

# Fractional exclusion and braid statistics in one dimension: a study via dimensional reduction of Chern-Simons theory<sup>‡</sup>

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**Abstract.** The relation between braid and exclusion statistics is examined in one-dimensional systems, within the framework of Chern-Simons statistical transmutation in gauge invariant form with an appropriate dimensional reduction. If the matter action is anomalous, as for chiral fermions, a relation between braid and exclusion statistics can be established explicitly for both mutual and nonmutual cases. However, if it is not anomalous, the exclusion statistics of emergent low energy excitations is not necessarily connected to the braid statistics of the physical charged fields of the system. Finally, we also discuss the bosonization of one-dimensional anyonic systems through T-duality.

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## 1. Introduction

In quantum mechanics of identical particles, there are two ways, a priori rather different, to define the statistics.

One is the *braid*(or exchange) statistics, which can be defined through the monodromy of the many-body wave-function as follows: For  $\mathbb{C}$ -valued wave functions, when one performs a positively oriented exchange between two particles, the wave function acquires a phase factor  $e^{i(1-\alpha)\pi}$ , with  $\alpha$  as the braid-statistics parameter and in space-dimensions  $d < 3$ . It is in general arbitrary. In fact considering all possible oriented exchanges in an  $n$ -particle wave function these phase factors provide an abelian representation of the braid group  $B_n$ [1, 2]. Obviously, fermions(bosons) correspond to  $\alpha = 0(\alpha = 1)$ ; for  $\alpha \neq 0, 1$  the statistics is called fractional. In one or two dimensional space, the existence of *braid* statistics is well known [3, 4, 5], and the corresponding particle is dubbed as *anyon*. The first example of what will be called a braid statistics of fields in  $d = 1$  appears for the free fields in Ref. [6]. The earliest examples in interacting theories can be found in nuce in Ref. [7], and are discussed in details in Ref. [8] and in Ref. [9]. According to Wilczek[5], the anyon in 2D can be viewed as a charged particle(fermion or boson) bound to a flux with the statistics coded through the Aharonov-Bohm effect. The flux-binding changing the braid statistics can be realized through a minimal coupling to a Chern-Simons gauge field and this procedure is called statistical transmutation[10]. For  $d \geq 3$  only  $\alpha = 0$  or 1 are allowed, since the orientation of the exchange is irrelevant and the braid group collapses to the permutation group.

The other kind of statistics can be defined through state counting, which is known as (Haldane's) *exclusion* statistics and can be viewed as an effective interaction among particles occupying identical or different states in the Hilbert space. The exclusion statistics is characterized by a parameter  $g$ , first introduced by Haldane [11], which measures the change rate of the dimension  $D$  of Hilbert space with respect to the total particle number  $N$  when an additional particle is introduced. For a single species of particles, there is a linear relation between  $D$  and  $N$ ,  $\Delta D = (g - 1)\Delta N$ . Again,  $g = 0$  for fermions and  $g = 1$  for bosons. It turns out that  $1/(1 - g)$  gives also the maximum average occupation number for the quantum states below the Fermi energy[12]. The fractional *exclusion* statistics can be viewed as a generalized Pauli's exclusion principle and it exists in arbitrary dimensions[11]. Following the nomenclature in Refs.[13], we call *exclusons* the low-energy quasiparticles/quasiholes obeying fractional exclusion statistics. The best known examples of exclusons arise in a number of one dimensional systems solvable by the thermodynamic Bethe ansatz, such as the Yang-Yang  $\delta$ -function gas [14, 15, 16] or the Calogero-Sutherland model [17, 18, 19, 20, 21]. In two dimensions there are also a few strongly correlated systems whose low-lying excitations exhibit the fractional exclusion statistics, [11, 12, 22], including the "Laughlin vortices" of the fractional quantum Hall effect. For relativistic elementary particles the fermion(boson) exclusion statistics can be derived from the antisymmetry(symmetry) of the many-body wave-function and therefore the two ways of defining the statistics can be identified.

Both types of statistics are naturally related to interactions. In the Chern-Simons theory of anyons, the braid statistics arises simply as a charge-current interaction among particles mediated by the statistical gauge fields (see [23]). Although the anyon model is easy to construct using Chern-Simons statistical transmutation, even the 2D free anyon gas is extremely difficult to solve due to the highly entangled motions of anyons. The exclusion statistics is also a consequence of interactions. It is actually an emergent phenomenon in the low-energy behavior of an interacting system consisting of physical particles with prescribed braid statistics (usually either fermion or boson). The exclusion statistics can be directly applied to calculate the thermodynamics of the exclusion gases [12]. As a typical example, the Yang and Yang's thermodynamic Bethe ansatz solution of the one dimensional repulsive  $\delta$ -potential boson gas can be reformulated as a *free* exclusion gas obeying nontrivial mutual exclusion statistics [24]. More generally, in one dimensional integrable models the thermodynamic Bethe ansatz equations can be reinterpreted exactly as statistical interactions between exclusions with identical or different momenta, determined by the two-body scattering phase shift [24]. Based on these observations, Wu and Yu proved that the low energy physics of an exclusion gas is equivalent to that of a one-component Luttinger liquid with Haldane's controlling parameter  $\lambda$  identified as the statistical parameter  $1 - g$  [13, 25], thus providing a unified description for various interactions using fractional exclusion statistics. Inspired by the success in one dimensional systems, there were attempts to generalize the Fermi liquid theory to a "Haldane" liquid with fractional exclusion statistics in higher dimension to provide a unified description to the low energy behavior of interacting systems [26]. The fractional statistics (both exchange and exclusion) has also been proposed to provide a better mean field theory for some strongly correlated systems, especially the cuprate superconductors [27, 28, 29, 30].

A natural question to ask is what is the relation between the two aspects of the fractional statistics, i.e. between  $\alpha$  and  $g$ . It turns out that there is no universal relationship, but relations appear in specific examples. In two dimensional quantum Hall systems with Hall conductance  $\sigma_h$ , we can derive a linear relation between  $g$  and  $\alpha$ ,  $g = 2\pi\sigma_h\alpha$  [23], clarifying and extending previous results. More generally we proved the existence of a relation in systems with chiral edge currents. Although there were many studies on the fractional statistics in one dimension, to the best of our knowledge, it is still lacking a satisfactory understanding of the relation between  $g$  and  $\alpha$ , probably in part because the exchange of two particles in one dimension necessarily involves scattering processes, thus there is no unique way to separate the braid statistics from the dynamical processes [24, 31, 32, 13, 25].

In this article, we examine the relation between braid and exclusion fractional statistics of particles moving in a straight line, including chiral cases. In the present study, we adopt the fermion-based Chern-Simons theory to unambiguously define the braid statistics. This is also the same framework adopted in our previous study on two dimensional fractional statistics [23], but to discuss braid statistics in one dimension we need a careful dimensional reduction following the technique developed in [33]. In this

setup, we restrict to impenetrable two-body interactions, to have a well-defined braid statistics. The main purpose of this article is then to study *whether the Chern-Simons statistical transmutation can induce non-trivial fractional exclusion statistics or not*. It turns out that if the gauge effective action of the matter system minimally coupled to the statistical gauge field is gauge invariant, the exclusion statistics of the emergent exclusions has nothing to do with the braid statistics of the physical charged fields of the model, but the statistical transmutation shifts the value of their Fermi momenta. However, if the gauge effective action of the matter system is anomalous(chiral case), then a precise relation between the braid and exclusion statistics emerges. Thus, our present study for one-dimensional systems together with our previous study for two-dimensional systems provides a systematic description of the relation between fractional abelian exchange statistics and fractional exclusion statistics in low dimensions( $d < 3$ ) in the same framework of Chern-Simons theory.

This article is organized as follows: in Sec. 2 the dimensional reduction of two dimensional Chern-Simons theory is introduced, and with this formalism we calculate the Green's functions of noninteracting anyons. In Sec. 3 we present the results in the presence of both fractional exchange and exclusion statistics for matter systems that exhibit a gauge-invariant effective action if minimally coupled to the Chern-Simons gauge field. In Secs. 4 and 5, we analyze the relation between the braid and exclusion statistics emerging for *noninteracting* anyonic systems for both nonmutual and mutual statistics. Finally, in Sec. 6 we sketch a derivation of the corresponding bosonization formulas for one-dimensional anyonic system via T-duality. Throughout the paper we use the euclidean path-integral formalism.

## 2. Chern-Simons theory and dimensional reduction for noninteracting Anyons

We consider non-relativistic spinless fermions in one dimension with particle density  $\rho^0$  and Fermi velocity  $v_f$  set to unit for simplicity. In the scaling limit, the low energy physics is controlled by excitations near the two Fermi points  $\pm k_f$  with  $k_f = \pi\rho^0$  in the noninteracting case. This allows us to decompose the non-relativistic fermion field  $\Psi$  in the low-energy region into right and left movers as  $\Psi(x) \sim \psi_R(x)e^{ik_f x^1} + \psi_L(x)e^{-ik_f x^1}$ , with the dynamics of a massless Dirac fermion described by the spinor doublet  $\psi = (\psi_R, \psi_L)^t$ . For convenience we also use  $\psi_{1,2}$  to denote  $\psi_{R,L}$ , respectively.

To implement the braid statistics, we first embed the 1+1 dimensional spacetime of the matter field with coordinates  $x^0$  and  $x^1$ , into the 2+1 dimensional spacetime as the plane at  $x^2 = 0$ . Then we couple minimally the matter fields to a 2+1 dimensional statistical gauge field  $A_\mu$ , whose dynamics is described by the Chern-Simons action. The total action in Euclidean space with metric tensor  $g_{\mu\nu} = \text{diag}(-1, -1, -1)$  consists of the following two terms:

$$S_f[\psi, \bar{\psi}|A] = \int d^2x \bar{\psi} i \gamma^\mu D_\mu \psi - i \int d^2x \rho^0 A_0, \quad (1)$$

$$S_{cs}[A] = \frac{i}{4\pi\alpha} \int d^3x \epsilon^{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda, \quad (2)$$

where  $\rho^0$  is the expectation value of the fermion density with respect to the noninteracting vacuum, and the covariant derivatives are defined as  $D_\mu = \partial_\mu + iA_\mu(x, x^2 = 0)$  with  $x = (x^0, x^1)$  and  $\mu = 0, 1$ . The  $\gamma$ -matrices  $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$ , and  $\bar{\psi} = -i\psi^\dagger \gamma^0$ .

Being the matter field one-dimensional, one needs to carry out the dimensional reduction from three to two dimensions, implemented following the technique developed in Ref.[33]. Since the  $A_0$  field has no dynamics, it can be integrated out leading to the following flux-binding constraint,

$$F_{12}(x, x_2) = -2\pi i\alpha \delta(x^2)[j^0(x) + i\rho^0], \quad (3)$$

where  $j^\mu =: \bar{\psi} \gamma^\mu \psi$  : is the (normal ordered) Dirac current. (3) has the following solution:

$$\begin{aligned} A_1(x, x^2) &= i\pi\alpha \text{sgn}(x^2)[j^0(x) + i\rho^0] + \partial_1 f(x, x^2), \\ A_2(x, x^2) &= \partial_2 f(x, x^2), \end{aligned} \quad (4)$$

where  $f(x, x_2)$  is an arbitrary gauge function and the sign function is taken antisymmetric, i.e.,  $\text{sgn}(0) = 0$ . Inserting Eq. (4) into the remaining term of the Chern-Simons action and in Eq. (1), we obtain a free Dirac fermion coupled to a pure gauge field  $\partial_\mu f$ , and by choosing the gauge  $A_2 = 0$  the gauge function  $f$  can be absorbed by a redefinition of the fermion field, as we assume henceforth. It follows that the energy spectrum and the correlators of the Fermi field  $\psi$  are controlled by local *fermionic* excitations, unaffected by flux binding. This is due to the triviality of the total partition function of the gauge field, leaving only the free fermion as the final result.

However, the correlators of the  $\psi$  field alone are not gauge invariant, hence they are unphysical. Therefore, the states obtained from the vacuum acting with the Fermi field  $\psi$  do not belong to the physical Hilbert space of the theory with action given by Eqs. (1) and (2). In fact it is known that in gauge theories the physical charged excitations can be created/annihilated by gauge invariant non-local fields acting on the physical Hilbert space (see Ref.[34] and references therein). In the present case for  $\alpha \neq 0, 1$  these fields obey non-trivial braid statistics and can be constructed as follows: The action  $S_f$  is invariant under gauge transformation  $\psi \rightarrow e^{-i\Lambda} \psi$  and  $A_\mu \rightarrow A_\mu + \partial_\mu \Lambda$ ; correspondingly the gauge-invariant anyon field is given by:

$$\tilde{\psi}_{R,L}(x) \equiv \psi_{R,L}(x) e^{-i \int_{P_x} A_\mu d\ell^\mu}, \quad (5)$$

where the path  $P_x$  is a straight line from  $x^1$  to  $\infty$  with fixed time  $x^0$ . To avoid an ill-defined crossing with the world lines of the fermions, we shift  $P_x$  slightly from  $x^2 = 0$  to  $x^2 = \epsilon$  with  $\epsilon$  an infinitesimal positive number. More precisely we take the limit  $\epsilon \searrow 0$  on the correlation functions and technically one needs a compensating current at infinity joining all the paths  $P$  appearing in the correlators, but we do not discuss this matter here, referring for details to Refs.[28, 33]. The exponential in the r.h.s. of Eq. (5) will be called gauge string. The low-energy anyon Green's function including

the Fermi momenta is given by:

$$G_{ab}^\alpha(x, y) = \langle \tilde{\psi}_a(x) \tilde{\psi}_b^\dagger(y) \rangle e^{-i\pi\rho^0[(-1)^a x^1 - (-1)^b y^1]} \quad (6)$$

with  $a = 1, 2$  and the same for  $b$ , where the expectation value is now referred to zero density. With the above prescription the braiding effect is captured by the gauge strings.

To calculate the Green's function of the physical particles  $G_{ab}^\alpha(x, y)$ , one needs to average Eq. (6) over the statistical gauge field  $A_\mu$  weighted by the Chern-Simons action, following Ref.[33]. We first insert Eq. (4) into Eq. (5), yielding

$$\tilde{\psi}_a(x) = \psi_a(x) e^{-i\pi\alpha\rho^0 x^1 + \pi\alpha \int_{x^1}^\infty j^0(x^0, z^1) dz^1}. \quad (7)$$

Then the Green's function defined by Eq. (6) reads

$$\begin{aligned} G_{ab}^\alpha(x - y) &= \frac{1}{Z_{0f}} e^{-i\pi\rho^0[(-1)^a x^1 - (-1)^b y^1 + \alpha(x^1 - y^1)]} \\ &\times \int \mathcal{D}\psi \mathcal{D}\bar{\psi} \psi_a(x) \psi_b^\dagger(y) e^{\int d^2x \bar{\psi} i\gamma^\mu [\partial_\mu - i\pi\alpha\delta_\mu^0 \partial_0 Q_{x,y}(z)] \psi} \end{aligned} \quad (8)$$

where  $Z_{0f} \equiv \int \mathcal{D}[\psi, \bar{\psi}] e^{S_f[\psi, \bar{\psi}|0]}$  is the partition function of free Fermi field and

$$Q_{x,y}(z) \equiv \theta(z^1 - x^1) \theta(z^0 - x^0) - \theta(z^1 - y^1) \theta(z^0 - y^0) \quad (9)$$

with  $\theta(x)$  being the step function. To calculate the Green's function Eq. (8), we follow Schwinger's formalism[35], which leads to the exact correlation of the Fermi fields in the presence of gauge field,

$$\begin{aligned} G_{ab}(x - y | \mathcal{A}) &\equiv \frac{1}{Z_{0f}} \int \mathcal{D}\psi \mathcal{D}\bar{\psi} \psi_a(x) \psi_b^\dagger(y) e^{\int d^2z \bar{\psi} i\gamma^\mu (\partial_\mu + i\mathcal{A}_\mu) \psi} \\ &= \frac{\delta_{ab}}{2\pi} \frac{e^{i\Theta_{x,y}[\mathcal{A}]} e^{S_{eff}[\mathcal{A}]}}{(x^0 - y^0) + i(-1)^a(x^1 - y^1)}, \end{aligned} \quad (10)$$

where the functionals  $\Theta_{x,y}^a$  and  $S_{eff}$  are given by

$$\begin{aligned} \Theta_{x,y}^a[\mathcal{A}_\mu] &= \int d^2z [\Delta^{-1}(z - x) - \Delta^{-1}(z - y)] \\ &\times [\partial^\mu \mathcal{A}_\mu(z) - i(-1)^a \epsilon_{\mu\nu} \partial^\mu \mathcal{A}^\nu(z)], \end{aligned} \quad (11)$$

$$S_{eff}[\mathcal{A}_\mu] = \frac{1}{2\pi} \int d^2z \epsilon^{\mu\nu} \partial_\mu \mathcal{A}_\nu(z) \Delta^{-1} \epsilon^{\sigma\tau} \partial_\sigma \mathcal{A}_\tau(z), \quad (12)$$

with  $\Delta \equiv -\partial_\mu \partial^\mu$  being the two dimensional Laplacian. Comparing Eq. (8) with the Schwinger's formula Eq. (10), by identifying  $\mathcal{A}_\mu = -\pi\alpha\delta_\mu^0 \partial_0 Q_{x,y}$ , one immediately finds the gauge-invariant anyon's Green's function

$$G_{ab}^\alpha(x - y) = \frac{\delta_{ab}}{2\pi} \frac{e^{-i[\alpha + (-1)^a][\pi\rho^0(x^1 - y^1) + \arg(x - y)]}}{|x - y|^{1 + (-1)^a \alpha + \alpha^2/2}}. \quad (13)$$

where we denote by  $\arg(x)$  the argument of the complex number  $x^0 + ix^1$ .

The Green's function Eq. (13) indicates that there is no correlation between the left- and right-handed branches. When  $\alpha = 0$  one easily recovers the free fermion result, and when  $\alpha = 1$ , the Green's function of the right-handed branch has the well-known form  $|x - y|^{-1/2}$  of the one-dimensional hardcore bosons[36]. The braid statistics of the

physical particles is reflected in the numerator of Eq. (13). If we exchange  $x$  and  $y$  with increasing the argument of  $x - y$  by  $\pi$ , an additional phase  $e^{-i\alpha\pi}$  appears besides the Fermi statistical factor.

The free fermions obey the the Pauli's exclusion principle, and each particle occupies a volume of  $2\pi/L$  for a finite system of length  $L$  in the (pseudo-) momentum space. This gives rise to a finite Fermi area determined by the fermion density  $\mathcal{S} = 2\pi\rho$ . For the general exclusions with non-mutual statistical interaction  $\lambda \equiv 1 - g$ , the occupied volume per particle is modified to  $2\pi\lambda/L$ , then the "Fermi area" is also changed with  $\delta\mathcal{S}/(\delta\rho) = 2\pi\lambda$ . Since in our calculation the particle density is kept invariant when the interactions are switched on, the exclusion statistics is eventually reflected in the change of the Fermi area. We read the Fermi momenta off from the coefficient of  $(x^1 - y^1)$  in the phase factor. Indeed, the Fermi points are shifted by the Chern-Simons coupling from  $\pm\pi\rho^0$  to  $(\pm 1 - \alpha)\pi\rho^0$ . However, the Fermi area is still  $2\pi\rho^0$ . Hence there is no sign of non-trivial exclusion statistics.

### 3. Fractional braid and exclusion statistics in anyon "Luttinger" Liquids

In the previous section we have considered the one dimensional free anyon systems, where the elementary excitations still obey the conventional Pauli exclusion principle in spite of their anyonic nature. Once the two-body interaction is turned on in one dimensional fermion gases, the system may become in the low-energy limit an *exclusion* gas (we adopt the jargon invented in Refs.[13]) subject to the fractional exclusion statistics[11, 24, 13], for which the low-energy physics can be described by a Luttinger liquid theory [13, 25], as quoted in the introduction. Since the Luttinger liquid theory is applicable to a wide class of one-dimensional systems, one may expect that the fractional exclusion statistics is ubiquitous in one dimension.

In fact the exclusion statistics can also be introduced in the anyon systems. By minimally coupling a one-dimensional Luttinger liquid to a Chern-Simon gauge field, the two types of statistics can be realized simultaneously. A standard way to introduce a non-trivial exclusion statistics is to add to the action Eq. (1) an interaction term in the Luttinger-Thirring form  $(\kappa\pi/2)j_\mu(x)j^\mu(x)$ , which one can rewrite (up to a UV renormalization) by introducing a vector Hubbard-Stratonovich(H.S.) field  $B_\mu$  as

$$\int d^2x \left[ \frac{1}{2\kappa\pi} B_\mu B^\mu(x) - B_\mu j^\mu(x) \right]. \quad (14)$$

As result, the total partition function has the following form

$$Z_T = \int \mathcal{D}A \mathcal{D}B \mathcal{D}\psi \mathcal{D}\bar{\psi} e^{S_f[\psi, \bar{\psi}|A+B] + S_{cs}[A] + \frac{1}{2\pi\kappa} \int d^2z B_\mu B^\mu}, \quad (15)$$

and the Green's function of the anyon fields reads

$$\begin{aligned} & G_{ab}^{\alpha, \kappa}(x - y) \\ &= \frac{1}{Z_T} e^{-i\pi\rho^0[(-1)^a x^1 - (-1)^b y^1]} \end{aligned}$$

$$\times \int \mathcal{D}A \mathcal{D}B \mathcal{D}\psi \mathcal{D}\bar{\psi} \tilde{\psi}_a(x) \tilde{\psi}_b^\dagger(y) e^{S_f[\psi, \bar{\psi}|A+B] + S_{cs}[A] + \frac{1}{2\pi\kappa} \int d^2z B_\mu B^\mu}. \quad (16)$$

The procedure is then similar to that given in Sec. 2, except that an additional procedure of integrating over the H.S. auxiliary field  $B$  is needed now. We first integrate over  $A_0$  leading to the same constraint, Eq. (4), which is then substituted into Eq. (1) and Eq. (2). Now we obtain

$$\begin{aligned} & G_{ab}^{\alpha, \kappa}(x-y) \\ &= \frac{\delta_{ab}}{2\pi} \frac{e^{-i\pi\rho^0(-1)^a(x^1-y^1)}}{(x^0-y^0) + i(-1)^a(x^1-y^1)} \\ & \times \frac{\int \mathcal{D}B e^{\int d^2z \frac{B_\mu B^\mu}{2\pi\kappa} + i\Theta_{x,y}^a[B+A] + S_{eff}[B+A] - i \int d^2z \rho^0(B_0 + \mathcal{A}_0)}}{\int \mathcal{D}B e^{\int d^2z \frac{B_\mu B^\mu}{2\pi\kappa} + S_{eff}[B] - i \int d^2z \rho^0 B_0}}, \end{aligned} \quad (17)$$

where  $\mathcal{A}_\mu = -\pi\alpha\delta_\mu^0\partial_0 Q_{x,y}(z)$  as before. Using Eq. (11) and Eq. (12), integrating over  $B$ -field is straightforward, though a little bit tedious, and it leads to the following Green's function in the presence of both Chern-Simons term and Thirring interaction:

$$G_{ab}^{\alpha, \kappa}(x-y) = \frac{\delta_{ab}}{2\pi} e^{-i[\alpha+(-1)^a][\frac{\pi\rho^0(x^1-y^1)}{1+\kappa} + \arg(x-y)]} |x-y|^{-\frac{(1+\kappa)^2 + [\alpha+(-1)^a]^2}{2(1+\kappa)}}. \quad (18)$$

The Green's function Eq. (18) goes back to the standard power-law form of the Luttinger liquid theory, when  $\alpha = 0$ . The Haldane's controlling parameter can be read off from the decaying exponent  $\lambda = 1/(1 + \kappa)$ , which is also the exclusion parameter according to Y.S. Wu and Y.Yu[13]. One can also identify the exclusion statistics by fixing the particle number and directly measuring the occupied area in the pseudo-momentum space after the interactions are switched on. As explained in the previous section, the change of the Fermi area is in fact a direct consequence of the nontrivial exclusion statistics. One can read the left and right Fermi wavevectors from the Green's function which are  $\pm\pi\rho^0/(1 + \kappa)$ , respectively. The corresponding Fermi area is then  $2\pi\rho^0/(1 + \kappa)$  indicating a statistical interaction  $\lambda = 1/(1 + \kappa)$ . This Fermi area is not changed even when the corresponding fields becomes anyonic with braid parameter  $\alpha$ , implying that the braiding effect is not necessarily connected to the exclusion statistics. This conclusion is in fact quite general, since it is obtained in the framework of one-dimensional interacting Dirac fermions coupled to the Chern-Simons gauge field and the Dirac Fermion describes the low energy physics of a large class of one dimensional models. Our result is also consistent with that given in Ref.[37].

Before ending this section, we discuss the periodicity of  $\alpha$  from the point of view of Chern-Simons theory. As well known, for a finite number of non-relativistic particles in the first-quantization formalism, there is a period 2 for the braid parameter  $\alpha$ , since binding  $4\pi$ -flux does not change the exchange statistics. However, we notice that in the low-energy description in the thermodynamic limit, the shift  $\alpha \rightarrow \alpha + 2n(n \in \mathbb{Z})$  is not trivial due to the coupling to the average particle density, which in fact corresponds to multiple particle-hole excitations between the two Fermi points with current  $2nk_f(\kappa_f = \pi\rho^0)$ [38]. In general cases with more sophisticated dispersion and interaction, the generic form of the Green's function for the physical anyons obeying the

same braid statistics  $1 - \alpha$  is actually a sum of  $G_{RR}^{\alpha+2n}$ ,  $n \in \mathbb{Z}$ , with  $\alpha$  restricted in the range  $[0, 2)$ ,

$$\tilde{G}^{\alpha,\kappa}(x-y) = \sum_{n \in \mathbb{Z}} \frac{C_n}{2\pi} e^{-i(\alpha+2n+1)[\pi\rho^0\lambda(x^1-y^1)+\arg(x-y)]} |x-y|^{-\frac{\lambda^{-1}+(\alpha+2n+1)^2\lambda}{2}}, \quad (19)$$

where  $C_n$ 's are some regularization parameters depending on the details of UV limit of specific models. Notice that the left- and right- handed fermions fall in the sectors with  $n = -1$  and  $n = 0$ , respectively. One may interpret the Green's function Eq. (19) as an anyon version in the Haldane's harmonic-fluid theory of one-dimension quantum gas[38, 39]. Indeed, by setting  $\alpha = 0$  or  $\alpha = 1$ , it reduces to the well-known results for fermions and bosons, respectively.

#### 4. Fractional statistics for a chiral anyon system

The results given in Sec. 2 and 3 show unambiguously that there is no direct relation between braid and exclusion statistics with or without interactions, if the one-dimensional matter field couples to the statistical field gauge invariantly. However, if the system is anomalous like chiral fermion, the fractional exclusion statistics can be induced by braiding the free particles(chiral fermion) through the Chern-Simons statistical transmutation, which is closely connected with our previous study in two-dimensional systems[23], as we demonstrate in this section.

We consider the simple case of chiral fermions. It has been shown long time ago that one cannot couple gauge-invariantly the chiral fermions to a gauge field  $A_\mu$ [40]. A way out is to consider the Dirac operator acting on the full mode-space of a (1+1)-dimensional, two-component Dirac field and restrict the gauge field to its chiral component  $(A_\mu \pm i\epsilon_{\mu\nu}A^\nu)/2 \equiv A_\mu^\pm/2$ . The corresponding action has the following form:

$$S_f^{c\pm}[\psi, \bar{\psi}|A] = \int d^2x \bar{\psi} i\gamma^\mu \left( \partial_\mu + i\frac{A_\mu^\pm}{2} \right) \psi - i \int d^2x A_0^\pm \rho_\pm^0, \quad (20)$$

where  $\rho_\pm^0$  is the density of right- and left- handed fermion, respectively. Following Ref. [41], we integrate over the chiral fermion field and obtain the effective action of the gauge field

$$S_{eff}^{c\pm}[A_\mu] = S_{eff}\left[\frac{A^\pm}{2}\right] + \frac{c}{8\pi} \int d^2x A_\mu A^\mu - i \int d^2x A_0^\pm \rho_\pm^0, \quad (21)$$

where  $S_{eff}[A]$  is given in Eq. (12) and a local quadratic term of  $A_\mu$  is added, reflecting a finite renormalization ambiguity due to the lack of gauge invariance with the coefficient  $c$ , a priori an arbitrary real constant. Here we take the "minimal choice"  $c = 1$ .

The effective actions  $S_{eff}^{c\pm}[A_\mu]$  for gauged chiral fermions are anomalous: performing a gauge transformation,  $A^\mu \rightarrow A^\mu + \partial^\mu \Lambda$ , we have

$$S_{eff}^{c\pm}(A_\mu) \rightarrow S_{eff}^{c\pm}(A_\mu) \pm \frac{i}{4\pi} \int d^2x \Lambda \epsilon_{\mu\nu} \partial^\mu A^\nu(x). \quad (22)$$

A remedy for such an inconsistency is to take the chiral fermion system as the boundary of a bulk system with Hall conductance  $\pm 1/(2\pi)$ . The bulk effective action then reads

$$S_{bulk}^{\pm}[A] = \pm \frac{i}{4\pi} \int d^3x \theta(x^2) \epsilon^{\mu\nu\lambda} A_{\mu} \partial_{\nu} A_{\lambda} - i \int d^3x \theta(x^2) \rho_B^0 A_0, \quad (23)$$

where  $\theta(x^2)$  is the Heaviside step function and  $\rho_B^0$  is the expectation value of the fermion density in the bulk w.r.t. the *noninteracting* vacuum. The bulk action is also gauge variant due to the existence of the boundary. In fact under the gauge transformation, one finds

$$S_{bulk}^{\pm}[A] \rightarrow S_{bulk}^{\pm}[A] \mp \frac{i}{4\pi} \int d^2x \Lambda \epsilon_{\mu\nu} \partial^{\mu} A^{\nu}(x). \quad (24)$$

Comparing Eq. (24) and Eq. (22), we observe the anomaly of the bulk effective action restricted in the region  $x^2 > 0$  cancel that of the matter field on its boundary. The whole action is then gauge invariant.

Next, we couple the matter field to the statistical gauge field with support in the whole space. After integrating out  $A_0$  we obtain the following constraint:

$$i\delta(x^2)[j^{\pm}(x) + i\rho_{\pm}^0] - \rho_B^0 \theta(x^2) = -\frac{F_{12}}{2\pi\alpha} \mp \frac{F_{12}}{2\pi} \theta(x^2) \pm \frac{1}{4\pi} \delta(x^2) A_1(x, x^2) \quad (25)$$

where  $j^{\pm}(x) \equiv [j^0(x) \pm ij^1(x)]/2$  is the chiral current on the edge.

Before proceeding to the analysis of the statistics of chiral fermion, we first examine the fractional statistics of the bulk fermion. In the bulk region  $x^2 > 0$ , the constraint reduces to  $F_{12} = 2\pi\alpha\rho_B^0/(1 \pm \alpha)$ . Note that the particle density  $\rho_B(\alpha)$  in the ground state depends on  $\alpha$  and the flux density reads  $F_{12} = 2\pi\alpha\rho_B(\alpha)$  if the bulk system is incompressible. Therefore, we find the relation between  $\rho_B(\alpha)$  and the bare particle density  $\rho_B^0$  for the bulk

$$\rho_B(\alpha) = \rho_B^0/(1 \pm \alpha). \quad (26)$$

which indicates that the Haldane's statistical interaction  $g$  is  $\mp\alpha$ . The relation between  $g$  and  $\alpha$  for a Hall insulator with a general Hall conductance  $\sigma_h$  has been given in a different way in our previous study[23]. Inspired by the bulk result, one may expect that the edge mode may also exhibit non-trivial fractional *exclusion* statistics. Let us prove it by focusing only on the right-handed anyon.

The gauge invariant Green's function for right-handed fermion in the presence of Chern-Simons term reads

$$G_{++}^{\alpha}(x-y) = \frac{\int \mathcal{D}\psi \mathcal{D}\bar{\psi} \mathcal{D}A \tilde{\psi}_R(x) \tilde{\psi}_R^{\dagger}(y) e^{ik_f(x^1-y^1)} e^{S_f^{c+}[\psi, \bar{\psi}|A] + S_{bulk}^+[A] + S_{c.s.}[A]}}{\int \mathcal{D}\psi \mathcal{D}\bar{\psi} \mathcal{D}A e^{S_f^{c+}[\psi, \bar{\psi}|A] + S_{bulk}^+[A] + S_{c.s.}[A]}}. \quad (27)$$

One can notice that there is only one Fermi wavevector  $k_f = 2\pi\rho_0^+$  for the chiral fermions, unlike for the Dirac fermions where two Fermi points exist. In Eq. (27), the left-handed fermion is also present, however it is only an auxilliary free field to make the fermion measure well defined[41]. In fact, the final result of  $G_{++}^{\alpha}(x-y)$  defined in Eq. (27) is completely anti-analytic without interference from the left-handed fermions. In order to implement the  $\alpha$ -braiding, we put the gauge string of the anyon field  $\tilde{\psi}_{\alpha}(x)$  outside the bulk to avoid the entanglement with the bulk fermions.

To calculate the Green's function, we follow the same procedure given in previous two sections. First, we solve the constraint Eq. (25) in the gauge  $A_2 = 0$ , leading to the following solution of the statistical gauge field for the right-handed anyon

$$A_1(x, x^2) = i\pi\alpha \text{sgn}(x^2)[j^+(x) + i\rho_+^0] - \theta(x^2)x^2 2\pi\alpha\rho_B(\alpha), \quad (28)$$

where we take  $\theta(0) = 0$ . Substituting Eq. (28) into Eq. (27) and using the Schwinger formula Eq. (10), the Green's function can be written as

$$\begin{aligned} & G_{++}^\alpha(x - y) \\ &= \frac{\int \mathcal{D}\psi \mathcal{D}\bar{\psi} \psi_R(x) \psi_R^\dagger(y) e^{i2\pi\rho_+^0(x^1 - y^1)(1 + \frac{\alpha}{2})} e^{\int d^2z \bar{\psi} i\gamma^\mu (\partial_\mu + i\mathcal{A}_\mu)\psi}}{\int \mathcal{D}\psi \mathcal{D}\bar{\psi} e^{\int d^2z \bar{\psi} i\gamma^\mu \partial_\mu \psi}} \\ &= \frac{\delta_{ab}}{2\pi} \frac{1}{(x^0 - y^0) - i(x^1 - y^1)} e^{i\Theta_{x,y}^1[\mathcal{A}]} e^{S_{eff}[\mathcal{A}]}, \end{aligned} \quad (29)$$

where the auxiliary field  $\mathcal{A}_\mu$  defined by  $\mathcal{A}_0(z) = \frac{\pi\alpha}{2}\partial_0 Q_{x,y}(z)$  and  $\mathcal{A}_1(z) = i\frac{\pi\alpha}{2}\partial_0 Q_{x,y}(z)$ . The functionals  $\Theta_{x,y}^1[\mathcal{A}]$  and  $S_{eff}[\mathcal{A}]$  are given in Eq. (11) and Eq. (12), respectively, which can be calculated straightforwardly. Thus we obtain the Green's function for the right-handed anyon:

$$G_{++}^\alpha(x - y) = \frac{1}{2\pi} \frac{e^{i2\pi\rho_+^0(1 + \alpha/2)(x^1 - y^1)}}{[x^0 - y^0 - i(x^1 - y^1)]^{(1 + \alpha/2)^2}}. \quad (30)$$

This result shows that the Fermi momentum  $k_f$  is shifted from  $2\pi\rho_+^0$  to  $2\pi\rho_+^0(1 + \alpha/2)$  by the Chern-Simons coupling. Since the chiral fermion has only one Fermi wavevector  $k_f$ , its Fermi area is in fact linearly dependent on  $k_f$ . Therefore, the shift of  $k_f$  actually implies a non-trivial *exclusion* statistics. Notice that the Green's function has a power-law dependence on the anti-holomorphic coordinates  $x^0 - ix^1$  with a fractional exponent  $(1 + \alpha/2)^2$  which also reflects the braiding effect. The present system is in fact a chiral ‘‘Luttinger’’ anyon liquid.

There is also an essential difference between the chiral and Dirac anyon gases, namely, the braid phase factor arising from an oriented exchange of the coordinates  $x$  and  $y$  for the chiral anyon depends quadratically on the Chern-Simons coupling parameters  $\alpha$ . This spoils the periodicity of the braid statistics on  $\alpha$  seen in the Dirac anyon system, as discussed in the previous sections. This can be attributed to the lack of the backscattering channel for chiral anyons. Indeed, the existence of two Fermi points is crucial for the Dirac fermions to form different sectors with multiple particle-hole excitations carrying a current of  $2nk_f (n \in \mathbb{Z})$ , and the shift of  $\alpha$  by  $2n$  simply transfers one sector to the other with the same braid statistics. Furthermore, in a conventional Luttinger liquid, a Galileian boost can be used to pump one particle from one Fermi point to the other, and to increase the total current by  $2k_f$  [13] without changing the total particle numbers. However, for a single branch of chiral fermion with one Fermi point, such a boost not only increases the total current but also changes the total particle number as it is connected with a bulk particle reservoir. Thus, with the particle number constraint, it is simply not allowed to make such a boost, and one can not expect the existence of different sectors.

We would like to stress that the present chiral ‘‘Luttinger’’ anyon liquid is induced by pure Chern-Simons coupling, unlike the edge mode in the fractional quantum Hall system where the chiral Luttinger liquid is due to the interaction of electrons in the lowest Landau level.

*Remark:* One may also implement the braid and exclusion statistics in an analogous way to fractional quantum Hall systems, and it turns out  $\alpha = \lambda^{-1}$ . Therefore, the relation between  $\alpha$  and  $g$  in the anomalous systems is not universal, depending on how the braid statistics is implemented. Here, we just sketch such a different implementation, and more details with a general physical interpretation will be discussed elsewhere. With the same methods given in Sec. 3, one can easily prove that a Luttinger fermion with Haldane parameter  $\lambda$  coupled chirally to a gauge field has an effective action given by  $\lambda S_{eff}^{c\pm}(A_\mu)$ . Then one can cancel its anomaly by adding a Chern-Simons action for the gauge field in the half-space-time with a braid parameter matching with  $\lambda$ . One can then construct a chiral field by attaching to, e.g.,  $\psi_R(x)$  a phase string:  $\psi_R(x) \exp[-i(\lambda^{-1} - 1)\pi \int_{x^1}^\infty dy^1 j^0(x^0, y^1)]$ . One finds that this field is indeed chiral with both braid and fractional exclusion statistics.

## 5. Mutual statistics of multiple species of chiral anyons

In this section we consider the mutual statistics among multiple chiral anyon species. Multiple chiral edge modes may exist in the integer quantum Hall insulators with Hall conductance  $\sigma_h \geq 2$ . It may also occur in the fractional quantum Hall systems in the hierarchical theory, where the quantum state at some filling is not of the simple Laughlin type, leading to many branches of edge excitations[42, 43, 44].

We consider the following action of right-handed fermions with  $N_f$  flavors, each of which couples with the same statistical gauge field with different statistical charges  $q_a$ ,

$$S_{N_f}^{c+}[\{\psi_a, \bar{\psi}_a\}|A] = \sum_{a=1}^{N_f} \int d^2x \left[ \bar{\psi}_a i\gamma^\mu \left( \partial_\mu + iq_a \frac{A_\mu^+}{2} \right) \psi_a - iq_a A_0^+ \rho_{a+}^0 \right]. \quad (31)$$

As explained in Sec. 4, for each right-handed fermion, we need to add the fermion with opposite chirality which, however, does not couple to the gauge field and serves as an auxiliary field to give a well-defined fermionic integration measure. The anomalous effective action of statistical gauge field is simply  $\nu_t S_{eff}^{c+}[A_\mu]$  with  $\nu_t \equiv \sum_{a=1}^{N_f} q_a^2$  (here we temporarily ignore the density term purposely). To cancel the gauge anomaly of the chiral edge modes and make the whole theory gauge invariant, we need a bulk Chern-Simons term in the upper half plane with  $x^2 > 0$

$$S_{bulk}^+[A] = \frac{i\nu_t}{4\pi} \int d^3x \theta(x^2) \epsilon^{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda - i \sum_{a=1}^{N_f} \int d^3x \theta(x^2) q_a \rho_{B,a}^0 A_0, \quad (32)$$

where  $\rho_{B,a}^0$  is the bulk density of the  $a$ -flavor fermion w.r.t. the non-interacting vacuum.

Next, we add the Chern-Simons action of Eq. (2) in the whole space to implement

the braid statistics. Integrating out  $A_0$  one finds a simple extension of Eq. (25),

$$\begin{aligned} & i\delta(x^2) \sum_{a=1}^{N_f} q_a [j_a^+(x) + i\rho_{a,+}^0] - \sum_{a=1}^N \theta(x^2) q_a \rho_{B,a}^0 \\ &= \frac{F_{12}}{2\pi\alpha} - \frac{\nu_t F_{12}}{2\pi} \theta(x^2) + \frac{\nu_t}{4\pi} \delta(x^2) A_1(x, x^2), \end{aligned} \quad (33)$$

where  $j_a^+(x) \equiv (j_a^0 + ij_a^1)/2$  is the particle density of right-handed fermions and the charge density is  $q_a j_a^+$ . For the bulk region, this constraint has a simple form  $F_{12} = 2\pi\alpha(1 + \alpha\nu_t) \sum_{a=1}^{N_f} q_a \rho_{B,a}^0$ . As we proved in Ref.[23], the bulk anyon also obeys the mutual fractional exclusion statistics induced by braiding particles, and the particle density is shifted to a  $\alpha$ -dependent value  $\rho_{B,a}(\alpha)$  for each flavor. The corresponding total flux density is then  $2\pi\alpha \sum_{a=1}^{N_f} q_a \rho_{B,a}(\alpha)$ , therefore we obtain a relation between  $\rho_{B,a}^0$  and  $\rho_{B,a}(\alpha)$  as following

$$\sum_{a=1}^{N_f} q_a \rho_{B,a}^0 = (1 - \alpha\nu_t) \sum_{a=1}^{N_f} q_a \rho_{B,a}(\alpha). \quad (34)$$

This result can also be derived using our previous results on the mutual statistics in two-dimensional Hall insulator consisting of multiple species of anyons, where we proved the parameters  $g_{ab}$  of mutual exclusion satisfy  $g_{ab} = 2\pi\sigma_{h,a}\alpha q_a q_b$  for all flavors in the presence of Chern-Simons coupling(see the appendix of Ref. [23]).

We now turn to the mutual statistics of one-dimensional chiral anyons. The solution of Eq. (33) is similar to that of Eq. (28), and in the gauge  $A_2 = 0$  we have:

$$A_1(x, x^2) = i\pi\alpha \text{sgn}(x^2) \sum_{a=1}^{N_f} q_a [j_a^+(x) + \rho_{a,+}^0], \quad (35)$$

where we omit the bulk density term, since it is not necessary for the discussion on the mutual statistics of the edge anyons. Following the same procedure given for the single chiral fermion, it is straightforward to derive the Green's function of the chiral fermions

$$\begin{aligned} & G_{ab}^{\alpha,+}(x-y) \\ &= \frac{\delta_{ab}}{(Z_{0f})^{N_f}} \int \mathcal{D}\psi \mathcal{D}\bar{\psi} e^{i2\pi\rho_a^+(x^1-y^1) + i\pi\alpha q_a(x^1-y^1) \sum_{c=1}^{N_f} q_c \rho_{c,+}^0} \\ & \quad \times \psi_{aR}(x) \psi_{aR}^\dagger(y) e^{\sum_{c=1}^{N_f} \int d^2z \bar{\psi}_c i\gamma^\mu (\partial_\mu + iq_c \mathcal{A}_\mu) \psi_c} \\ &= \frac{\delta_{ab}}{2\pi} \frac{e^{i2\pi(\rho_a^+ + \frac{\alpha q_a}{2} \sum_{c=1}^{N_f} q_c \rho_{c,+}^0)(x^1-y^1)}}{[x^0 - y^0 - i(x^1 - y^1)]^{1+\alpha q_a^2 + \frac{q_a^2}{4} \sum_{c=1}^{N_f} q_c^2 \alpha^2}}, \end{aligned} \quad (36)$$

where  $\mathcal{A}_0 = \pi\alpha q_a \partial_0 Q_{x,y}(z)/2$  and  $\mathcal{A}_1 = i\pi\alpha q_a \partial_0 Q_{x,y}(z)/2$  coming from the the solution of the constraint Eq. (35).

There is no correlation between anyons with different flavors. However the Green's function gets modifications from other flavors of chiral fermions: (1) The anomalous exponent of the Green's function  $G_{aa}^{\alpha,+}(x-y)$  for flavor  $a$  receives the contributions from other chiral fermions, which is  $\sum_{c \neq a} (q_a q_c \alpha/2)^2$ . This reflects the interaction induced by the flux binding between different flavors of chiral fermions. (2) The Fermi momentum

$k_f^a$ , which reflects the occupation status of flavor  $a$  particle in the momentum space, is also modified by other chiral fermions. The change of the Fermi momentum  $k_f^a$  w.r.t.  $\rho_b^+$  can be calculated straightforwardly:

$$\frac{\partial k_f^a}{\partial \rho_b^+} = 2\pi\delta_{ab} + \pi\alpha q_a q_b. \quad (37)$$

where the second term unambiguously shows the mutual exclusion statistics consistent with our previous results for the two-dimensional Hall systems[23]. The present mutual exclusion statistics is indeed induced only by the mutual exchange statistics, as we do not add any interactions.

## 6. T-duality and Bosonization

In this section we review some standard formulas of one-dimensional bosonization in the Euclidean path-integral formalism following Ref.[45], and we apply them to the one-dimensional "Luttinger" anyons given in Sec. 3 and chiral anyons in Sec. 4. Then, the results given in the previous sections can be reproduced.

We introduce the zero-mass Gaussian measure in euclidean 1+1 spacetime with mean zero and covariance  $(4\pi\lambda)(-\Delta)^{-1}$ , where  $\lambda > 0$ . This measure is written formally as

$$\frac{1}{Z} \int \mathcal{D}\phi e^{\frac{1}{8\pi\lambda} \int d^2x \partial_\mu \phi(x) \partial^\mu \phi(x)}, \quad (38)$$

assuming the scalar field  $\phi$  to vanish at infinity and  $Z$  is the partition function of free boson. Denoting by  $\langle \cdot \rangle_\lambda$  the corresponding expectation value, the Gaussian measure can be more rigorously defined by:

$$\langle e^{i \int d^2x \phi(x) f(x)} \rangle_\lambda = e^{2\pi\lambda \int d^2x d^2y f(x) \Delta^{-1}(x,y) f(y)}, \quad (39)$$

if  $f$  is a test function whose Fourier transform vanishes at the origin, and  $\langle e^{i \int d^2x \phi(x) f(x)} \rangle_\lambda = 0$  if  $f$  is real with non-vanishing Fourier transform at the origin. We now introduce the two main composite fields we will use in the theory with expectation value  $\langle \cdot \rangle_\lambda$ . The first is the vertex, which is just an imaginary exponential of the field  $\phi$  normal ordered, formally defined by  $: e^{i\beta\phi(x)} := e^{i\beta\phi(x)} (2\pi)^{\beta^2} e^{-2\pi\lambda\beta^2\Delta^{-1}(x,x)}$ , for  $\beta \in \mathbb{R}$ . The second one is the disorder field. Let us consider the vector potential  $V_\mu^x(y)$  of a magnetic vortex of charge 1 at the point  $x$ . It satisfies  $\epsilon^{\mu\nu} \partial_\mu V_\nu^x(y) = \delta(x-y)$  and locally on  $\mathbb{R}^2 \setminus \{x\} \sim \mathbb{C} \setminus \{x\}$  can be written as  $V_\mu^x(y) = \partial_\mu \arg(x-y)/(2\pi)$ . The expectation value of a product of  $N$  disorder fields  $D(x^j, \zeta^j)$ ,  $\zeta^j \in \mathbb{R}$ ,  $j = 1 \dots N$  is given, up to a UV renormalization, by

$$\left\langle \prod_{j=1}^N D(x^j, \zeta^j) \right\rangle_\lambda = \frac{\int \mathcal{D}\phi e^{-\frac{1}{8\pi\lambda} \int d^2z (\partial_\mu \phi + \sum_{j=1}^N \zeta^j V_\mu^{x^j})^2(z)}}{\int \mathcal{D}\phi e^{-\frac{1}{8\pi\lambda} \int d^2z (\partial_\mu \phi)^2(z)}}. \quad (40)$$

If both vertex and disorder fields are present one gets

$$\left\langle \prod_{j=1}^N D(x^j, \zeta^j) \prod_{\ell=1}^M : e^{i\beta^\ell \phi(y^\ell)} : \right\rangle_\lambda$$

$$= \frac{\int \mathcal{D}\phi e^{-\frac{1}{8\pi\lambda} \int d^2z (\partial_\mu \phi + \sum_{j=1}^N \zeta^j V_\mu^{x^j})^2(z)} \prod_{\ell=1}^M e^{i\beta^\ell [\phi(y^\ell) + \sum_{j=1}^N \zeta^j \arg(x^j - y^\ell)]}}{\int \mathcal{D}\phi e^{-\frac{1}{8\pi\lambda} \int d^2z (\partial^\mu \phi)^2(z)}} \quad (41)$$

Performing the integration over  $\phi$  one evaluates the version of Eq. (41) relevant for the bosonization of anyons' two-point function obtaining:

$$\langle D(x, \zeta) : e^{i\beta\phi(x)} : D(y, -\zeta) : e^{-i\beta\phi(y)} : \rangle_\lambda = e^{-\left(\frac{\zeta^2}{2\lambda} + 2\lambda\beta^2\right) \ln|x-y|} e^{-i2\zeta\beta \arg(x-y)} \quad (42)$$

Next we show the bosonization as a special version of T-duality (see e.g. Ref.[46]) in one dimension, as has been realized in Ref.[47] and independently in Refs.[48, 49]. The basic idea underlying T-duality is the following: Consider a quantum field theory expressed in euclidean formalism in terms of charged fields  $\chi, \chi^*$  whose action  $S(\chi, \chi^*)$  is invariant under an abelian (e.g. U(1)) global gauge transformation

$$\chi(x) \rightarrow e^{i\xi} \chi(x), \chi^*(x) \rightarrow e^{-i\xi} \chi^*(x).$$

Then, we promote the global gauge invariance to a local gauge invariance by introducing a minimal coupling between  $\chi, \chi^*$  and a U(1)-gauge field  $C_\mu$ . We then integrate over  $C_\mu$  with the zero-field constraint  $\epsilon^{\mu\nu} \partial_\mu C_\nu = 0$ , so that the original theory is recovered. The Lagrange-multiplier field enforcing the constraint for  $C_\mu$  is the boson field of the corresponding T-dual theory. Bosonization is just T-duality in case when  $\chi$  is the fermion field  $\psi$ . Let us show this procedure for the partition function of "Luttinger" anyons with formal calculation. Their partition function Eq. (15) can be rewritten as:

$$Z_T = \int \mathcal{D}A e^{S_{cs}[A]} \delta(\partial^\nu A_\nu) \int \mathcal{D}C \delta(\epsilon^{\mu\nu} \partial_\mu C_\nu) \delta(\partial^\nu C_\nu) \int \mathcal{D}B e^{\int d^2z \frac{1}{2\kappa\pi} B_\mu B^\mu} \int \mathcal{D}\psi \mathcal{D}\bar{\psi} e^{S_f[\psi, \bar{\psi}|A+B+C]}, \quad (43)$$

where  $\delta(\partial^\nu C_\nu)$  and  $\delta(\partial^\nu A_\nu)$  are just possible gauge-fixing for  $C_\mu$  and  $A_\mu$ , respectively. For Dirac fermion,  $S_f$  is given by Eq. (1), and one can integrate the fermion fields and H.S. field firstly leading to

$$Z_T = \int \mathcal{D}A e^{S_{cs}[A]} \delta(\partial^\nu A_\nu) \int \mathcal{D}C \delta(\epsilon^{\mu\nu} \partial_\mu C_\nu) \delta(\partial^\nu C_\nu) \times e^{\frac{1}{1+\kappa} \{S_{eff}[A+C] - i \int d^2z (A_0 + C_0) \rho^0\}}$$

with  $S_{eff}[A]$  given in Eq. (12). We then represent the gauge-invariant constraint on  $C$  as:

$$\delta(\epsilon^{\mu\nu} \partial_\mu C_\nu) = \int \mathcal{D}\phi e^{-i \int d^2z \frac{1}{2\pi} \phi(z) \epsilon^{\mu\nu} \partial_\mu C_\nu(z)}, \quad (44)$$

where the Lagrange-multiplier  $\phi$  is a real scalar field and the factor  $2\pi$  has been introduced for later convenience. Changing variable from  $C_\mu + A_\mu \rightarrow C_\mu$  and subsequently integrating over  $C$ , one obtains

$$Z_T = \int \mathcal{D}A e^{S_{cs}(A)} \delta(\partial^\nu A_\nu) \int \mathcal{D}\phi e^{\int d^2z \mathcal{L}_B[\phi, A]}, \quad (45)$$

$$\mathcal{L}_B[\phi, A] = \frac{1+\kappa}{8\pi} \partial_\mu \phi \partial^\mu \phi + \frac{\rho^0}{2} \epsilon_{0\nu} \partial^\nu \phi + \frac{i}{2\pi} \epsilon^{\mu\nu} \partial_\mu \phi A_\nu$$

For  $A = 0$ , one recognizes the bosonized Luttinger liquid action with  $\lambda = 1/(1 + \kappa)$ , plus a density term. To keep the calculation well defined, we provide the particle density with a support in a finite system of length  $L$ , and eventually take the thermodynamic limit. Then, one can shift  $\phi(x) \rightarrow \phi(x) - 2\pi\lambda \int_{-\infty}^{x^1} dz^1 \rho^0(z^1)$ , and the boson Lagrangian acquires a standard Gaussian form given in Eq. (38)

$$\mathcal{L}_B[\phi, A] \rightarrow \frac{1}{8\pi\lambda} \partial_\mu \phi \partial^\mu \phi - \frac{\pi\lambda(\rho^0)^2}{2},$$

where the additional quadratic density term reminds us of the charge excitation given in Haldane's Luttinger liquid theory[42]. When calculating the correlation function of vertex operators, the previous shift contributes an additional phase factor given below,

$$\frac{\int \mathcal{D}\phi e^{i\beta[\phi(x) - \phi(y)]} e^{\int d^2z \mathcal{L}[\phi, 0]}}{\int \mathcal{D}\phi e^{\int d^2z \mathcal{L}[\phi, 0]}} = e^{-i2\pi\lambda\beta\rho^0(x^1 - y^1)} \langle : e^{i\beta\phi(x)} :: e^{-i\beta\phi(y)} : \rangle_\lambda \quad (46)$$

which is useful to identify the exclusion statistics via the change of ‘‘Fermi’’ area as shown in previous sections, where this  $\rho^0$ -related phase factor is calculated alternatively in the fermion formalism.

In a similar way, one can obtain the bosonized anyon two-point function with the identifications:

$$\begin{aligned} \psi_a(x) &\rightarrow D(x, 1) : e^{i\frac{(-1)^\alpha + \alpha}{2}\phi(x)} :, \\ \psi_a^\dagger(x) &\rightarrow D(x, -1) : e^{-i\frac{(-1)^\alpha + \alpha}{2}\phi(x)} :. \end{aligned} \quad (47)$$

In the following, we sketch how to derive via duality the bosonization formulas Eq. (47) in the simplest case with  $\kappa = 0$ , and the general case can be easily handled by inserting the  $B$  fields following the procedure outlined in Sec. 3 (for more details see Ref.[48]). First, for  $\alpha = 0$ , the Green's function of non-interacting fermion fields can be written in terms of auxiliary field  $C_\mu$  and boson field  $\phi$ ,

$$\begin{aligned} &\langle \psi_a(x) \psi_a^\dagger(y) \rangle \\ &= \frac{\int \mathcal{D}\phi \mathcal{D}C \delta(\partial^\nu C_\nu) e^{S_{eff}[C] - i \int d^2z \frac{1}{2\pi} \phi \epsilon^{\mu\nu} \partial_\mu C_\nu} \psi_a(x) \psi_b^\dagger(y)}{\int \mathcal{D}\phi \mathcal{D}C \delta(\partial^\nu C_\nu) e^{S_{eff}[C] - \int d^2z \frac{i}{2\pi} \phi \epsilon^{\mu\nu} \partial_\mu C_\nu}} \\ &= \frac{e^{-i(-1)^\alpha \arg(x-y)} \int \mathcal{D}\phi \mathcal{D}C \delta(\partial^\nu C_\nu) e^{i\Theta_{x,y}^\alpha[C] + S_{eff}[C] - \int d^2z \frac{i}{2\pi} \phi \epsilon^{\mu\nu} \partial_\mu C_\nu}}{|x-y| \int \mathcal{D}\phi \mathcal{D}C \delta(\partial^\nu C_\nu) e^{S_{eff}[C] - \int d^2z \frac{i}{2\pi} \phi \epsilon^{\mu\nu} \partial_\mu C_\nu}} \\ &= e^{-i(-1)^\alpha \arg(x-y) - \pi \Delta^{-1}(x-y)} \\ &\quad \times \frac{\int \mathcal{D}\phi e^{-\int d^2z \frac{1}{8\pi} (\partial_\mu \phi)^2(z)} : e^{i\frac{(-1)^\alpha}{2}\phi(x)} :: e^{-i\frac{(-1)^\alpha}{2}\phi(y)} :}{\int \mathcal{D}\phi e^{-\int d^2z \frac{1}{8\pi} (\partial_\mu \phi)^2(z)}} \\ &= \langle D(x, 1) : e^{i\frac{(-1)^\alpha}{2}\phi(x)} : D(y, -1) : e^{-i\frac{(-1)^\alpha}{2}\phi(y)} : \rangle, \end{aligned} \quad (48)$$

where we used Schwinger's formula Eq. (10) and we set  $\rho^0 = 0$  since it is irrelevant for the present purpose. For the anyon field, we need to attach the gauge string. Using Eq. (4) and  $j^0(x) = -i\partial_1\phi/(2\pi)$  in the bosonization form, one can easily justify Eq. (47).

The bosonization for the chiral fermion based on T-duality is similar. Comparing the Lagrangian for the chiral fermion given in Eq. (20) with that for the Dirac fermion,

we find the bosonized form for chiral fermion which is obtained by replacing  $A_\mu$  in Eq. (45) with  $A_\mu^\pm/2$ ,

$$\mathcal{L}_B^\pm = \frac{1}{8\pi} \partial_\mu \phi \partial^\mu \phi + \frac{\rho_\pm^0}{2} \epsilon_{0\nu} \partial_\nu \phi + \frac{i}{4\pi} \epsilon^{\mu\nu} \partial_\mu \phi A_\nu^\pm + \frac{c}{4\pi} A_\mu A^\mu, \quad (49)$$

where the local quadratic term of  $A_\mu$  is added again due to the finite renormalization ambiguity. The boson Lagrangian  $\mathcal{L}_B^\pm$  can reproduce the same effective action  $S_{eff}^\pm[A]$  as the Lagrangian of chiral fermion, though it is not the minimal version. To get the minimal form of boson Lagrangian  $\mathcal{L}_B^\pm$ , one can simply put the chiral constraint on the fermion current, then it is easy to reproduce in this T-dual formalism the chiral bosonization of Refs.[50, 51].

## 7. Conclusions and Remarks

As a summary, we adopt the Chern-Simons gauge theory with suitable dimensional reduction to clarify the relation between the braid and fractional exclusion statistics in one dimension. The same framework has also been used in our previous study on the two-dimensional case[23], thus, completing a systematic study on the two aspects of the fractional statistics in low dimensions  $d \leq 2$ .

For Dirac fermions, the flux-binding does not necessarily induce the nontrivial fractional exclusion statistics, which is also consistent with the result given in Ref.[37]. Here we would like to mention that, for the exactly solvable Calogero-Sutherland model, which exhibits explicitly the fractional exclusion statistics as derived from its energy spectrum, one can actually assign arbitrary braid statistics to its manybody wavefunction without changing the energy spectrum due to the impenetrable  $x^{-2}$  interaction. Indeed, the fermion and boson solutions of this model were given in Ref.[20], and another ‘‘natural’’ anyonic Jastrow-Laughlin type wavefunction together with the corresponding correlation functions was constructed in Refs.[31, 32]. Therefore, in this model, the two statistics are not necessarily connected.

For chiral fermions, however, binding flux to the fermion field can induce a well-defined fractional exclusion statistics for both single and multiple species of particles. Since the chiral fermion is simply a boundary system for a Hall insulator, and this result is consistent with our study on the two dimensional cases[23] where we proved the braid statistics together with Hall response can result in a nontrivial exclusion statistics. Both the one and two dimensional results suggest that the time-reversal breaking in the original system before coupling to a Chern-Simons term is somehow necessary for connecting the braid and fractional exclusion statistics for dimensions  $d \leq 2$ .

Since the fractional statistics are naturally related to interactions, we hope our study may shed light on the application of fractional statistics to strongly correlated condensed matter system within the framework of Chern-Simons gauge theory. In particular, using the tomographic decomposition[52] one can analyze the long-range behaviour (scaling limit) of two-dimensional fermionic systems in terms of one-dimensional systems labelled by the rays of the two-dimensional Fermi surface. As an application of the formalism

developed here, we are presently considering the two-dimensional  $t - J$  model, relevant for the high Tc cuprates.

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