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OGUS REALIZATION OF 1-MOTIVES

F. ANDREATTA, L. BARBIERI-VIALE AND A. BERTAPELLE

ABSTRACT. After introducing the Ogus realization of 1-motives we prove that it is a fully faithful functor. More precisely, following a framework introduced by Ogus, considering an enriched structure on the de Rham realization of 1-motives over a number field, we show that it yields a full functor by making use of an algebraicity theorem of Bost.

INTRODUCTION

The Ogus realization of motives over a number field is considered as an analogue of the Hodge realization over the complex numbers and the ℓ -adic realization over fields which are finitely generated over the prime field. The fullness of these realizations, along with the semi-simplicity of the essential image of pure motives, is a longstanding conjecture which implies the Grothendieck standard conjectures (*e.g.* see [1, §7.1]).

The named conjecture on fullness is actually a theorem if we restrict to the category of abelian varieties up to isogenies regarded as the semi-simple abelian \mathbb{Q} -linear category of *pure* 1-motives (*e.g.* see [1, Prop. 4.3.4.1 & Thm. 7.1.7.5]). A natural task is then to extend this theorem to *mixed* 1-motives up to isogenies. For the Hodge realization this goes through Deligne’s result on the algebraicity of the effective mixed polarizable Hodge structures of level ≤ 1 (see [15, §10.1.3]). For the ℓ -adic realization, the fullness follows from the Tate conjectures for abelian varieties (proven by Faltings) and the fullness for 1-motives is proven by Jannsen (see [18, §4]).

The main task of this paper is to show that there is a suitable version of Ogus realization for 1-motives such that the fullness can be achieved: this is Theorem 3.3.5 below. This result for abelian varieties relies on a theorem of Bost as explained by André (see [1, §7.4.2]). For pure 0-motives is our Lemma 3.3.4. However, in the mixed case, our theorem doesn’t follow directly from Bost’s theorem nor André’s arguments (see Example 3.3.6).

For a number field K , recall that the Ogus category $\mathbf{Og}(K)$ is the \mathbb{Q} -linear abelian category whose objects are finite dimensional K -vector spaces V such that the v -adic completion V_v is endowed, for almost every unramified place v of K , of a bijective semilinear endomorphism F_v . Actually, we here introduce an enriched version of the Ogus category denoted by $\mathbf{FOg}(K)$ whose objects $\mathcal{V} \in \mathbf{Og}(K)$ are endowed with an increasing finite exhaustive weight filtration $W \cdot \mathcal{V}$ (see Section 1 for details). We provide a realization functor

$$T_{\mathbf{Og}}: \mathcal{M}_{1,\mathbb{Q}} \rightarrow \mathbf{FOg}(K)$$

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where $\mathcal{M}_{1,\mathbb{Q}}$ is the abelian \mathbb{Q} -linear category of 1-motives up to isogenies (see Proposition 3.2.3). With the notation adopted below, this $T_{\text{Og}}(\mathbf{M}_K)$ of a 1-motive \mathbf{M}_K (see Definition 3.2.1), is given by $V := T_{\text{dR}}(\mathbf{M}_K)$ the de Rham realization (see Definition 2.3.1), as a K -vector space, so that $V_v \simeq T_{\text{dR}}(\mathbf{M}_{\mathcal{O}_{K_v}}) \otimes_{\mathcal{O}_{K_v}} K_v$ and $F_v := (\Phi_v \otimes \text{id})^{-1}$ where the σ_v^{-1} -semilinear endomorphism Φ_v (Verschiebung) on $T_{\text{dR}}(\mathbf{M}_{\mathcal{O}_{K_v}})$ is obtained via the canonical isomorphism $T_{\text{dR}}(\mathbf{M}_{\mathcal{O}_{K_v}}) \simeq T_{\text{cris}}(\mathbf{M}_{k_v})$ given by the comparison with the crystalline realization of the special fiber \mathbf{M}_{k_v} for every unramified place v of good reduction for \mathbf{M}_K (see [2, §4, Cor. 4.2.1]).

In the proof of the fullness of T_{Og} for arbitrary 1-motives, with nontrivial discrete part, the main ingredient is the fact that the decomposition of V_v as a sum of pure F - K_v -isocrystals is realized geometrically via the p -adic logarithm (see Section 2, in particular Lemma 2.2.4, and the key Lemma 3.3.1).

Note that the essential image of T_{Og} is contained in the category $\mathbf{FOg}(K)_{(1)}$ of effective objects of level ≤ 1 (see Definition 1.4.4). After Theorem 3.3.5 it is clear that the Ogus realization of pure 1-motives in $\mathcal{M}_{1,\mathbb{Q}}$ is semi-simple. However, the characterization of the essential image is an open question (unless we are in the case of Artin motives, cf. [20]).

Further directions of investigation are related to Voevodsky motives. Recall that the derived category of 1-motives $D^b(\mathcal{M}_{1,\mathbb{Q}})$ can be regarded as a reflective triangulated full subcategory of effective Voevodsky motives $\text{DM}_{\text{gm}}^{\text{eff}}$ (see [6, Thm. 6.2.1 & Cor. 6.2.2]). In fact, there is a functor $\text{LAlb}^{\mathbb{Q}}: \text{DM}_{\text{gm}}^{\text{eff}} \rightarrow D^b(\mathcal{M}_{1,\mathbb{Q}})$ which is a left adjoint to the inclusion. Thus, associated to any algebraic K -scheme X , we get a complex of 1-motives $\text{LAlb}^{\mathbb{Q}}(X) \in D^b(\mathcal{M}_{1,\mathbb{Q}})$. Whence, by taking i -th homology, we get $L_i \text{Alb}^{\mathbb{Q}}(X)_K \in \mathcal{M}_{1,\mathbb{Q}}$ which, most likely, is the geometric avatar of level ≤ 1 Ogus i -th homology of X , i.e., the Ogus realization $T_{\text{Og}}(L_i \text{Alb}^{\mathbb{Q}}(X)) \in \mathbf{FOg}(K)_{(1)}$ is the largest quotient of level ≤ 1 of the i -th Ogus homology of X , according with the framework of Deligne's conjecture (see [6, §14] and compare with [2, Conj. C]). This is actually the case for the underlying K -vector spaces by the corresponding result for the mixed realization but Ogus realization (see [6, Thm. 16.3.1]).

Notation. We here denote by K a number field. For any finite place v of K , i.e., any prime ideal of the ring of integers \mathcal{O}_K of K , we let K_v be the completion of K with respect to the valuation v and \mathcal{O}_{K_v} its ring of integers. Let \mathfrak{p}_v be the unique maximal ideal of \mathcal{O}_{K_v} , k_v the residue field of \mathcal{O}_{K_v} , p_v its characteristic and $n_v := [k_v : \mathbb{F}_{p_v}]$. If v is unramified, we let σ_v be the canonical Frobenius map on K_v and σ_v will also denote the Frobenius maps on \mathcal{O}_{K_v} and on k_v .

1. OGUS CATEGORIES

1.1. The plain Ogus category $\mathbf{Og}(K)$. As in [1, §7.1.5] let $\mathbf{Og}(K)$ be the \mathbb{Q} -linear abelian category whose objects are finite dimensional K -vector spaces V such that the v -adic completion $V_v = V \otimes_K K_v$ is equipped for almost every unramified place v of K , of a bijective semilinear endomorphism F_v , i.e., for any $\alpha \in K_v, x, y \in V_v$ we have $F_v(\alpha x + y) = \sigma_v(\alpha)F_v(x) + F_v(y)$. Morphisms in $\mathbf{Og}(K)$ are K -linear maps compatible with the F_v 's for almost all v .

To give a more precise definition of the above category we have to present it as a 2-colimit category. Let \mathcal{P} denote the set of unramified places of K . For any cofinite

subset \mathcal{P}' of \mathcal{P} , let $\mathcal{C}_{\mathcal{P}'}$ denote the \mathbb{Q} -linear category whose objects are of the type $(V, (V_v, F_v)_{v \in \mathcal{P}'}, (g_v)_{v \in \mathcal{P}'})$ where V is a finite dimensional K -vector space, V_v are finite dimensional K_v -vector spaces, F_v is a bijective σ_v -semilinear endomorphism on V_v , and $g_v: V \otimes_K K_v \rightarrow V_v$ is an isomorphism of K_v -vector spaces. Morphisms are morphisms of vector spaces (on K and on K_v) which respect the given structures (F_v and g_v). Clearly, for any cofinite subset \mathcal{P}'' of \mathcal{P}' , we have a canonical restriction functor $i_{\mathcal{P}', \mathcal{P}''}: \mathcal{C}_{\mathcal{P}'} \rightarrow \mathcal{C}_{\mathcal{P}''}$ which simply forgets the data for $v \in \mathcal{P}' \setminus \mathcal{P}''$.

1.1.1. Definition. Let

$$\mathbf{Og}(K) := 2\text{-}\varinjlim_{\mathcal{P}' \subseteq \mathcal{P}} \mathcal{C}_{\mathcal{P}'}$$

be the 2-colimit category. If $\mathcal{V}_1 \in \mathcal{C}_{\mathcal{P}_1}$ and $\mathcal{V}_2 \in \mathcal{C}_{\mathcal{P}_2}$ then

$$\mathrm{Hom}_{\mathbf{Og}(K)}(\mathcal{V}_1, \mathcal{V}_2) = \varinjlim_{\mathcal{P}_3 \subseteq \mathcal{P}_1 \cap \mathcal{P}_2} \mathrm{Hom}_{\mathcal{C}_{\mathcal{P}_3}}(i_{\mathcal{P}_1, \mathcal{P}_3}(\mathcal{V}_1), i_{\mathcal{P}_2, \mathcal{P}_3}(\mathcal{V}_2)).$$

1.1.2. Remark. Given a positive integer n we denote by $\mathcal{P}_n \subset \mathcal{P}$ the subset of places v such that n is invertible in \mathcal{O}_{K_v} . Since any cofinite subset \mathcal{P}' in \mathcal{P} contains \mathcal{P}_n , for any n divisible by p_v with $v \in \mathcal{P} \setminus \mathcal{P}'$, we may equivalently get $\mathbf{Og}(K)$ as $2\text{-}\varinjlim_n \mathcal{C}_{\mathcal{P}_n}$.

For a given $\mathcal{V} := (V, (V_v, F_v)_{v \in \mathcal{P}'}, (g_v)_{v \in \mathcal{P}'})$ in $\mathcal{C}_{\mathcal{P}'}$ and an integer $n \in \mathbb{Z}$ we define the twist $\mathcal{V}(n) := (V, (V_v, p_v^{-n} F_v)_{v \in \mathcal{P}'}, (g_v)_{v \in \mathcal{P}'})$.

1.2. F - K_v -isocrystals. Recall that a *pure* F - K_v -isocrystal (H, ψ) of integral weight i relative to k_v in the sense of [11, II, §2.0] is a K_v -vector space H together with a K_v -linear endomorphism ψ such that the eigenvalues of ψ are Weil numbers of weight i relative to k_v . Namely, they are algebraic numbers such that they and all their conjugates have archimedean absolute value equal to $p_v^{in_v/2}$. In particular, ψ is an isomorphism. The trivial space $H = 0$ is assumed to be a pure F - K_v -isocrystal of any weight. We say that (H, ψ) is a *mixed* F - K_v -isocrystal with integral weights relative to k_v if it admits a finite increasing K_v -filtration

$$0 = W_m H \subseteq \cdots \subseteq W_i H \subseteq W_{i+1} H \subseteq \cdots \subseteq W_n H = H$$

respected by ψ such that $\mathrm{gr}_i^W H := W_i H / W_{i-1} H$ is pure of weight i for any $m < i \leq n$. A morphism $(H, \psi) \rightarrow (H', \psi')$ of F - K_v -isocrystals is a homomorphism of K_v -vector spaces $f: H \rightarrow H'$ such that $f \circ \psi' = \psi \circ f$.

1.2.1. Lemma. *Let H be a K_v -vector space and $\psi: H \rightarrow H$ an endomorphism. The following are equivalent:*

- (i) (H, ψ) is a mixed F - K_v -isocrystal.
- (i') (H, ψ) admits a unique finite increasing K_v -filtration $0 = W_m H \subseteq \cdots \subseteq W_n H = H$ respected by ψ such that $\mathrm{gr}_i^W H := W_i H / W_{i-1} H$ is pure of weight i for each $m < i \leq n$.
- (ii) All eigenvalues of ψ are Weil numbers of integral weight relative to k_v .
- (iii) H has a unique decomposition $\bigoplus_{i=m+1}^n H_i$ by ψ -stable vector subspaces so that $(H_i, \psi|_{H_i})$ is a pure F - K_v -isocrystal of weight i .

Proof. (i) \Rightarrow (ii). Let $W_\bullet H$ be the filtration of H and let $\bar{\psi}_i$ denote the endomorphism of $\mathrm{gr}_i^W H$ induced by ψ . The eigenvalues of $\bar{\psi}_i$ are Weil numbers of integral weight by hypothesis. Since the characteristic polynomial of ψ is the product of the characteristic polynomials of $\psi|_{W_{i-1} H}$ and $\bar{\psi}_i$, one proves recursively that all eigenvalues of ψ are

Weil numbers (of integral weight).

(ii) \Rightarrow (iii). If all eigenvalues of ψ are in K_v , then H is the direct sum of its generalized eigenspaces. Let H_i be the direct sum of generalized eigenspaces associated to eigenvalues of weight i . Then $(H_i, \psi|_{H_i})$ is a pure F - K_v -isocrystal of weight i and $H = \bigoplus_i H_i$. In the general case, let L/K_v be a finite Galois extension containing all the eigenvalues of ψ and let k_L be its residue field. Observe that $(H \otimes_{K_v} L, \psi \otimes \text{id})$ is not an F - L -isocrystal in general. Indeed if an eigenvalue α of ψ has weight i (relative to k_v), i.e., $|\alpha| = p_v^{n_v i/2}$ then $|\alpha| = p_v^{n_v r i/(2r)}$, with $p_v^{n_v r} = |k_L|$, would have weight i/r relative to k_L and i/r might not be an integer. However, the decomposition result works the same if we consider rational weights. Hence $H \otimes_{K_v} L = \bigoplus_i (H \otimes_{K_v} L)_{i/r}$ where the L -linear subspace $(H \otimes_{K_v} L)_{i/r}$ is the direct sum of generalized eigenspaces associated to eigenvalues of modulus $p_v^{i n_v/2} = p_v^{i n_v r/(2r)}$. Since all conjugates of a Weil number of weight i have the same weight, the conjugate of any eigenvalue α of ψ has the same weight. Hence the action of $\text{Gal}(L/K_v)$ on $H \otimes_{K_v} L$ respects the decomposition and $(H \otimes_{K_v} L)_{i/r}$ descends to a K_v -linear subspace H_i of H which is a pure F - K_v -isocrystal of weight i .

Assume now $H = \bigoplus_{i=m+1}^n H'_i$ is another decomposition as in (iii). In order to prove that $H'_i = H_i$ we may assume $L = K_v$. Now H_i is sum of the generalized eigenspaces associated to eigenvalues of weight i . Since the eigenvalues of $\psi|_{H'_i}$ have weight i , it is $H'_i \subseteq H_i$. One concludes then by dimension reason.

(iii) \Rightarrow (i). Let $\bigoplus_{i=m+1}^n H_i$ be a decomposition as in (iii) and set $W_m H = 0$ and $W_i H = \bigoplus_{j=m+1}^i H_j$ for $m < i \leq n$. This filtration makes H a mixed F - K_v -isocrystal. Let $W' H$ be another filtration making (H, ψ) a F - K_v -isocrystal. One proves recursively that the eigenvalues of $\psi|_{W'_i H}$ have weights $\leq i$. In particular, the image of $W'_i H$ in $\text{gr}_{i+1}^W H$ is trivial. Hence $W'_i H \subseteq W_i H$. The reverse inclusion is proved analogously. Hence $W \cdot H = W' \cdot H$. \square

It follows from the lemma above that any morphism of F - K_v -isocrystals respects the decompositions in pure F - K_v -isocrystals, thus the filtrations.

1.3. The enriched Ogus category $\mathbf{FOg}(K)$. We say that an object \mathcal{V} of $\mathbf{Og}(K)$ is pure of weight i if for almost all unramified places v the K_v -vector space V_v and the K_v -linear operator $F_v^{n_v}$ on V_v define a pure F - K_v -isocrystal of weight i relative to k_v .

1.3.1. Definition. Let $\mathbf{FOg}(K)$ be the category whose objects are objects \mathcal{V} in $\mathbf{Og}(K)$ endowed with an increasing finite exhaustive filtration

$$0 = W_m \mathcal{V} \subseteq \cdots \subseteq W_i \mathcal{V} \subseteq W_{i+1} \mathcal{V} \subseteq \cdots \subseteq W_n \mathcal{V} = \mathcal{V}$$

in $\mathbf{Og}(K)$ such that for every $i > m$ the graded $\text{gr}_i^W \mathcal{V} := W_i \mathcal{V} / W_{i-1} \mathcal{V}$ is pure of weight i . In particular, $(V_v, F_v^{n_v})$ is a mixed F - K_v -isocrystal for almost all v . Morphisms are morphisms in $\mathbf{Og}(K)$.

Morphisms in $\mathbf{FOg}(K)$ actually respect the filtration by the following:

1.3.2. Lemma. *With the notation above we have the following properties.*

- (1) *Given an object \mathcal{V} of $\mathbf{Og}(K)$ there is at most one filtration $W \cdot \mathcal{V}$ on \mathcal{V} such that $(\mathcal{V}, W \cdot \mathcal{V})$ is an object of $\mathbf{FOg}(K)$.*

- (2) Let $(\mathcal{V}, W \cdot \mathcal{V})$, $(\mathcal{V}', W \cdot \mathcal{V}')$ be objects of $\mathbf{FOg}(K)$. Then any morphism $\mathcal{V} \rightarrow \mathcal{V}'$ in $\mathbf{Og}(K)$ respects the filtration and is strict.
- (3) $\mathbf{FOg}(K)$ is a \mathbb{Q} -linear abelian category.
- (4) Given an object $(\mathcal{V}, W \cdot \mathcal{V})$ of $\mathbf{FOg}(K)$ for almost all v we have that \mathcal{V}_v has a unique decomposition as a direct sum of pure F - K_v -isocrystal of different weights.

Proof. (1) and (4) follow from Lemma 1.2.1. Since the morphisms between pure F - K_v -isocrystals of different weights are trivial, it follows that morphisms between mixed F - K_v -isocrystal with integral weights respect the filtrations and are strict. Using the fact that g_v induces an isomorphism $W \cdot V_v \simeq W \cdot V \otimes_K K_v$ and that the map $K \rightarrow K_v$ is faithfully flat, we get (2). Assertion (3) follows easily from (2). \square

Note that for $(\mathcal{V}, W \cdot \mathcal{V}) \in \mathbf{FOg}(K)$ and an integer $n \in \mathbb{Z}$ we have that

$$(\mathcal{V}, W \cdot \mathcal{V})(n) := (\mathcal{V}(n), W \cdot {}_{+2n}\mathcal{V}(n)) \in \mathbf{FOg}(K)$$

such that $\mathrm{gr}_i^W \mathcal{V}(n) := W_{i+2n}\mathcal{V}(n)/W_{i+2n-1}\mathcal{V}(n)$ is pure of weight i (cf. Definition 1.1.1).

1.4. The weight filtration on $\mathbf{FOg}(K)$. Consider the Serre subcategories $\mathbf{FOg}(K)_{\leq n}$ of $\mathbf{FOg}(K)$ given by objects $(\mathcal{V}, W \cdot \mathcal{V})$ of $\mathbf{FOg}(K)$ with $W_n \mathcal{V} = \mathcal{V}$. We get a filtration

$$\cdots \rightarrow \mathbf{FOg}(K)_{\leq n} \xrightarrow{\iota_n} \mathbf{FOg}(K)_{\leq n+1} \xrightarrow{\iota_{n+1}} \cdots$$

Recall that a filtration of an abelian category by Serre subcategories is a weight filtration (in the sense of [6, Def. D.1.14]) if it is separated, exhaustive and split, *i.e.*, all the inclusion functors ι_n have exact right adjoints.

1.4.1. Lemma. $\mathbf{FOg}(K)_{\leq n} \subset \mathbf{FOg}(K)$ is a weight filtration.

Proof. In fact, the filtration is clearly separated, *i.e.*, $\cap \mathbf{FOg}(K)_{\leq n} = 0$, and exhaustive, *i.e.*, $\cup \mathbf{FOg}(K)_{\leq n} = \mathbf{FOg}(K)$. The claimed adjoints are given by $(\mathcal{V}, W \cdot \mathcal{V}) \mapsto (W_n \mathcal{V}, W \cdot {}_{\leq n}\mathcal{V})$ where $W_i {}_{\leq n}\mathcal{V} = W_i \mathcal{V}$ for $i < n$ and $W_i {}_{\leq n}\mathcal{V} = W_n \mathcal{V}$ for $i \geq n$ and they are exact by Lemma 1.3.2 (2). \square

We have that

$$\mathrm{gr}_i^W \mathcal{V} = W_i \mathcal{V} / W_{i-1} \mathcal{V} \in \mathbf{FOg}(K)_i := \mathbf{FOg}(K)_{\leq i} / \mathbf{FOg}(K)_{\leq i-1}.$$

Note that these categories $\mathbf{FOg}(K)_i$ are not necessarily semi-simple.

We may introduce a notion of effectivity following [1, 7.4.2] or [6, 17.4.4].

1.4.2. Definition. An object $(V, (V_v, F_v)_{v \in \mathcal{P}'}, (g_v)_{v \in \mathcal{P}'})$ of $\mathbf{Og}(K)$ is said to be *l-effective* if there exists a \mathcal{O}_K -lattice L of V such that the image under g_v of $L \otimes_{\mathcal{O}_K} \mathcal{O}_{K_v}$ in V_v is preserved by the F_v for almost all $v \in \mathcal{P}'$. Denote $\mathbf{Og}(K)^{\mathrm{eff}} \subset \mathbf{Og}(K)$ the full subcategory of l-effective objects.

Similarly define $\mathbf{FOg}(K)^{\mathrm{eff}} \subset \mathbf{FOg}(K)$ as the full subcategory given by objects $(\mathcal{V}, W \cdot \mathcal{V})$ such that \mathcal{V} is in $\mathbf{Og}(K)^{\mathrm{eff}}$.

Moreover, we say that an object $(\mathcal{V}, W \cdot \mathcal{V})$ of $\mathbf{FOg}(K)$ is *e-effective* if the eigenvalues of $F_v^{n_v}$ on V_v are algebraic integers for almost all v .

1.4.3. Remarks. (a) An object $(V, (V_v, F_v)_{v \in \mathcal{P}'}, (g_v)_{v \in \mathcal{P}'})$ of $\mathbf{Og}(K)$ is l-effective if, and only if, every \mathcal{O}_K -lattice L of V satisfies the condition in Definition 1.4.2. Indeed any two \mathcal{O}_K -lattices L, L' of V coincide over $\mathcal{O}_K[1/n]$ for n sufficiently divisible. Hence if L satisfies the condition in Definition 1.4.2 for any $v \in \mathcal{P}'' \subset \mathcal{P}'$, the same does L' for any $v \in \mathcal{P}'' \cap \mathcal{P}_n$.

(b) Since any $\mathcal{O}_K[1/n]$ -lattice L of V is isomorphic to the base change along $\mathcal{O}_K \rightarrow \mathcal{O}_K[1/n]$ of a \mathcal{O}_K -lattice, an object $(V, (V_v, F_v)_{v \in \mathcal{P}'}, (g_v)_{v \in \mathcal{P}'})$ of $\mathbf{Og}(K)$ is l-effective if, and only if, there exists a positive integer n and a $\mathcal{O}_K[1/n]$ -lattice L of V such that the image under g_v of $L \otimes_{\mathcal{O}_K} \mathcal{O}_{K_v}$ in V_v is preserved by the F_v for all $v \in \mathcal{P}_n$.

(c) If \mathcal{V} in $\mathbf{Og}(K)$ is l-effective, the same is any subobject $\mathcal{V}' \subset \mathcal{V}$ in $\mathbf{Og}(K)$. As a consequence given $(\mathcal{V}, W \cdot \mathcal{V}) \in \mathbf{FOg}(K)^{\text{eff}}$ all $W_i \mathcal{V}$ and $\text{gr}_i^W \mathcal{V}$ are in $\mathbf{FOg}(K)^{\text{eff}}$ as well.

According with [6, Def. 14.3.2] we also set a subcategory of level ≤ 1 objects:

1.4.4. Definition. Call Artin-Lefschetz objects those objects of $\mathbf{FOg}(K)_{-2}$ which are $(\mathcal{V}, W \cdot \mathcal{V})(1)$ for $(\mathcal{V}, W \cdot \mathcal{V})$ both l-effective and e-effective of weight zero. Denote $\mathbf{FOg}(K)_{\mathbb{L}}$ the Serre subcategory of $\mathbf{FOg}(K)_{-2}$ given by Artin-Lefschetz objects. Denote $\mathbf{FOg}(K)_{(1)}$ the full subcategory of $\mathbf{FOg}(K)$ given by those $(\mathcal{V}, W \cdot \mathcal{V})$ that are e-effective of weights $\{-2, -1, 0\}$ such that $W_{-2} \mathcal{V}$ is an Artin-Lefschetz object and $(\mathcal{V}, W \cdot \mathcal{V})(-1)$ is l-effective.

1.5. The Bost-Ogus category $\mathbf{BOg}(K)$. Let $\mathbf{BOg}(K)$ denote the \mathbb{Q} -linear category whose objects are finite dimensional K -vector spaces V such that the reduction modulo \mathfrak{p}_v of V is equipped with a σ_v -semilinear endomorphism for almost every unramified place v of K . Morphisms are morphisms of K -vector spaces which respect the extra structure. The category $\mathbf{BOg}(K)$ is the category denoted $\mathbf{Frob}_{\text{ae}}(K)$ in [9, 2.3.2].

Also this category can be better described as a 2-colimit category. Recall that, given a positive integer n , \mathcal{P}_n denotes the subset of \mathcal{P} consisting of those v such that n is invertible in \mathcal{O}_{K_v} (see Remark 1.1.2). Let $\mathcal{L}_{\mathcal{P}_n}$ be the category whose objects are of the type $(V, L, ({}^b F_v)_{v \in \mathcal{P}_n})$ where V is a finite dimensional K -vector space, L is an $\mathcal{O}_K[1/n]$ -lattice in V and ${}^b F_v$ is a σ_v -semilinear endomorphism on $L \otimes_{\mathcal{O}_K} k_v$. A morphism in $\mathcal{L}_{\mathcal{P}_n}$ is the data of a homomorphism of lattices (and by K -linearization of vector spaces) which respects the given ${}^b F_v$ for all v . We then define

$$\mathbf{BOg}(K) := 2\text{-}\varinjlim_{\mathcal{P}_n \subseteq \mathcal{P}} \mathcal{L}_{\mathcal{P}_n}.$$

Note that there is a functor (see also [1, 7.4.2])

$$(1.1) \quad \Psi: \mathbf{FOg}(K)^{\text{eff}} \rightarrow \mathbf{BOg}(K)$$

defined as follows. Consider an object $(\mathcal{V}, W \cdot \mathcal{V})$ in $\mathbf{FOg}(K)^{\text{eff}}$ and assume \mathcal{V} in $\mathbf{Og}(K)^{\text{eff}}$ is represented by an object $(V, (V_v, F_v)_{v \in \mathcal{P}'}, (g_v)_{v \in \mathcal{P}'})$ of $\mathcal{C}_{\mathcal{P}'}$. Let L be any \mathcal{O}_K -lattice of V . By Remarks 1.4.3 (a) and 1.1.2 there exists a positive integer n such that $\mathcal{P}_n \subset \mathcal{P}'$ and the image under g_v of $L \otimes_{\mathcal{O}_K} \mathcal{O}_{K_v}$ in V_v is preserved by F_v for all $v \in \mathcal{P}_n$. Let $\Psi(\mathcal{V}, W \cdot \mathcal{V})$ be represented by $(V, L \otimes_{\mathcal{O}_K} \mathcal{O}_K[1/n], ({}^b F_v)_{v \in \mathcal{P}_n})$ in $\mathcal{L}_{\mathcal{P}_n}$ where ${}^b F_v$ is the reduction modulo \mathfrak{p}_v of the mapping on $L \otimes_{\mathcal{O}_K} \mathcal{O}_{K_v}$ induced by F_v . Note that a different choice of a lattice of V will provide the same lattice over $\mathcal{O}_K[1/n]$ for n sufficiently divisible and hence the functor is well defined. Furthermore, remark that the functor Ψ is not full (*cf.* Remark 3.3.7).

2. LOGARITHMS AND UNIVERSAL EXTENSIONS

2.1. The p th power operation. Recall from [13, II §7 n.2 p. 273] and [14, Exp. VII_A, §6] that given a field k of characteristic $p > 0$ and a k -group scheme G one can define a p th power operation $x \mapsto x^{[p]}$ on $\text{Lie}(G)$ as follows. Recall that

$$\text{Lie}(G) = \text{Ker}(G(k[\varepsilon]/(\varepsilon^2)) \rightarrow G(k))$$

and for any $x \in \text{Lie}(G)$ write $e^{\varepsilon x}$ for the corresponding element in $G(k[\varepsilon]/(\varepsilon^2))$. Let $k[\varsigma, \pi] \subseteq k[\varepsilon_1, \dots, \varepsilon_p]/(\varepsilon_1^2, \dots, \varepsilon_p^2)$, with $\varsigma = \sum_{i=1}^p \varepsilon_i$, $\pi = \prod_{i=1}^p \varepsilon_i$, be the subalgebra generated by the elementary symmetric polynomials in ε_i . Observe that $\varsigma^p = 0$, $\varsigma\pi = 0$ and $\pi^2 = 0$. Then $e^{\varepsilon_1 x} e^{\varepsilon_2 x} \dots e^{\varepsilon_p x}$ makes sense as element in

$$\text{Ker}(G(k[\varepsilon_1, \dots, \varepsilon_p]/(\varepsilon_1^2, \dots, \varepsilon_p^2)) \rightarrow G(k))$$

where we use the multiplicative notation for the group law on G . Since $e^{\varepsilon_1 x} e^{\varepsilon_2 x} \dots e^{\varepsilon_p x}$ is invariant by permutations of the ε_i 's (see [13, II §4, 4.2 (6) p. 210]), it is indeed an element of $\text{Ker}(G(k[\varsigma, \pi]) \rightarrow G(k))$. Consider now the canonical projection $k[\varsigma, \pi] \rightarrow k[\pi]$, $\varsigma \mapsto 0$. It induces a map

$$G(k[\varsigma, \pi]) \rightarrow G(k[\pi]).$$

Let $e^{\pi y}$ be the image of $e^{\varepsilon_1 x} e^{\varepsilon_2 x} \dots e^{\varepsilon_p x}$ via this map. We further have that $e^{\pi y}$ is mapped to the unit section via the map $G(k[\pi]) \rightarrow G(k)$ induced by $\pi \mapsto 0$. We have

$$\begin{array}{ccc} G(k[\varsigma, \pi]) & \longrightarrow & G(k) \\ \downarrow & \nearrow & \\ G(k[\pi]) & & \end{array} \quad \begin{array}{ccc} e^{\varepsilon_1 x} e^{\varepsilon_2 x} \dots e^{\varepsilon_p x} & \longrightarrow & 1 \\ \downarrow & \nearrow & \\ e^{\pi y} & & \end{array}$$

Hence $y \in \text{Lie}(G) = \text{Ker}(G(k[\pi]) \rightarrow G(k))$ and we define $x^{[p]} := y$. The map

$$(2.1) \quad {}^{[p]}: \text{Lie}(G) \rightarrow \text{Lie}(G) \quad x \mapsto x^{[p]}$$

endows $\text{Lie}(G)$ with a structure of Lie p -algebra over k (see [13, II, §7 Prop. 3.4 p. 277]). If G is commutative, then $[x, y] = 0$ in $\text{Lie}(G)$ (see [14, Exp. II, Def. 4.7.2]) and hence ${}^{[p]}$ is p -linear, *i.e.*, $(x + y)^{[p]} = x^{[p]} + y^{[p]}$, $(\lambda x)^{[p]} = \lambda^p x^{[p]}$ for $\lambda \in k$ and $x, y \in \text{Lie}(G)$. Up to the usual identification of $\text{Lie}(G)$ with the invariant derivations of G , the p th power operation maps a derivation D to D^p (*cf.* [14, Exp. VII_A, §6.1], [13, II, §7, Prop. 3.4 p. 277]).

Let σ denote the Frobenius map on k . For the sake of exposition we provide a proof of the following well known fact (*cf.* [14, Exp. VII_A §4]).

2.1.1. Lemma. *Let G be a commutative algebraic k -group. Then the p th power operation (2.1) is a σ -semilinear map and coincides with the map on Lie algebras associated to the Verschiebung $\text{Ver}_G: G^{(p)} \rightarrow G$.*

Proof. The first claim is obvious. By functoriality of the Verschiebung (see [14, Exp. VII_A, 4.3]) and the fact that $\text{Lie}(\text{Fr}_G) = 0$ with $\text{Fr}_G: G \rightarrow G^{(p)}$ the (relative) Frobenius, it suffices to show the second claim replacing G by the kernel of Fr_G : we thus assume G to be finite (infinitesimal). By [7, Thm. 3.1.1] we can embed G as a closed subgroup-scheme of an abelian variety. By functoriality of the Verschiebung and of the p th power operation we can further reduce to the case of abelian varieties. The latter follows

from Example 2.1.2 (b) below as duality on abelian varieties exchanges Frobenius with Verschiebung (cf. [16, Prop. 7.34]). \square

2.1.2. Examples. (a) It follows from [13, II, §7 Exemples 2.2 p. 273] that the p th power operation on $\mathrm{Lie}(\mathbb{G}_a)$ is the zero map $x \mapsto 0$, while the p th power operation on $\mathrm{Lie}(\mathbb{G}_m)$ is given by $x \mapsto x^p$.

(b) When $G = A$ is an abelian variety and A^* is the dual abelian variety, it is known that there is a natural isomorphism $\mathrm{Lie}(A) \simeq H^1(A^*, \mathcal{O}_{A^*})$ and the p th power operation on $\mathrm{Lie}(A)$ corresponds to the Frobenius map on $H^1(A^*, \mathcal{O}_{A^*})$, *i.e.*, the σ -semilinear map induced by the Frobenius homomorphism $\alpha \mapsto \alpha^p$ on \mathcal{O}_{A^*} (see [22, §15, Thm. 3]).

2.2. The logarithms. Let k be a finite field of characteristic p . Let $W(k)$ be the ring of Witt vectors over k , K_0 its quotient field, and set $W_n(k) := W(k)/(p^n)$. Let G be a group scheme of finite type over $W(k)$. As claimed in [17, III, 5.4.1] we have that

$$G(W(k)) = \varprojlim G(W_n(k)).$$

(The separatedness hypothesis in *loc. cit.* can be ignored since $W(k)$ is local.)

Consider the canonical homomorphism of groups

$$(2.2) \quad \rho: G(W(k)) \rightarrow G(k)$$

induced by the closed immersion $\mathrm{Spec}(k) \rightarrow \mathrm{Spec}(W(k))$. Note that as k is a finite field, the k -valued points $G(k)$ of G form a finite, and hence torsion, group. In particular, if we denote Γ the kernel of ρ in (2.2), we obtain an isomorphism of \mathbb{Q} -vector spaces

$$(2.3) \quad G(W(k)) \otimes_{\mathbb{Z}} \mathbb{Q} \simeq \Gamma \otimes_{\mathbb{Z}} \mathbb{Q}.$$

Assume from now on that G is a *smooth* and *commutative* $W(k)$ -group scheme and let G_n be the base change of G to $S_n = \mathrm{Spec}(W_n(k))$; in particular G_1 denotes the special fiber of G . Let \mathcal{J} be the ideal sheaf of the unit section of G_n and recall that $\mathcal{O}_{G_n}/\mathcal{J}^N$ is a finite and free $W_n(k)$ -module for any $N > 0$. Hence $D = \varinjlim_N \mathrm{Hom}_{W_n(k)\text{-mod}}(\mathcal{O}_{G_n}/\mathcal{J}^N, W_n(k))$ is a coalgebra with a $W_n(k)$ -algebra structure induced by the group structure on G_n . The flatness of D over $W_n(k)$ ensures that the PD-structure on $pW_n(k)$ extends uniquely to a PD-structure (γ_m) on pD and hence there exist two mutually inverse maps

$$\exp: pD \rightarrow (1 + pD)^*, \quad \log: (1 + pD)^* \rightarrow pD,$$

defined by $\exp(x) = \sum_{m \geq 0} \gamma_m(x)$ and $\log(1 + x) = \sum_{m \geq 1} (-1)^{m-1} (m-1)! \gamma_m(x)$ ([21, III, 1.6]). Let $\mathrm{Cospec}(D)(W(k)) \subset D$ denote the subgroup of $W_n(k)$ -algebra homomorphisms. Then $p\mathrm{Cospec}(D)(W(k)) = \mathrm{Cospec}(D)(W(k)) \cap pD$ consists of those homomorphisms whose reduction modulo p is the homomorphism associated to the unit section of the special fiber of G . Let $\mathrm{Prim}(D) \subset D$ consists of the primitive elements of the coalgebra D , *i.e.*, those $x \in D$ such that $\Delta(x) = x \otimes 1 + 1 \otimes x$ with Δ the comultiplication of D , and let $p\mathrm{Prim}(D) = \mathrm{Prim}(D) \cap pD$. Then by [21, III, 2.2.5] (see also [3, §5.2]) there is an isomorphism of groups \exp_{G_n} which makes the following

diagram

$$(2.4) \quad \begin{array}{ccc} \text{Ker}(\text{Lie}(G_n) \rightarrow \text{Lie}(G_1)) & \xrightarrow{\cong} & p\text{Prim}(D) \subseteq pD \\ \downarrow \text{exp}_{G,n} & & \downarrow \text{exp} \\ \text{Ker}(G_n(W_n(k)) \rightarrow G_n(k)) & \xrightarrow{\cong} & p(\text{Cospec}(D)(W(k)) \subset (1+pD)^* \end{array}$$

commute. The vertical arrow on the left can also be written as an isomorphism

$$\text{exp}_{G,n}: p\text{Lie}(G_n) \xrightarrow{\cong} \text{Ker}(G(W_n(k)) \rightarrow G(k)).$$

Finally, taking the limit over n , one gets the exponential isomorphism for G

$$\text{exp}_G: p\text{Lie}(G) \xrightarrow{\cong} \Gamma.$$

Let $\log_G: \Gamma \xrightarrow{\cong} p\text{Lie}(G)$ denote the inverse of exp_G . We set (cf. [23, §2.4, p. 169]):

2.2.1. Definition. The logarithm is the isomorphism of \mathbb{Q} -vector spaces

$$\log_{G,\mathbb{Q}}: G(W(k)) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\cong} \text{Lie}(G) \otimes_{W(k)} K_0$$

obtained by composing (2.3) with $\log_G \otimes \text{id}_{\mathbb{Q}}$ and recalling that $p\text{Lie}(G) \otimes_{\mathbb{Z}} \mathbb{Q} \simeq \text{Lie}(G) \otimes_{\mathbb{Z}} \mathbb{Q} \simeq \text{Lie}(G) \otimes_{W(k)} K_0$ where the isomorphism $p\text{Lie}(G) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\cong} \text{Lie}(G) \otimes_{\mathbb{Z}} \mathbb{Q}$ is the \mathbb{Q} -linearization of the inclusion $p\text{Lie}(G) \rightarrow \text{Lie}(G)$.

2.2.2. Examples. (a) Let $G = \mathbb{G}_{a,W(k)}$. Then (2.2) is the reduction map $W(k) \rightarrow k$ and hence $\Gamma = pW(k)$. Recall that $\text{Lie}(\mathbb{G}_{a,W_n(k)}) = \text{Ker}(W_n(k) + W_n(k)\varepsilon \rightarrow W_n(k), a + b\varepsilon \mapsto a) \simeq W_n(k)\varepsilon$ with $\varepsilon^2 = 0$ and $D = \text{Hom}_{\text{cont}}(W_n(k)[[Z]], W_n(k))$. Diagram (2.4) becomes

$$\begin{array}{ccc} pW_n(k)\varepsilon & \xrightarrow{\cong} & p\text{Prim}(D) & & b\varepsilon & \xrightarrow{\quad} & f_b \\ \downarrow \text{exp}_{\mathbb{G}_{a,n}} & & \downarrow \text{exp} & & & & \downarrow \\ pW_n(k) & \xrightarrow{\cong} & p(\text{Cospec}(D)(W(k))) & & \text{exp}(f_b)(Z) & \longleftarrow & \text{exp}(f_b) \end{array}$$

where $f_b(1) = 0$, $f_b(Z^r) = rb$ for $r \geq 1$. Since $\gamma_m(f_b)(Z) = 0$ for $m \neq 1$ and $\gamma_1(f_b) = f_b$, one gets that $\text{exp}(f_b)(Z) = b$ and hence, up to the obvious identifications, we may consider $\text{exp}_{\mathbb{G}_a}$ and $\log_{\mathbb{G}_a}$ as the identity maps on $W_n(k)$. Finally $\text{Lie}_{\mathbb{G}_a,\mathbb{Q}}$ is the identity of K_0 , up to the usual identifications of $G(W(k))$ and $\text{Lie}(\mathbb{G}_a)$ with $W(k)$, i.e., $\log_{\mathbb{G}_a,\mathbb{Q}}: W(k) \otimes_{\mathbb{Z}} \mathbb{Q} \rightarrow \varepsilon W(k) \otimes_{W(k)} K_0$ in Definition 2.2.1 is $x \mapsto \varepsilon x$.

(b) Let $G = \mathbb{G}_{m,W(k)} = \text{Spec}(W(k)[X^{\pm 1}])$. Then $\Gamma = 1 + pW(k) \subset W(k)^*$, $\text{Lie}(\mathbb{G}_{m,W_n(k)}) \simeq 1 + pW_n(k)\varepsilon$ with $\varepsilon^2 = 0$, and D is as in (a) as a co-algebra (with $Z = X - 1$). Diagram (2.4) becomes

$$\begin{array}{ccc} 1 + pW_n(k)\varepsilon & \xrightarrow{\cong} & p\text{Prim}(D) & & 1 + b\varepsilon & \xrightarrow{\quad} & f_b \\ \downarrow \text{exp}_{\mathbb{G}_m,n} & & \downarrow \text{exp} & & & & \downarrow \\ 1 + pW_n(k) & \xrightarrow{\cong} & p(\text{Cospec}(D)(W(k))) & & 1 + \text{exp}(f_b)(Z) & \longleftarrow & \text{exp}(f_b) \end{array}$$

where $f_b(1) = 0$, $f_b(Z^r) = rb$ for $r \geq 1$. Since $\gamma_m(f_b)$ maps Z to 0 if $m = 0$ and to $b^m/m!$ if $m \geq 1$ then $\text{exp}(f_b)(Z) = \sum_{m \geq 1} b^m/m!$ and $\text{exp}_{\mathbb{G}_m,n}(1 + b\varepsilon) = \sum_{m \geq 0} b^m/m!$. Hence, up to the obvious identifications, $\text{exp}_{\mathbb{G}_m}$ is the exponential map $pW(k) \rightarrow 1 + pW(k)$ and

$\log_{\mathbb{G}_m}$ is the usual p -adic logarithm. Finally the isomorphism $\log_{\mathbb{G}_m, \mathbb{Q}}: W(k)^* \otimes_{\mathbb{Z}} \mathbb{Q} \rightarrow 1 + (W(k)\varepsilon \otimes_{W(k)} K_0)$ in Definition 2.2.1 is given by

$$x \otimes 1 \mapsto 1 + \frac{\log(1+y)\varepsilon}{(p_v^{n_v} - 1)}$$

where $x^{p_v^{n_v} - 1} = 1 + y$, $y \in pW(k)$.

2.2.3. Remark. Note that for any $W_n(k)$ -scheme T , one can define a functorial (in T and G) isomorphism $\exp_{G,n}: p\underline{\text{Lie}}(G)(T) \xrightarrow{\sim} \text{Ker}(G(T) \rightarrow G(T_0))$ where T_0 is the reduction of T modulo p and $\underline{\text{Lie}}(G)$ is the Lie algebra scheme of G (see [3, §5.2]). In particular any map $\exp_{G,n}$ (and thus \exp_G) behaves well with respect to finite unramified extension of $W(k)$. Further, for any finite Galois extension k'/k the map $\log_{G,\mathbb{Q}}$ in Definition 2.2.1 can be obtained by descent from the analogous isomorphism over $W(k')$.

Let $u: L \rightarrow G$ be a morphism of $W(k)$ -group schemes where $L = \mathbb{Z}^r$ and G is a smooth and commutative $W(k)$ -group scheme with connected fibers. Define δ_u as the K_0 -linear extension of the composition

$$(2.5) \quad L(W(k)) \xrightarrow{u \otimes 1} G(W(k)) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\log_{G,\mathbb{Q}}} \text{Lie}(G) \otimes_{W(k)} K_0$$

2.2.4. Lemma. *Let $u: L \rightarrow G$ be a morphism of $W(k)$ -group schemes where L is a lattice (i.e., isomorphic to \mathbb{Z}^r over some finite unramified extension of $W(k)$) and G is a smooth, connected and commutative $W(k)$ -group scheme. Then there is a unique morphism of K_0 -vector spaces*

$$\delta_u: \text{Lie}(L \otimes \mathbb{G}_a) \otimes_{W(k)} K_0 \rightarrow \text{Lie}(G) \otimes_{W(k)} K_0$$

which is functorial in u and in $W(k)$ and agrees with the one in (2.5) for L constant. Moreover, if u is given by the obvious inclusion $\text{id} \otimes 1: L \rightarrow L \otimes \mathbb{G}_a$, then δ_u is the identity of $\text{Lie}(L \otimes \mathbb{G}_a) \otimes_{W(k)} K_0$.

Proof. It follows by Remark 2.2.3. The last assertion follows from the fact that, up to the usual identifications, $\log_{\mathbb{G}_a, \mathbb{Q}}$ in Definition 2.2.1 is the identity map as computed in Example 2.2.2 (a). \square

2.2.5. Remark. For any formal $W(k)$ -group scheme $\widehat{G} = \varprojlim_n G_n$ where G_n is a smooth commutative group scheme of finite type over $W_n(k)$, one can define in a similar way the logarithm $\log_{\widehat{G}, \mathbb{Q}}: \widehat{G}(W(k)) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\sim} \text{Lie}(\widehat{G}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ with $\text{Lie}(\widehat{G}) := \varprojlim_n \text{Lie}(G_n)$. This construction is again functorial, i.e., given a morphism $g = \varprojlim_n g_n: \widehat{G} \rightarrow \widehat{H}$ then $\log_{\widehat{H}, \mathbb{Q}} \circ (g \otimes 1) = (\text{Lie}(g) \otimes \text{id}) \circ \log_{\widehat{G}, \mathbb{Q}}: \widehat{G}(W(k)) \otimes_{\mathbb{Z}} \mathbb{Q} \rightarrow \text{Lie}(\widehat{H}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. Further, $\log_{G, \mathbb{Q}}$ in Definition 2.2.1 equals $\log_{\widehat{G}, \mathbb{Q}}$ with \widehat{G} the p -adic completion of G .

As in Lemma 2.2.4, given a lattice L over $W(k)$ with base change L_n to $W_n(k)$ and a compatible system of morphisms of $W_n(k)$ -group schemes $\{u_n: L_n \rightarrow G_n\}_{n \in \mathbb{N}}$ over $W_n(k)$ we get a natural morphism of K_0 -vector spaces $\delta_u: \text{Lie}(L \otimes \mathbb{G}_a) \otimes_{W(k)} K_0 \rightarrow \text{Lie}(\widehat{G}) \otimes_{W(k)} K_0$.

2.3. Universal extensions. Let $\mathcal{M}_1(S)$ be the category of (Deligne) 1-motives over a scheme S (cf. [6, §1.2 & App. C]). If $S = \text{Spec}(K)$, let $\mathcal{M}_{1,\mathbb{Q}}$ denote the \mathbb{Q} -linear category of 1-motives up to *isogenies* over K , *i.e.*, the category whose objects are

$$\mathbf{M}_K = [\mathbf{u}: \mathbf{L}_K \rightarrow \mathbf{G}_K]$$

in $\mathcal{M}_1 := \mathcal{M}_1(\text{Spec}(K))$ and whose morphisms are given by $\text{Hom}_{\mathcal{M}_1}(\mathbf{M}_K, \mathbf{N}_K) \otimes_{\mathbb{Z}} \mathbb{Q}$. See [6, Prop. 1.2.6] for a proof that $\mathcal{M}_{1,\mathbb{Q}}$ is an abelian category. A morphism (h_{-1}, h_0) in \mathcal{M}_1 becomes an isomorphism in $\mathcal{M}_{1,\mathbb{Q}}$ if and only if it is an isogeny, *i.e.*, h_{-1} is injective with finite cokernel and h_0 is an isogeny (see [6, Lemma 1.2.7]). Recall that the canonical weight filtration of 1-motives yields a weight filtration on $\mathcal{M}_{1,\mathbb{Q}}$ (see [6, Prop. 14.2.1]). We let \mathbf{A}_K be the maximal abelian quotient of \mathbf{G}_K and let

$$\mathbf{M}_{\text{ab},K} := [\mathbf{u}_{\mathbf{A}_K}: \mathbf{L}_K \rightarrow \mathbf{A}_K]$$

denote the (Deligne) 1-motive induced by \mathbf{M}_K via $\mathbf{L}_K \rightarrow \mathbf{G}_K \rightarrow \mathbf{A}_K$.

Let $\mathcal{M}_{0,\mathbb{Q}}$ be the abelian category of Artin motives identified with the full subcategory of $\mathcal{M}_{1,\mathbb{Q}}$ whose objects are of the type $[\mathbf{L}_K \rightarrow 0]$, *i.e.*, 1-motives which are pure of weight zero.

Let $\mathcal{M}_1^{\text{a},l}$ be the larger category of generalized 1-motives with *additive factors* over K whose objects are two terms complexes (in degree $-1, 0$) $\mathbf{M}_K = [\mathbf{u}_K: \mathbf{L}_K \rightarrow \mathbf{E}_K]$ where \mathbf{L}_K is a lattice, \mathbf{E}_K is a commutative connected algebraic K -group and \mathbf{u}_K is a morphism of algebraic K -groups. Note that $\mathcal{M}_1^{\text{a},l}$ is a full subcategory of the category \mathcal{M}_1^{a} considered in [4, §1].

Let $\mathbf{M} = [\mathbf{u}: \mathbf{L} \rightarrow \mathbf{G}] \in \mathcal{M}_1(S)$ be a 1-motive over S . Recall (*e.g.* see [2, §2]) that there exists the *universal* \mathbb{G}_a -extension of \mathbf{M} which we denote by

$$\mathbf{M}^{\natural} := [\mathbf{u}^{\natural}: \mathbf{L} \rightarrow \mathbf{G}^{\natural}].$$

It is an extension of \mathbf{M} by a vector group $\mathbb{V}(\mathbf{M})$ such that the homomorphism of push-out

$$\text{Hom}_{\mathcal{O}_S}(\mathbb{V}(\mathbf{M}), W) \longrightarrow \text{Ext}(\mathbf{M}, W)$$

is an isomorphism for all vector groups W over S (where $\text{Hom}_{\mathcal{O}_S}(-, -)$ means homomorphisms of vector groups). If $S = \text{Spec} K$ then $\mathbf{M}_K^{\natural} \in \mathcal{M}_1^{\text{a},l}$ is a 1-motive with additive factors over K (and Ext is taken in $\mathcal{M}_1^{\text{a},l}$).

2.3.1. Definition. The de Rham realization of \mathbf{M} is

$$T_{\text{dR}}(\mathbf{M}) := \text{Lie}(\mathbf{G}^{\natural}).$$

It is easy to check that $\mathbb{V}(\mathbf{M})$ has to be the vector group associated to $\text{Ext}(\mathbf{M}, \mathbb{G}_a)^{\vee}$. Furthermore $\mathbb{V}(\mathbf{M})$ is canonically isomorphic to the vector group associated to the sheaf $\omega_{\mathbf{G}^*}$ of invariant differentials of the semiabelian scheme \mathbf{G}^* Cartier dual of \mathbf{M}_{ab} . We have a push-out diagram

$$(2.6) \quad \begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{V}(\mathbf{G}) = \omega_{\mathbf{A}^*} & \longrightarrow & \mathbf{A}^{\natural} \times_{\mathbf{A}} \mathbf{G} & \longrightarrow & \mathbf{G} \longrightarrow 0 \\ & & \downarrow i & & \downarrow & & \parallel \\ 0 & \longrightarrow & \mathbb{V}(\mathbf{M}) = \omega_{\mathbf{G}^*} & \longrightarrow & \mathbf{G}^{\natural} & \xrightarrow{\rho} & \mathbf{G} \longrightarrow 0 \\ & & \downarrow \bar{\tau} & & \downarrow \tau & & \\ & & \mathbf{L} \otimes \mathbb{G}_a & \xlongequal{\quad} & \mathbf{L} \otimes \mathbb{G}_a & & \end{array}$$

with $A^\natural := \text{Pic}^{\natural,0}(A^*)$ the universal extension of A . Note that the lifting u^\natural of u composed with τ gives the map $L \rightarrow L \otimes \mathbb{G}_a$, $x \mapsto x \otimes 1$; see [5, §2] for details.

Any morphism of 1-motives $\varphi: M \rightarrow N$ provides a morphism $\varphi^\natural: M^\natural \rightarrow N^\natural$ that maps term by term the elements of the corresponding diagrams (2.6); it provides a morphism such that the induced morphism $\mathbb{V}(M) \rightarrow \mathbb{V}(N)$ corresponds to the pull-back on invariant differentials along the induced morphism obtained via Cartier duality from $M_{\text{ab}} \rightarrow N_{\text{ab}}$.

2.3.2. Remark. Assume $S = \text{Spec}(W(k))$, and recall the morphism $u^\natural: L \rightarrow G^\natural$ defining M^\natural . By Lemma 2.2.4 we have a morphism of K_0 -vector spaces

$$\delta_{u^\natural}: \text{Lie}(L \otimes \mathbb{G}_a) \otimes_{W(k)} K_0 \rightarrow \text{Lie}(G^\natural) \otimes_{W(k)} K_0$$

which is a section of $\text{Lie}(\tau) \otimes \text{id}_{K_0}$ with τ as in (2.6). We will show next that these K_0 -vector spaces are endowed with a structure of an F - K_0 -isocrystal and that δ_{u^\natural} commutes with the Frobenius, see Lemma 3.3.1.

3. FULLNESS OF THE OGUS REALIZATION

3.1. Models. By writing $K = \varinjlim_n \mathcal{O}_K[1/n]$ any scheme X_K of finite type over K admits a model $X[1/n]$ of finite presentation over $\mathcal{O}_K[1/n]$ for n sufficiently divisible, *i.e.*, large enough in the preorder given by divisibility (see [17, IV, 8.8.2(ii) p. 28]). Furthermore this model is essentially unique, *i.e.*, models $X[1/n]$ and $X[1/m]$ of X_K become isomorphic on $\mathcal{O}_K[1/N]$, with N a suitable multiple of m and n (see [17, IV, 8.8.2.5 p. 32]).

Any algebraic K -group G_K thus admits a model G which is a group scheme of finite presentation over $S = \text{Spec}(\mathcal{O}_K[1/n])$, for n sufficiently divisible, and G is essentially unique. Note that G may be assumed to be smooth, see [17, IV, Proposition 17.7.8(ii)]. Let $G_{\mathcal{O}_{K_v}}$ denote the base change of G to \mathcal{O}_{K_v} , when it makes sense, and let G_{k_v} be its special fiber.

A morphism of algebraic K -groups $f_K: G_K \rightarrow G'_K$ extends to an S -morphism of schemes between the models (see [17, IV, 8.8.1.1 p. 28]). Up to inverting finitely many primes, one can assume that this is indeed a morphism of S -group schemes.

Let $\mathbf{GCC}(K)$ be the category of commutative connected algebraic K -groups and $\mathbf{GCC}(K)_\mathbb{Q}$ its localization at the class of isogenies. One can define a functor

$$(3.1) \quad \mathbf{Lie}: \mathbf{GCC}(K) \rightarrow \mathbf{BOg}(K)$$

which associates to any commutative connected algebraic K -group G_K the object in $\mathbf{BOg}(K)$ (see §1.5) represented by the triple $(V, L, ({}^bF_v)_{v \in \mathcal{P}_n})$ in $\mathcal{L}_{\mathcal{P}_n}$ defined as follows. Set $V := \text{Lie}(G_K)$, $L := \text{Lie}(G)$ and bF_v on $\text{Lie}(G_{k_v}) \cong L \otimes_{\mathcal{O}_K} k_v$ the canonical σ_v -semilinear homomorphism described in (2.1) and induced by the Verschiebung $\text{Ver}_{G_{k_v}}$ (see Lemma 2.1.1).

Note that L is a $\mathcal{O}_K[1/n]$ -lattice in $\text{Lie}(G_K)$ via the isomorphism $L \otimes_{\mathcal{O}_K} K \cong \text{Lie}(G_K)$ and that $\text{Lie}(G_{k_v}) \cong L \otimes_{\mathcal{O}_K} k_v$ is the canonical isomorphism (see [14, Exp. II, Prop. 3.4 and §3.9.0]).

We have:

3.1.1. Theorem (Bost). *The functor $\mathbf{Lie}: \mathbf{GCC}(K)_\mathbb{Q} \rightarrow \mathbf{BOg}(K)$ is fully faithful.*

Proof. The difficult part in the proof is the fullness. See [9, Thm. 2.3 & Cor. 2.6] and [10] for details. On the other hand, the proof of the faithfulness is immediate. Let $f_K: G_K \rightarrow G'_K$ be a morphism of (commutative connected) algebraic K -groups and let $f: G \rightarrow G'$ be a morphism of $\mathcal{O}_K[1/n]$ -group schemes which extends it, for n sufficiently divisible. If $\text{Lie}(f_K) = 0$ then $f_K = 0$, see [13, II, §6, n. 2, Prop. 2.1 (b)]. \square

Now consider a 1-motive \mathbf{M}_K over K and models \mathbf{M} over $S = \text{Spec}(\mathcal{O}_K[1/n])$, i.e., 1-motives \mathbf{M} over S such that the base change to K is isomorphic to \mathbf{M}_K , for n sufficiently divisible. We have:

3.1.2. Lemma. *Any 1-motive $\mathbf{M}_K = [\mathbf{u}_K: \mathbf{L}_K \rightarrow \mathbf{G}_K]$ over K admits a model $\mathbf{M} = [\mathbf{u}: \mathbf{L} \rightarrow \mathbf{G}]$ over an $S = \text{Spec}(\mathcal{O}_K[1/n])$ for n sufficiently divisible. This model is essentially unique, i.e., any two such models are isomorphic over a $\text{Spec}(\mathcal{O}_K[1/m])$ with $n|m$.*

Proof. Let \mathbf{T}_K be the maximal torus in \mathbf{G}_K and let \mathbf{A}_K be the maximal abelian quotient of \mathbf{G}_K . Let \mathbf{T} be the model of \mathbf{T}_K over S . We may assume that it is a torus. Indeed \mathbf{T}_K becomes split over a finite Galois extension K' of K and, up to enlarging n , we may assume that $S' = \text{Spec}(\mathcal{O}_{K'}[1/n])$ is étale over $S = \text{Spec}(\mathcal{O}_K[1/n])$. Now, \mathbf{T} is a torus since its base change to S' is a split torus by the essential unicity of the model. Further, by Cartier duality, the group of characters of any torus over K admits a model over an S which is étale locally a split lattice of fixed rank. In particular \mathbf{L}_K extends to such a model \mathbf{L} over S . On the other side, we may assume that the model \mathbf{A} of \mathbf{A}_K is an abelian scheme over S [8, §1.2, Thm. 3], hence \mathbf{A} is the global Néron model of \mathbf{A}_K over S . Finally, by [8, §10.1, Prop. 4 & 7] \mathbf{G}_K admits a global Néron lft-model \mathcal{G} over S and the identity component $\mathbf{G} := \mathcal{G}^0$ is a smooth model of \mathbf{G}_K of finite type over S .

Recall that \mathbf{G}_K is the Cartier dual of the 1-motive $\mathbf{M}_{\text{ab},K}^* = [\mathbf{u}_{\mathbf{A}_K}^*: \mathbf{Y}_K \rightarrow \mathbf{A}_K^*]$ where \mathbf{Y}_K is the group of characters of \mathbf{T}_K . By the above discussion on models of abelian varieties and lattices and by the universal property of Néron models [8, §1.2, Definition 1], the 1-motive $\mathbf{M}_{\text{ab},K}^*$ extends uniquely to a 1-motive $\mathbf{M}_{\text{ab}}^* = [\mathbf{u}_{\mathbf{A}}^*: \mathbf{Y} \rightarrow \mathbf{A}^*]$ over S and, by the essential unicity of the model of \mathbf{G}_K , \mathbf{G} is the Cartier dual of \mathbf{M}_{ab}^* , in particular it is extension of the abelian scheme \mathbf{A} by the torus \mathbf{T} .

Clearly \mathbf{u}_K extends uniquely to a morphism $\mathbf{u}: \mathbf{L} \rightarrow \mathcal{G}$. It remains to check that up to enlarging n , \mathbf{u} factors through the open subscheme $\mathbf{G} = \mathcal{G}^0$ of \mathcal{G} . Since the formation of lft Néron models is compatible with étale base change we may assume that \mathbf{T} is a split torus over S and $\mathbf{L} \simeq \mathbb{Z}^r$. Fix a basis of \mathbb{Z}^r and let $e_i: S \rightarrow \mathbb{Z}^r, i = 1, \dots, r$, denote the corresponding morphisms. It is sufficient to check that up to enlarging n each map $\mathbf{u} \circ e_i$ (which corresponds to a K -rational point of \mathbf{G}_K) factors through \mathbf{G} . This follows from [17, IV, 8.8.1.1]. \square

3.2. The Ogus realization. For a 1-motive $\mathbf{M}_K = [\mathbf{u}_K: \mathbf{L}_K \rightarrow \mathbf{G}_K]$ over K consider its model $\mathbf{M} = [\mathbf{u}: \mathbf{L} \rightarrow \mathbf{G}]$ over a $S = \text{Spec}(\mathcal{O}_K[1/n])$ given in Lemma 3.1.2. For any $v \in \mathcal{P}_n$ let $\mathbf{M}_{\mathcal{O}_{K_v}}$ be the base change to \mathcal{O}_{K_v} of the model \mathbf{M} and let \mathbf{M}_{k_v} be the reduction of $\mathbf{M}_{\mathcal{O}_{K_v}}$ modulo \mathfrak{p}_v .

3.2.1. Definition. Let $V := T_{\text{dR}}(\mathbf{M}_K)$ be the de Rham realization (see Definition 2.3.1) as a K -vector space. Let $V_v := T_{\text{dR}}(\mathbf{M}_{\mathcal{O}_{K_v}}) \otimes_{\mathcal{O}_{K_v}} K_v$ together with the induced isomorphism obtained by $g_v: V \otimes_K K_v \rightarrow V_v$ and compatibility of T_{dR} with base change.

Consider a model \mathbf{M} over $S = \mathrm{Spec}(\mathcal{O}_K[1/n])$ as in Lemma 3.1.2. For every $v \in \mathcal{P}_n$ consider the canonical isomorphism $T_{\mathrm{dR}}(\mathbf{M}_{\mathcal{O}_{K_v}}) \simeq T_{\mathrm{cris}}(\mathbf{M}_{k_v})$ as stated in [2, Cor. 4.2.1]. Via this identification $T_{\mathrm{dR}}(\mathbf{M}_{\mathcal{O}_{K_v}})$ is endowed with a σ_v^{-1} -semilinear endomorphism Φ_v (Verschiebung). Let F_v be the σ_v -semilinear endomorphism $(\Phi_v \otimes \mathrm{id}_{K_v})^{-1}$ on V_v . In this way, we have associated to \mathbf{M}_K an object $\mathcal{V} := (V, (V_v, F_v)_{v \in \mathcal{P}_n}, (g_v)_{v \in \mathcal{P}_n})$ in $\mathbf{Og}(K)$. The usual weight filtration on 1-motives induces a weight filtration $W_\bullet \mathcal{V}$ on \mathcal{V} in $\mathbf{Og}(K)$. We shall denote

$$T_{\mathrm{Og}}(\mathbf{M}_K) := (\mathcal{V}, W_\bullet \mathcal{V}).$$

Note that Φ_v is, in general, not invertible on $T_{\mathrm{dR}}(\mathbf{M}_{\mathcal{O}_{K_v}})$ and this F_v is the Frobenius of [2, §4.1] divided by p_v (see [2, §4.3 (4.b)]), *i.e.*, $p_v F_v$ is the map associated to the Verschiebung $\mathbf{M}_{\mathcal{O}_{K_v}}^{(p_v)} \rightarrow \mathbf{M}_{\mathcal{O}_{K_v}}$. This choice is made so that the weight filtration on 1-motives and on the isocrystals are compatible. We have:

3.2.2. Lemma. $T_{\mathrm{Og}}(\mathbf{M}_K) \in \mathbf{FOg}(K)$.

Proof. In fact $W_0 \mathcal{V} / W_{-1} \mathcal{V} = T_{\mathrm{Og}}([\mathbf{L}_K \rightarrow 0])$ so that the underlying K -vector space is $T_{\mathrm{dR}}([\mathbf{L}_K \rightarrow 0]) = \mathbf{L}_K \otimes K$ and similarly for $W_{-2} \mathcal{V} = T_{\mathrm{Og}}([0 \rightarrow \mathbf{T}_K])$ with underlying vector space $Y_K \otimes K$ where Y_K the cocharacter group of the torus \mathbf{T}_K . For \mathbf{T} the S -torus which is a model of \mathbf{T}_K let Y be the group of cocharacters of \mathbf{T} . According to [2, §4.1] for almost all unramified places v the σ_v -semilinear homomorphism F_v on $T_{\mathrm{cris}}([\mathbf{L}_{k_v} \rightarrow 0]) = \mathbf{L} \otimes W(k_v)$ (respectively on $T_{\mathrm{cris}}([0 \rightarrow \mathbf{T}_{k_v}]) = Y \otimes W(k_v)$) is $1 \otimes \sigma_v$ (respectively the map $1 \mapsto 1 \otimes p_v^{-1} \sigma_v$). Hence $W_0 \mathcal{V} / W_{-1} \mathcal{V}$ and $W_{-2} \mathcal{V}$ are pure of weight 0 and -2 respectively (in the sense of §1.3).

On the other hand $W_{-1} \mathcal{V} / W_{-2} \mathcal{V} = T_{\mathrm{Og}}(\mathbf{A}_K)$ and \mathbf{A} is a model over S of the abelian variety \mathbf{A}_K . Thanks to [2, Thm. A, p. 111] for every unramified place v of good reduction for \mathbf{A}_K we can identify $T_{\mathrm{cris}}([0 \rightarrow \mathbf{A}_{k_v}])$ and the σ_v -semilinear homomorphism F_v with the covariant Dieudonné module or equivalently with the crystalline homology of the reduction \mathbf{A}_{k_v} of \mathbf{A} . The latter defines a pure F - K_v -isocrystal of weight -1 thanks to a homological version of [19, Cor. 1 (2)]. \square

Recall that a functor between abelian categories with a weight filtration respects the splittings (in the sense of [6, Def. D.2.2]) if it takes pure objects to pure objects of the same weight.

3.2.3. Proposition. *There is a functor*

$$T_{\mathrm{Og}}: \mathcal{M}_{1, \mathbb{Q}} \rightarrow \mathbf{FOg}(K)$$

which associates to a 1-motive \mathbf{M}_K the object $T_{\mathrm{Og}}(\mathbf{M}_K)$ in $\mathbf{FOg}(K)$ provided by Lemma 3.2.2. This functor respects the splittings and its essential image is contained in $\mathbf{FOg}(K)_{(1)}$ (see Definition 1.4.4).

Proof. It follows from the proof of the Lemma 3.2.2 and Remark 1.4.3 (b). In fact, $T_{\mathrm{Og}}(\mathbf{M}_K)$ is l-effective and e-effective in weight 0 and Artin-Lefschetz in weight -2 . Moreover, it is e-effective in weight -1 by [19, Cor. 1 (2)]. Let $L := T_{\mathrm{dR}}(\mathbf{M})$ be the Lie algebra of the universal extension of the model \mathbf{M} over $\mathrm{Spec}(\mathcal{O}_K[1/n])$ as in Lemma 3.1.2. It is a $\mathcal{O}_K[1/n]$ -lattice in $V = T_{\mathrm{dR}}(\mathbf{M}_K)$ and $L \otimes \mathcal{O}_{K_v} \cong T_{\mathrm{dR}}(\mathbf{M}_{\mathcal{O}_{K_v}})$ is preserved by $p_v F_v$ as remarked above. Hence $T_{\mathrm{Og}}(\mathbf{M}_K)(-1) \in \mathbf{FOg}(K)^{\mathrm{eff}}$. \square

3.2.4. **Definition.** The Bost-Ogus realization

$$T_{\text{BOg}}: \mathcal{M}_{1, \mathbb{Q}} \rightarrow \mathbf{BOg}(K)$$

associates to a 1-motive M_K an object $T_{\text{BOg}}(M_K) := \Psi T_{\text{Og}}(M_K)(-1)$ in $\mathbf{BOg}(K)$ where Ψ is the functor defined in (1.1).

Note that if $M_K = [L_K \rightarrow G_K]$, then $T_{\text{BOg}}(M_K) = \mathbf{Lie}(G_K^\natural)$ with \mathbf{Lie} as in (3.1).

3.3. **The main Theorem.** For a 1-motive $M_K = [u_K: L_K \rightarrow G_K]$ over K consider the universal extension M^\natural of a model M over $S = \text{Spec}(\mathcal{O}_K[1/n])$ and the morphism τ in (2.6). For every unramified place v in \mathcal{P}_n let $M_{\mathcal{O}_{K_v}}^\natural = [L_{\mathcal{O}_{K_v}} \rightarrow G_{\mathcal{O}_{K_v}}^\natural]$ be the base change of M^\natural to \mathcal{O}_{K_v} . Recall the K_v -linear map

$$\delta_v: \mathbf{Lie}(L_{\mathcal{O}_{K_v}} \otimes \mathbb{G}_a) \otimes_{\mathcal{O}_{K_v}} K_v \rightarrow \mathbf{Lie}(G_{\mathcal{O}_{K_v}}^\natural) \otimes_{\mathcal{O}_{K_v}} K_v$$

considered in Remark 2.3.2.

3.3.1. **Lemma.** *The homomorphism δ_v is the unique section of $\mathbf{Lie}(\tau_{K_v})$ in the category of F - K_v -isocrystals.*

Proof. By Remark 2.3.2 δ_v is a K_v -linear section of $\mathbf{Lie}(\tau_{K_v})$. Since $p_v F_v$ on $T_{\text{dR}}(M_{K_v})$ is the K_v -linear extension of the Frobenius $f_{\mathcal{O}_{K_v}}$ on $T_{\text{cris}}(M_{k_v})$ discussed in [2, §4.1], it suffices to show that δ_v is Frobenius equivariant and, hence, provides a splitting of F - K_v -isocrystals. Now Lemma 1.2.1(iii) and the fact that morphisms of pure F - K_v -isocrystals of different weight are trivial imply that such a splitting is unique.

Recall that $f_{\mathcal{O}_{K_v}}$ on $T_{\text{cris}}(M_{k_v})$ is defined by the Verschiebung $M_{k_v}^{(p)} \rightarrow M_{k_v}$ thanks to the crystalline nature of universal extensions (see [2, §4], [3, Thm. 2.1]). More precisely, for any $n \in \mathbb{N}$ there is a canonical lift $V_n^\natural: \sigma_v^* M_{W_n(k_v)}^\natural \rightarrow M_{W_n(k_v)}^\natural$ of the Verschiebung on $M_{k_v}^\natural$ where σ_v also denotes the Frobenius on $\text{Spec}(W_n(k_v))$, and the homomorphism $\mathbf{Lie}(V_n^\natural): T_{\text{dR}}(M_{W_n(k_v)}) \otimes_{\sigma_v} W_n(k_v) \rightarrow T_{\text{dR}}(M_{W_n(k_v)})$ defines the Frobenius $f_{W_n(k_v)}$ on $T_{\text{dR}}(M_{W_n(k_v)})$. Now, the construction is compatible with the truncation maps. Hence the morphisms V_n^\natural , $n \geq 1$, provide a morphism V^\natural of complexes of formal schemes over $\mathcal{O}_{K_v} = W(k_v)$

$$\begin{array}{ccc} \sigma_v^* L_{\mathcal{O}_{K_v}} & \xrightarrow{(V^\natural)_{-1}} & L_{\mathcal{O}_{K_v}} \\ \downarrow \sigma_v^* \hat{u} & & \downarrow \hat{u} \\ \sigma_v^* \hat{G}_{\mathcal{O}_{K_v}}^\natural & \xrightarrow{(V^\natural)_0} & \hat{G}_{\mathcal{O}_{K_v}}^\natural. \end{array}$$

Note that since the Verschiebung on L_{k_v} can be identified with the multiplication by p_v , the morphism $(V^\natural)_{-1}$ maps $x \in L_{\mathcal{O}_{K_v}} \times_{\mathcal{O}_{K_v}, \sigma_v} \text{Spf}(\mathcal{O}_{K_v})$ to $p_v x \in L_{\mathcal{O}_{K_v}}$. By the functoriality of (the formal) δ_v in Remark 2.2.5 we then get a commutative diagram

$$\begin{array}{ccc} \mathbf{Lie}(L_{\mathcal{O}_{K_v}} \otimes \mathbb{G}_a) \otimes_{\mathcal{O}_{K_v}} K_v \otimes_{\sigma_v} K_v & \xrightarrow{x \otimes 1 \otimes 1 \rightarrow p_v x \otimes 1} & \mathbf{Lie}(L_{\mathcal{O}_{K_v}} \otimes \mathbb{G}_a) \otimes_{\mathcal{O}_{K_v}} K_v \\ \downarrow \delta_v \otimes \text{id} & & \downarrow \delta_v \\ \mathbf{Lie}(G_{\mathcal{O}_{K_v}}^\natural) \otimes_{\mathcal{O}_{K_v}} K_v \otimes_{\sigma_v} K_v & \xrightarrow{\quad\quad\quad} & \mathbf{Lie}(G_{\mathcal{O}_{K_v}}^\natural) \otimes_{\mathcal{O}_{K_v}} K_v. \end{array}$$

Since the upper (respectively, lower) horizontal arrow defines the Frobenius $f_{\mathcal{O}_{K_v}}$ on $T_{\text{dR}}([L_{K_v} \rightarrow 0])$ (respectively, on $T_{\text{dR}}(M_{K_v})$), the result follows. \square

3.3.2. Lemma. *The functor T_{Og} is faithful.*

Proof. Let $\varphi: \mathbf{M}_K \rightarrow \mathbf{N}_K$ be a morphism of 1-motives such that $T_{\text{Og}}(\varphi) = 0$. In particular $T_{\text{dR}}(\varphi) = 0$. Then $n\varphi = 0$ for a suitable n . Indeed, let $\varphi_{\mathbb{C}}$ be the base change of φ to \mathbb{C} . Then $T_{\text{dR}}(\varphi_{\mathbb{C}}) = 0$ implies $T_{\mathbb{C}}(\varphi_{\mathbb{C}}): T_{\mathbb{Z}}(\mathbf{M}_{\mathbb{C}}) \otimes_{\mathbb{Z}} \mathbb{C} \rightarrow T_{\mathbb{Z}}(\mathbf{N}_{\mathbb{C}}) \otimes_{\mathbb{Z}} \mathbb{C}$ is the zero map by [15, 10.1.8]. Hence $nT_{\mathbb{Z}}(\varphi_{\mathbb{C}}) = 0$ for a suitable n . Then one concludes that $n\varphi_{\mathbb{C}} = 0$ by [15, 10.1.3] and hence $n\varphi = 0$. \square

3.3.3. Remark. One could provide an alternative proof of Lemma 3.3.2 using an argument similar to the one adopted for the faithfulness in Theorem 3.1.1.

3.3.4. Lemma. *The functor T_{Og} restricted to $\mathcal{M}_{0,\mathbb{Q}}$ is full.*

Proof. First consider two 1-motives $\mathbf{M}_K = [\mathbb{Z}^r \rightarrow 0]$, $\mathbf{N}_K = [\mathbb{Z}^s \rightarrow 0]$. Write e_1, \dots, e_r for the standard basis of \mathbb{Z}^r and similarly for \mathbb{Z}^s . We use the same letters for the induced bases on the de Rham realizations. Any morphism $\psi: T_{\text{dR}}(\mathbf{M}_K) \rightarrow T_{\text{dR}}(\mathbf{N}_K)$ corresponds to an $s \times r$ matrix $C = (c_{ij}) \in M_{s,r}(K)$, i.e., $\psi(e_j) = \sum_i c_{ij} e_i$. Observe that $c_{ij} \in \mathcal{O}[1/n]$ for n sufficiently divisible and hence $C \in M_{s,r}(\mathcal{O}_{K_v})$ for $v \in \mathcal{P}_n$. If ψ is a morphism in $\mathbf{Og}(K)$, the compatibility with the F_v 's implies that $\sigma_v(C) = C$ where $\sigma_v(C) = (\sigma_v(c_{ij}))$. Indeed

$$F_v(\psi(e_j)) = F_v\left(\sum_i c_{ij} e_i\right) = \sum_i \sigma_v(c_{ij}) e_i,$$

and

$$\psi(F_v(e_j)) = \psi(e_j) = \sum_i c_{ij} e_i.$$

Now $\sigma_v(c_{ij}) = c_{ij}$ for all $v \in \mathcal{P}_n$, is equivalent to $c_{ij}^{p_v} \equiv c_{ij} \pmod{p_v}$ for all $v \in \mathcal{P}_n$. We conclude by Kronecker's theorem that $c_{ij} \in \mathbb{Q}$; hence $C \in M_{s,r}(\mathbb{Q})$. Let m be the a positive integer such that $mC \in M_{s,r}(\mathbb{Z})$. Then $m\psi = \text{Lie}(\varphi \otimes \text{id})$ with $\varphi \in \text{Hom}(\mathbf{M}_K, \mathbf{N}_K) \simeq M_{s,r}(\mathbb{Z})$ the homomorphism which maps e_j to $\sum_i m c_{ij} e_i$ in degree -1 .

We conclude the proof of the fullness by Galois descent. Let $\mathbf{M}_K = [\mathbf{L}_K \rightarrow 0]$, $\mathbf{N}_K = [\mathbf{F}_K \rightarrow 0]$ be 1-motives in $\mathcal{M}_{0,\mathbb{Q}}$ and let $\psi: T_{\text{dR}}(\mathbf{M}_K) \rightarrow T_{\text{dR}}(\mathbf{N}_K)$ be a morphism in $\mathbf{FOg}(K)$; in particular the base change of ψ to K_v is compatible with the F_v 's for almost all places v . Let K' be a finite Galois extension of K such that $\mathbf{L}_{K'}$ and $\mathbf{F}_{K'}$ are split. Then, any unramified place v' of K' is above an unramified place v of K and $K'_{v'}/K_v$ is a finite unramified extension. Then by [2, Corollary 4.2.1] and the fact that the de Rham realization and the Verschiebung morphism are compatible with extension of the base, the formation of $T_{\text{Og}}(\mathbf{M}_K)$ behaves well with respect to base field extension. Let $\psi_{K'}: T_{\text{dR}}(\mathbf{M}_{K'}) \rightarrow T_{\text{dR}}(\mathbf{N}_{K'})$ be the morphisms in $\mathbf{FOg}(K')$ induced by ψ . Denote by the same letter also the associated morphism of vector groups $\mathbf{L}_{K'} \otimes \mathbb{G}_a \rightarrow \mathbf{F}_{K'} \otimes \mathbb{G}_a$. By above discussions we may assume that $\psi_{K'}$ comes from a morphism $\varphi_{K'}: \mathbf{L}_{K'} \rightarrow \mathbf{F}_{K'}$. Let us denote by ζ both an element in $\text{Gal}(K'/K)$ and the induced morphism on 1-motives. In order to check that $\varphi_{K'}$ descends to K , it suffices to check that $\zeta \circ \varphi_{K'} = \varphi_{K'} \circ \zeta$. Since we work over fields of characteristic 0, the morphism $\iota_{\mathbb{F}}: \mathbf{F}_{K'} \rightarrow \mathbf{F}_{K'} \otimes \mathbb{G}_a, x \mapsto x \otimes 1$ has trivial kernel. Hence it is sufficient to check that $\iota_{\mathbb{F}} \circ \zeta \circ \varphi_{K'} = \iota_{\mathbb{F}} \circ \varphi_{K'} \circ \zeta$. Now,

$$\iota_{\mathbb{F}} \circ \zeta \circ \varphi_{K'} = \zeta \circ \iota_{\mathbb{F}} \circ \varphi_{K'} = \zeta \circ \psi_{K'} \circ \iota_{\mathbb{L}} = \psi_{K'} \circ \zeta \circ \iota_{\mathbb{L}} = \psi_{K'} \circ \iota_{\mathbb{L}} \circ \zeta = \iota_{\mathbb{F}} \circ \varphi_{K'} \circ \zeta.$$

This concludes the proof. \square

3.3.5. Theorem. *The functor $T_{\text{Og}}: \mathcal{M}_{1,\mathbb{Q}} \rightarrow \mathbf{FOg}(K)$ is fully faithful.*

Proof. The faithfulness was proved in Lemma 3.3.2. For the fullness, let $\mathbf{M}_K = [\mathbf{u}_K: \mathbf{L}_K \rightarrow \mathbf{G}_K]$ and $\mathbf{N}_K = [\mathbf{v}_K: \mathbf{F}_K \rightarrow \mathbf{H}_K]$ be 1-motives. Suppose given a morphism $\psi: T_{\text{Og}}(\mathbf{M}_K) \rightarrow T_{\text{Og}}(\mathbf{N}_K)$ in $\mathbf{FOg}(K)$. By Definition 3.2.4 we also get a morphism $\psi': T_{\text{BOg}}(\mathbf{M}_K) \rightarrow T_{\text{BOg}}(\mathbf{N}_K)$ in $\mathbf{BOg}(K)$. Using Theorem 3.1.1 the morphism ψ' comes from a morphism $\tilde{g}_K: \mathbf{G}_K^{\natural} \rightarrow \mathbf{H}_K^{\natural}$, i.e., $\text{Lie}(\tilde{g}_K) = m\psi'$ in $\mathbf{BOg}(K)$ for a suitable $m \in \mathbb{N}$. We may assume $m = 1$. By Chevalley theorem (see [12, Lemma 2.3]) \tilde{g}_K yields a morphism on the semi-abelian quotients $g_K: \mathbf{G}_K \rightarrow \mathbf{H}_K$. Now consider the morphism $T_{\text{Og}}(g_K): T_{\text{Og}}(\mathbf{G}_K) \rightarrow T_{\text{Og}}(\mathbf{H}_K)$ and compare with the morphism induced by ψ on weight -1 parts as displayed in the following commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & T_{\text{Og}}(\mathbf{G}_K) & \longrightarrow & T_{\text{Og}}(\mathbf{M}_K) & \longrightarrow & T_{\text{Og}}(\mathbf{L}_K[1]) \longrightarrow 0 \\ & & \downarrow \psi_{-1} & & \downarrow \psi & & \downarrow \psi_0 \\ 0 & \longrightarrow & T_{\text{Og}}(\mathbf{H}_K) & \longrightarrow & T_{\text{Og}}(\mathbf{N}_K) & \longrightarrow & T_{\text{Og}}(\mathbf{F}_K[1]) \longrightarrow 0 \end{array}$$

Since by construction $T_{\text{BOg}}(g_K) = \psi'_{-1}$ in $\mathbf{BOg}(K)$ we deduce that $T_{\text{Og}}(g_K) = \psi_{-1}$ as well. In fact, $T_{\text{dR}}(g_K) = T_{\text{BOg}}(g_K) = T_{\text{Og}}(g_K)$ coincide on the underlying K -vector spaces via T_{dR} .

It follows from the Lemma 3.3.4 that there exists a morphism $f_K: \mathbf{L}_K \rightarrow \mathbf{F}_K$ such that $T_{\text{Og}}(f_K) = m\psi_0: T_{\text{Og}}(\mathbf{L}_K[1]) \rightarrow T_{\text{Og}}(\mathbf{F}_K[1])$ for an $m \in \mathbb{N}$. As above, we may assume $m = 1$.

Note that if the pair (f_K, g_K) gives a morphism of 1-motives $\varphi_K: \mathbf{M}_K \rightarrow \mathbf{N}_K$, i.e., if $g_K \circ \mathbf{u}_K = \mathbf{v}_K \circ f_K$, then $T_{\text{BOg}}(\varphi_K) = \psi'$ by construction. As the functor Ψ of (1.1) is faithful, the fact that $T_{\text{Og}}(\varphi_K) = \psi$ induces the zero morphism between the Bost-Ogus realizations implies that $T_{\text{Og}}(\varphi_K) = \psi$ in $\mathbf{FOg}(K)$.

It is then sufficient to check that, up to multiplication by a positive integer, we have $\tilde{g}_K \circ \mathbf{u}_K^{\natural} = \mathbf{v}_K^{\natural} \circ f_K$. After replacing K with a finite extension we may further assume that \mathbf{L} and \mathbf{F} are constant. Take v an unramified place of good reduction both for \mathbf{M}_K and \mathbf{N}_K such that \tilde{g}_K extends to a morphism $\tilde{g}: \mathbf{G}^{\natural} \rightarrow \mathbf{H}^{\natural}$ over $W(k_v) = \mathcal{O}_{K_v}$. In particular, \mathbf{u}_K^{\natural} and \mathbf{v}_K^{\natural} extend to morphisms $\mathbf{u}^{\natural}: \mathbf{L} \rightarrow \mathbf{G}^{\natural}, \mathbf{v}^{\natural}: \mathbf{F} \rightarrow \mathbf{H}^{\natural}$ over \mathcal{O}_{K_v} and hence are determined by the induced homomorphisms between the \mathcal{O}_{K_v} -rational points. It suffices then to prove that the following diagram

$$(3.2) \quad \begin{array}{ccc} \mathbf{L}(\mathcal{O}_{K_v}) & \xrightarrow{\mathbf{u}^{\natural}} & \mathbf{G}^{\natural}(\mathcal{O}_{K_v}) \\ \downarrow f & & \downarrow \tilde{g} \\ \mathbf{F}(\mathcal{O}_{K_v}) & \xrightarrow{\mathbf{v}^{\natural}} & \mathbf{H}^{\natural}(\mathcal{O}_{K_v}) \end{array}$$

commutes, up to multiplication by a positive integer. Consider the following diagram

$$\begin{array}{ccccccc} \mathbf{L}(\mathcal{O}_{K_v}) & \xrightarrow{\alpha_{\mathbf{L}}} & \text{Lie}(\mathbf{L} \otimes \mathbf{G}_a) \otimes K_v & \xrightarrow{\delta_v^{\mathbf{M}}} & \text{Lie}(\mathbf{G}^{\natural}) \otimes K_v & \xrightarrow{\log_{\mathbf{G}^{\natural}, \mathbb{Q}}^{-1}} & \mathbf{G}^{\natural}(\mathcal{O}_{K_v}) \otimes_{\mathbb{Z}} \mathbb{Q} \\ \downarrow f & & \downarrow \psi_0 \otimes K_v & & \downarrow \psi \otimes K_v & & \downarrow \tilde{g} \otimes \mathbb{Q} \\ \mathbf{F}(\mathcal{O}_{K_v}) & \xrightarrow{\alpha_{\mathbf{F}}} & \text{Lie}(\mathbf{F} \otimes \mathbf{G}_a) \otimes K_v & \xrightarrow{\delta_v^{\mathbf{N}}} & \text{Lie}(\mathbf{H}^{\natural}) \otimes K_v & \xrightarrow{\log_{\mathbf{H}^{\natural}, \mathbb{Q}}^{-1}} & \mathbf{H}^{\natural}(\mathcal{O}_{K_v}) \otimes_{\mathbb{Z}} \mathbb{Q} \end{array}$$

where the map α_L is the composition of the canonical map $L(\mathcal{O}_{K_v}) \rightarrow (L \otimes \mathbb{G}_a)(\mathcal{O}_{K_v}) \otimes_{\mathbb{Z}} \mathbb{Q}$, $x \mapsto (x \otimes 1) \otimes 1$ with $\log_{L \otimes \mathbb{G}_a, \mathbb{Q}}$ and similarly for α_F . Note that we here identify $T_{\text{dR}}(\mathbf{L}_K[1])_v$ with $\text{Lie}(L \otimes \mathbb{G}_a) \otimes K_v$ and $T_{\text{dR}}(\mathbf{F}_K[1])_v$ with $\text{Lie}(F \otimes \mathbb{G}_a) \otimes K_v$; we also identify $T_{\text{dR}}(\mathbf{M}_K)_v$ with $\text{Lie}(\mathbf{G}^{\natural}) \otimes K_v$ and $T_{\text{dR}}(\mathbf{N}_K)_v$ with $\text{Lie}(\mathbf{H}^{\natural}) \otimes K_v$. Hence the most left square commutes since by definition $T_{\text{Og}}(f_K) = \psi_0$. The commutativity of the square in the middle follows from Lemma 3.3.1 as ψ and ψ_0 are morphisms in $\mathbf{Og}(K)$ so that $\psi \otimes K_v$ and $\psi_0 \otimes K_v$ commutes with F_v . The last square on the right commutes by functoriality of the logarithm as $\psi \otimes K_v = \psi' \otimes K_v = \text{Lie}(\tilde{g}_K \otimes K_v)$ on the underlying K_v -vector spaces. Finally, note that the composition of the upper (respectively, lower) horizontal arrows is $u^{\natural} \otimes \text{id}_{\mathbb{Q}}$ (respectively, $v^{\natural} \otimes \text{id}_{\mathbb{Q}}$) by definition of δ_v in (2.5). Hence (3.2) commutes up to multiplication by a positive integer. \square

3.3.6. Example. Let $\mathbf{M}_K = [u_K: \mathbb{Z} \rightarrow \mathbb{G}_{m,K}]$ and $\mathbf{N}_K = [v_K: \mathbb{Z} \rightarrow \mathbb{G}_{m,K}]$ be two 1-motives over K . Set $a := u_K(1) \in K^*$ and $b := v_K(1) \in K^*$. Note that any morphism $(f_K, g_K): \mathbf{M}_K \rightarrow \mathbf{N}_K$ is of the type $f_K = m$, $g_K = r$ with $a^r = b^m$. In particular, for general $a, b \in K^*$ the unique morphism between \mathbf{M}_K and \mathbf{N}_K is the zero morphism. By Definition 2.3.1, it follows from (2.6) that

$$T_{\text{dR}}(\mathbf{M}_K) = T_{\text{dR}}(\mathbf{N}_K) = \text{Lie}(\mathbb{G}_{m,K}) \oplus \text{Lie}(\mathbb{G}_{a,K}) = K \oplus K.$$

The de Rham realisation of the two 1-motives yields objects in $\mathbf{Og}(K)$ with the same underlying structure of filtered K -vector spaces: the filtration being induced by the weight filtration $W_{-1}\mathbf{M}_K = [0 \rightarrow \mathbb{G}_{m,K}]$ and $\text{gr}_0^W \mathbf{M}_K = [\mathbb{Z} \rightarrow 0]$. As described in Definition 3.2.1 (cf. the proof of Lemma 3.2.2), for any unramified place v of K , $T_{\text{dR}}(\mathbf{M}_{K_v}) = K_v \oplus K_v$ is endowed with the σ_v -semilinear operator F_v such that for $(x, y) \in K_v \oplus K_v$ we have that $F_v(x, 0) = (p_v^{-1}\sigma_v(x), 0)$ and $\overline{F_v((x, y))} = \sigma_v(y)$ where the $\overline{(\quad)}$ stands for the class in the weight 0 quotient K_v ; the same holds for \mathbf{N}_K .

Note that, in general, we have several K -linear homomorphisms $T_{\text{dR}}(\mathbf{M}_K) \rightarrow T_{\text{dR}}(\mathbf{N}_K)$ that preserve the filtration and commute with the F_v 's on the graded pieces of the filtration but do not arise from morphisms of 1-motives (even up to isogeny). For example, take $a = 1$ and $b = 2$ and the identity map on $K \oplus K$. This example shows that knowing fullness for the pure weight parts is not enough to deduce the Theorem 3.3.5.

For general $a, b \in K^*$, knowing F_v on $T_{\text{dR}}(\mathbf{M}_K) \otimes K_v = K_v \oplus K_v$ is equivalent to give a Frobenius equivariant splitting $\delta_v^{\mathbf{M}}: K_v \rightarrow K_v \oplus K_v$ of the weight 0 quotient K_v thanks to Lemma 3.3.1. Now, assume $a, b \in \mathcal{O}_{K_v}$. Then, by Examples 2.2.2, we can write $\delta_v^{\mathbf{M}}(1) = (\log(a^{p_v^{n_v} - 1}})/p_v(p_v^{n_v} - 1) \oplus 1$. As a consequence, the identity map on $K \oplus K$ commutes with the sections $\delta_v^{\mathbf{M}}, \delta_v^{\mathbf{N}}$ if and only if $a^{p_v^{n_v} - 1} = b^{p_v^{n_v} - 1}$, thus if, and only if, $a^{p_v^{n_v} - 1} = b^{p_v^{n_v} - 1}$ (as K_v does not contain non-trivial p_v -roots of unity being absolutely unramified). We conclude that there exists a morphism $T_{\text{Og}}(\mathbf{M}_K) \rightarrow T_{\text{Og}}(\mathbf{N}_K)$ in $\mathbf{FOg}(K)$ which is identity on the underlying vector spaces if and only if $\mathbf{M}_K = \mathbf{N}_K$.

3.3.7. Remark. If we work with $\mathbf{BOg}(K)$ and even with the filtered analogue, it is not true that the functor T_{BOg} is full on 1-motives, in general. For example, take $\mathbf{M} = \mathbb{Z}[1]$ and note that $\text{End}_{\mathcal{M}_{1,\mathbb{Q}}}(\mathbf{M}) = \mathbb{Q}$ while $\text{End}_{\mathbf{BOg}(K)}(T_{\text{BOg}}(\mathbf{M})) = K$.

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