in A_d$. Clearly all the Sylow subgroups of G are d -generated. We claim that $d_P(G) = d + 2$. It follows from the main theorem in [4] that for every $k \in \mathbb{N}$, we have

$$P_{G_d}(k) = \left(1 - \frac{1}{2^k}\right) \prod_{p \neq 2} \left(1 - \frac{p}{p^k}\right) \cdots \left(1 - \frac{p^d}{p^k}\right).$$

In particular $P_{G_d}(d+1) = 0$ since the series

$$\sum_{p \neq 2} \left(\frac{p}{p^{d+1}} + \cdots + \frac{p^d}{p^{d+1}}\right) = \sum_{p \neq 2} \frac{p^{d+1} - 1}{(p-1)p^{d+1}}$$

is divergent. But then $d_P(G) > d + 1$, hence, by Theorem 3, we conclude $d_P(G) = d + 2$.

2. PROOF OF THE MAIN RESULT

Let G be a finite group and use the following notations:

- For a given prime p , $d_p(G)$ is the smallest cardinality of a generating set of a Sylow p -subgroup of G .
- For a given prime p and a positive integer t , $\alpha_{p,t}(G)$ is the number of complemented factors of order p^t in a chief series of G .
- For a given prime p , $\alpha_p(G) = \sum_t \alpha_{p,t}(G)$ is the number of complemented factors of p -power order in a chief series of G .
- $\beta(G)$ is the number of nonabelian factors in a chief series of G .

Lemma 4. *For every finite group G , we have:*

- (1) $\alpha_p(G) \leq d_p(G)$.
- (2) $\alpha_2(G) + \beta(G) \leq d_2(G)$.
- (3) If $\beta(G) \neq 0$, then $\beta(G) \leq d_2(G) - 1$.

Proof. We prove the three statements by induction on the order of G . Let N be a minimal normal subgroup of G .

- (1) If N is not complemented in G or N is not a p -group, then, by induction,

$$\alpha_p(G) = \alpha_p(G/N) \leq d_p(G/N) = d_p(G).$$

Otherwise $G = N \rtimes H$ for a suitable $H \leq G$ and $\alpha_p(G) = \alpha_p(G/N) + 1 = \alpha_p(H) + 1 \leq d_p(H) + 1 \leq d_p(G)$.

- (2) If N is abelian, we argue as in (1). Assume that N is nonabelian and let P be a Sylow 2-subgroup of G . By Tate's Theorem [3, p. 431], $P \cap N \not\leq \text{Frat } P$, and consequently $\beta(G) = \beta(G/N) + 1 \leq d_2(G/N) + 1 \leq d_2(G)$.
- (3) Suppose $\beta(G) \neq 0$. As before we may assume that N is nonabelian and this implies $d_2(G/N) + 1 \leq d_2(G)$. If $\beta(G/N) \neq 0$, then we easily conclude by induction. If $\beta(G/N) = 0$ then $\beta(G) = 1$ while $d_2(G) \geq 2$, since a Sylow 2-subgroup of a finite nonabelian simple group, and consequently of N , is never cyclic. \square

Notice that $\tau_G > n$ if and only if $\langle x_1, \dots, x_n \rangle \neq G$, so we have $P(\tau_G > n) = 1 - P_G(n)$, denoting by $P_G(n)$ the probability that n randomly chosen elements of G generate G . Clearly we have:

$$\begin{aligned} (2.1) \quad e(G) &= \sum_{n \geq 1} n P(\tau_G = n) = \sum_{n \geq 1} \left(\sum_{m \geq n} P(\tau_G = m) \right) \\ &= \sum_{n \geq 1} P(\tau_G \geq n) = \sum_{n \geq 0} P(\tau_G > n) = \sum_{n \geq 0} (1 - P_G(n)). \end{aligned}$$

Denote by $m_n(G)$ the number of index n maximal subgroups of G . We have (see [10, 11.6]):

$$(2.2) \quad 1 - P_G(k) \leq \sum_{n \geq 2} \frac{m_n(G)}{n^k}.$$

Using the notations introduced in [9, Section 2], we say that a maximal subgroup M of G is of type A if $\text{soc}(G/\text{Core}_G(M))$ is abelian, of type B otherwise, and we denote by $m_n^A(G)$ (respectively $m_n^B(G)$) the number of maximal subgroups of G of type A (respectively B) of index n . Given an irreducible G -group V , let $\delta_V(G)$ be the number of complemented factors G -isomorphic to V in a chief series of G and $q_V(G) = |\text{End}_G(V)|$. Moreover, for $n \in \mathbb{N}$, let \mathcal{A}_n be the set of the irreducible G -modules V with $\delta_V(G) \neq 0$ and $|V| = n$.

Lemma 5. *Let $n = p^t$ for some prime p . If $m_n^A(G) \neq 0$, then $\alpha_{p,t}(G) \neq 0$ and*

$$m_n^A(G) \leq \frac{n^{\alpha_{p,t}(G)+1}}{p-1}.$$

Proof. For a given $V \in \mathcal{A}_n$, let $m_V(G)$ be the number of maximal subgroups M of G with $\text{soc}(G/\text{Core}_G(M)) \cong_G V$. From [9, Section 2] and [2, Section 4] it follows that

$$m_V(G) \leq \frac{q_V(G)^{\delta_V(G)} - 1}{q_V(G) - 1} |\text{Der}(G/C_G(V), V)|.$$

By [6, Theorem 1], we have $|\text{Der}(G/C_G(V), V)| \leq |V|^{3/2}$. Moreover (see for example [14, Lemma 1]) $|\text{Der}(G/C_G(V), V)| \leq |V|$ if $G/C_G(V)$ is soluble, which happens in particular when $q_V(G) = n$ (indeed in this case $G/C_G(V)$ is isomorphic to a subgroup of the multiplicative group of the field of order $q_V(G)$). If $q_V(G) \neq n$, then $\dim_{\text{End}_G(V)} V \geq 2$, hence $n = |V| \geq q_V(G)^2$ and consequently

$$m_V(G) \leq \frac{q_V(G)^{\delta_V(G)} n^{3/2}}{q_V(G) - 1} \leq \frac{n^{\delta_V(G)/2} n^{3/2}}{p - 1} \leq \frac{n^{\delta_V(G)+1}}{p - 1}.$$

On the other hand, if $q_V(G) = n$, then

$$m_V(G) \leq \frac{q_V(G)^{\delta_V(G)} n}{q_V(G) - 1} \leq \frac{n^{\delta_V(G)} n}{p - 1} \leq \frac{n^{\delta_V(G)+1}}{p - 1}.$$

We conclude

$$\begin{aligned} m_n^A(G) &= \sum_{V \in \mathcal{A}_n} m_V(G) \leq \frac{n}{p - 1} \sum_{V \in \mathcal{A}_n} n^{\delta_V(G)} \leq \frac{n}{p - 1} \prod_{V \in \mathcal{A}_n} n^{\delta_V(G)} \\ &= \frac{n^{1 + \sum_{V \in \mathcal{A}_n} \delta_V(G)}}{p - 1} = \frac{n^{\alpha_{p,t}(G)+1}}{p - 1}. \quad \square \end{aligned}$$

Lemma 6. *If $m_n^B(G) \neq 0$, then $n \geq 5$, $\beta(G) \neq 0$ and $m_n^B(G) \leq \frac{\beta(G)(\beta(G)+1)n^2}{2}$.*

Proof. The condition $n \geq 5$ follows from the fact there there is no unsoluble primitive permutation group of degree $n < 5$. The remaining part of the statement follows from [9, Claim 2.4]. \square

Lemma 7. *Let $d = \max_p d_p(G)$ and let*

$$\mu_p(G) = \sum_{k \geq d+2} \left(\sum_{t \geq 1} \frac{m_{p^t}^A(G)}{p^{tk}} \right).$$

If $\alpha_p(G) = 0$, then $\mu_p(G) = 0$. Otherwise

$$\mu_p(G) \leq \begin{cases} \frac{1}{p^{d-\alpha_p(G)}} \frac{1}{(p-1)^2} \leq \frac{1}{(p-1)^2} & \text{if } p \text{ is odd,} \\ \frac{1}{2^{d-\alpha_2(G)}} \frac{1}{2} \leq \frac{1}{2} & \text{otherwise.} \end{cases}$$

Proof. First notice that, by Lemma 4, we have $\alpha_p(G) \leq d_p(G) \leq d$. Let $\theta_{p,t} = 0$ if $\alpha_{p,t}(G) = 0$, $\theta_{p,t} = 1$ otherwise. By Lemma 5 we have

$$\begin{aligned}
\sum_{k \geq d+2} \left(\sum_{t \geq 1} \frac{m_{p^t}^A(G)}{p^{tk}} \right) &\leq \frac{p}{p-1} \sum_{k \geq d+2} \left(\sum_{t \geq 1} \frac{p^{t\alpha_{p,t}(G)} \theta_{p,t}}{p^{tk}} \right) \\
&\leq \sum_{k \geq d+2} \frac{p}{p-1} \left(\sum_{t \geq 1} \frac{p^{\alpha_{p,t}(G)} \theta_{p,t}}{p^k} \right) \leq \frac{p}{p-1} \sum_{k \geq d+2} \left(\frac{p^{\sum_{t \geq 1} \alpha_{p,t}(G)}}{p^k} \right) \\
&\leq \frac{p}{p-1} \sum_{k \geq d+2} \frac{p^{\alpha_p(G)}}{p^k} \leq \frac{p}{p-1} \sum_{k \geq d+2} \frac{p^d}{p^k p^{d-\alpha_p(G)}} \\
&\leq \frac{p}{p^{d-\alpha_p(G)}(p-1)} \sum_{u \geq 2} \frac{1}{p^u} \leq \frac{1}{p^{d-\alpha_p(G)}} \frac{1}{(p-1)^2}.
\end{aligned}$$

Notice that, for $k > d \geq d_2(G) \geq \alpha_{2,t}(G)$, we have

$$\frac{m_{2^t}^A(G)}{2^{tk}} \leq \frac{2^{t\alpha_{2,t}(G)+1}}{2^{tk}} \leq \frac{2^{\alpha_{2,t}(G)}}{2^k} \quad \text{if } t > 1.$$

On the other hand,

$$m_2^A(G) = 2^{\alpha_{2,1}(G)} - 1 \leq 2^{\alpha_{2,1}(G)}.$$

Hence

$$\begin{aligned}
\sum_{k \geq d+2} \left(\sum_{t \geq 1} \frac{m_{2^t}^A(G)}{2^{tk}} \right) &\leq \sum_{k \geq d+2} \left(\sum_{t \geq 1} \frac{2^{\alpha_{2,t}(G)} \theta_{2,t}}{2^k} \right) \leq \sum_{k \geq d+2} \left(\frac{2^{\sum_{t \geq 1} \alpha_{2,t}(G)}}{2^k} \right) \\
&\leq \sum_{k \geq d+2} \frac{2^{\alpha_2(G)}}{2^k} \leq \sum_{k \geq d+2} \frac{2^d}{2^k 2^{d-\alpha_2(G)}} \leq \frac{1}{2^{d-\alpha_2(G)}} \sum_{u \geq 2} \frac{1}{2^u} \leq \frac{1}{2^{d-\alpha_p(G)}} \frac{1}{2}. \quad \square
\end{aligned}$$

Lemma 8. Let $d = \max_p d_p(G)$ and let

$$\mu^*(G) = \sum_{k \geq d+2} \left(\sum_{n \geq 5} \frac{m_n^B(G)}{n^k} \right).$$

If $\beta(G) = 0$, then $\mu^*(G) = 0$. Otherwise

$$\mu^*(G) \leq \frac{1}{4 \cdot 5^{d-(\beta(G)+1)}} \leq \frac{1}{4}.$$

Proof. Notice that, by Lemma 4, if $\beta(G) \neq 0$, then $d \geq d_2(G) \geq \beta(G) + 1 \geq 2$. We deduce from Lemma 6 that

$$\begin{aligned}
\sum_{k \geq d+2} \left(\sum_{n \geq 5} \frac{m_n^B(G)}{n^k} \right) &\leq \sum_{k \geq d+2} \left(\sum_{n \geq 5} \frac{\beta(G)(\beta(G)+1)n^2}{2n^k} \right) \\
&\leq \frac{\beta(G)(\beta(G)+1)}{2} \sum_{u \geq 2} \left(\sum_{n \geq 5} \frac{n^2}{n^{d+u}} \right) \leq \frac{\beta(G)(\beta(G)+1)}{2 \cdot 5^{d-2}} \sum_{u \geq 2} \left(\sum_{n \geq 5} \frac{1}{n^u} \right) \\
&\leq \frac{\beta(G)(\beta(G)+1)}{2 \cdot 5^{d-2}} \sum_{n \geq 5} \left(\sum_{u \geq 2} \frac{1}{n^u} \right) \leq \frac{\beta(G)(\beta(G)+1)}{2 \cdot 5^{d-2}} \sum_{n \geq 5} \frac{1}{n^2} \frac{n}{n-1} \\
&= \frac{\beta(G)(\beta(G)+1)}{2 \cdot 5^{d-2}} \sum_{n \geq 4} \frac{1}{n(n+1)} = \frac{\beta(G)(\beta(G)+1)}{2 \cdot 5^{\beta(G)-1} \cdot 5^{d-(\beta(G)+1)}} \cdot \frac{1}{4} \\
&\leq \frac{1}{4 \cdot 5^{d-(\beta(G)+1)}}. \quad \square
\end{aligned}$$

Lemma 9. *We have $\mu_2(G) + \mu^*(G) \leq 1/2$.*

Proof. By Lemma 4, $\alpha_2(G) + \beta(G) \leq d_2(G) \leq d$. If $d = \alpha_2(G)$ then $\beta(G) = 0$, and consequently

$$\mu_2(G) + \mu^*(G) = \mu_2(G) \leq \frac{1}{2}.$$

In the remain cases, we have

$$\mu_2(G) + \mu^*(G) \leq \frac{1}{2 \cdot 2^{d-\alpha_p(G)}} + \frac{1}{4 \cdot 5^{d-(\beta(G)+1)}} \leq \frac{1}{4} + \frac{1}{4} = \frac{1}{2}. \quad \square$$

Proof of Theorem 2. From (2.1), (2.2) and the last three lemmas, we deduce

$$\begin{aligned}
e(G) &= \sum_{k \geq 0} (1 - P_G(k)) \leq d + 2 + \sum_{k \geq d+2} (1 - P_G(k)) \\
&\leq d + 2 + \sum_p \left(\sum_{k \geq d+2} \left(\sum_{t \geq 1} \frac{m_{p^t}^A(G)}{p^{tk}} \right) \right) + \sum_{k \geq d+2} \left(\sum_{n \geq 5} \frac{m_n^B(G)}{n^k} \right) \\
&= d + 2 + \sum_p \mu_p(G) + \mu^*(G) \leq d + \frac{5}{2} + \sum_{p > 2} \frac{1}{(p-1)^2}. \quad \square
\end{aligned}$$

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