



UNIVERSITÀ
DEGLI STUDI
DI PADOVA

Università degli Studi di Padova

Padua Research Archive - Institutional Repository

Minimum time problem with impulsive and ordinary controls

Original Citation:

Availability:

This version is available at: 11577/3281580 since: 2018-10-19T17:17:43Z

Publisher:

American Institute of Mathematical Sciences

Published version:

DOI: 10.3934/dcds.2018252

Terms of use:

Open Access

This article is made available under terms and conditions applicable to Open Access Guidelines, as described at <http://www.unipd.it/download/file/fid/55401> (Italian only)

(Article begins on next page)

MINIMUM TIME PROBLEM WITH IMPULSIVE AND ORDINARY CONTROLS

MONICA MOTTA

Università di Padova
Via Trieste, 63
Padova 35121, Italy

ABSTRACT. Given a nonlinear control system depending on two controls u and v , with dynamics affine in the (unbounded) derivative of u and a closed target set \mathcal{S} depending both on the state and on the control u , we study the minimum time problem with a bound on the total variation of u and u constrained in a closed, convex set U , possibly with empty interior. We revisit several concepts of generalized control and solution considered in the literature and show that they all lead to the same minimum time function T . Then we obtain sufficient conditions for the existence of an optimal generalized trajectory-control pair and study the possibility of Lavrentiev-type gap between the minimum time in the spaces of regular (that is, absolutely continuous) and generalized controls. Finally, under a convexity assumption on the dynamics, we characterize T as the unique lower semicontinuous solution of a regular HJ equation with degenerate state constraints.

1. **Introduction.** Let us consider the problem of minimizing the time

$$t_{(x,u,v)} := \inf\{t \geq 0 : (x(t), u(t)) \in \mathcal{S}\} \quad (1)$$

over trajectory-control pairs (x, u, v) verifying

$$(u, v) \in BV(\mathbb{R}_+, U) \times \mathcal{M}(\mathbb{R}_+, V), \quad \text{Var}(u) \leq K \quad (K > 0); \quad (2)$$

$$\dot{x}(t) = f(x(t), u(t), v(t)) + \sum_{i=1}^m g_i(x(t), u(t), v(t)) \dot{u}_i(t), \quad t \geq 0, \quad (3)$$

$$x(0) = \bar{x}_0 \in \mathbb{R}^n, \quad u(0) = \bar{u}_0 \in U, \quad (4)$$

where the target $\mathcal{S} \subset \mathbb{R}^n \times U$ is a closed set with compact boundary, $V \subset \mathbb{R}^q$ is a compact set and $U \subset \mathbb{R}^m$ is a closed, convex set. Here $BV(\mathbb{R}_+, U)$ denotes the set of U -valued functions with bounded total variation $\text{Var}(u)$, and $\mathcal{M}(\mathbb{R}_+, V)$ is the set of Lebesgue measurable functions with values in V .

A solution x to (3) can be provided by the usual Carathéodory solution only if u is an absolutely continuous control and we call such solutions and controls *regular*. However, due to the unboundedness of \dot{u} , minimizing sequences of regular

2010 *Mathematics Subject Classification.* Primary: 49N25, 49L25; Secondary: 34H5, 49J15.

Key words and phrases. Impulsive optimal control problems, existence of optimal controls, Lavrentiev phenomenon, Hamilton Jacobi equations, lower semicontinuous viscosity solutions.

This research is partially supported by the INdAM-GNAMPA Project 2017 "Optimal impulsive control: higher order necessary conditions and gap phenomena"; and by the Padova University grant PRAT 2015 "Control of dynamics with reactive constraints".

trajectories may converge to a discontinuous map and the existence of minimizers for problem (1) can be guaranteed just on some larger class of *generalized or impulsive* controls and solutions. Optimization problems involving a dynamic as in (3), that is, affine in the derivative of a control but with nonlinear dependence on the state, are studied since the 80s, in relation with the control of mechanical systems by means of moving constraints (see [6] and the references therein). Nowadays the applications of optimal impulsive control extend to several branches of science and technology (see e.g. [5], [9], [14]). This motivates the interest in the subject and poses new theoretical problems as well, mainly in relation with the need of considering nonlinear models, ordinary together with impulsive controls and various types of constraints.

In this paper we focus on the minimum time problem for a general, non-commutative¹ control system, where the impulsive control u is constrained in a closed, convex set U , possibly with empty interior, and the closed target \mathcal{S} depends both on x and on u . It is well known that, for non-commutative systems, there is no canonical way to define a trajectory x associated to (u, v) as in (2) and several concepts of generalized control and solution have been proposed in the literature.

As a first result, in Section 2 after revisiting the notions of *space-time (control and) solution*, *graph completion solution* and *limit solution*, we show that the infimum of problem (1) obtained by minimizing the time over each one of these sets of solutions, is always the same value $T(\bar{x}_0, \bar{u}_0, K)$ (see Theorems 2.5, 2.7). In particular, in the *space-time* approach introduced in [7] and extended in [26] to v -dependent data –which is equivalent to the *method of discontinuous time substitution* considered by the russian school (see [21], [22], [25] and the references therein)–, by just identifying controls u and trajectories x with their graphs, (3) is embedded in the space-time system²

$$\begin{cases} t'(s) = \varphi'_0(s) \\ \xi'(s) = f(\xi(s), \varphi(s), \psi(s))\varphi'_0(s) + \sum_{i=1}^m g_i(\xi(s), \varphi(s), \psi(s))\varphi'_i(s) \end{cases}$$

where the new space-time control (φ_0, φ) is a $1 + m$ -dimensional, Lipschitz continuous map with φ_0 a non decreasing time parametrization. Considering space-time controls where φ'_0 is zero on some intervals, is a way to define generalized trajectories for the original control system in the extended, space-time setting. This definition gives rise to a set-valued notion of generalized solution $x(t) := \xi \circ \varphi_0^{-1}(t)$ to (3), (4) associated to a control (u, v) with $(u, v)(t) \in (\varphi, \psi) \circ \varphi_0^{-1}(t)$. Therefore φ describes a parametrization of a *completion of the graph* of u and by the choice of suitable selections, this allows to define univalued *graph completion* solutions (see Subsection 2.2). The concept of *limit solution* due to [3], is based on the observation that, in the non commutative case, different sequences of regular controls might converge to the same control, while the sequences of the corresponding regular solutions to (3), (4) converge to different limits. Such limits (suitably defined), are called limit solutions. In fact, in [3] the non drift terms g_1, \dots, g_m are independent of the ordinary control v , while here we consider instead a notion of *extended limit solution* to (3), (4) recently introduced in [31]. Notice that graph completion and limit solutions

¹The control system (3) is said *commutative or involutive* if g_1, \dots, g_m do not depend on v , are at least C^1 and the Lie brackets $[(e_i, g_i), (e_j, g_j)] \equiv 0$ for all $i \neq j, i, j = 1, \dots, m$.

²The apex " ' " denotes differentiation with respect to the new parameter s , in order to distinguish it from the time variable, t .

cannot be trivially identified: in particular, it is not obvious that a limit solution can be associated to a space-time control (see Subsection 2.3).

The present assumptions on the sets U and \mathcal{S} are quite a novelty in the graph completion approach, where usually $U = \mathbb{R}^m$ and $\mathcal{S} \subset \mathbb{R}^n$. The hypothesis that U is convex, is actually not a restriction, since it is necessary to guarantee that for any control $u \in BV(\mathbb{R}_+, U)$ there exists a graph completion of it *with the same variation*. This is essential when the variation of u is maximal, that is, $\text{Var}(u) = K$.

Furthermore, in Section 3, assuming convexity of the dynamics, we prove the existence of an optimal space-time trajectory-control pair and, by the results in Section 2, derive the existence of an optimal graph completion solution, which is an optimal limit solution too (see Theorem 3.1 and Corollary 1). In Section 4 we give some examples where a Lavrentiev-type gap between the infimum of the time over generalized and regular controls occurs. This phenomenon, already known in the calculus of variations (see e.g. [8]), is expected in the presence of endpoint constraints, since generalized trajectories may not be approximated by regular trajectories with the same endpoint (see, e.g., [23], [24], [40], [33], [28]). Then we provide new sufficient conditions to avoid it (see Theorem 4.4 and Propositions 2, 3). Let us point out that the no-gap requirement is mandatory in all the applications where only absolutely continuous controls u are implementable, as, for instance, the mechanical examples in [6]. Finally, in Section 5, under convexity of the dynamics and no controllability, following the approach of [13], we characterize the map $(\bar{x}_0, \bar{u}_0, K) \mapsto T(\bar{x}_0, \bar{u}_0, K)$ on $\mathbb{R}^n \times U \times \mathbb{R}_+$, as the unique lower semicontinuous (l.s.c.) solution of a regular Hamilton-Jacobi equation with state constraints, which are degenerate because of the assumptions on U .

Some bibliographical remarks just on those papers on the impulsive minimum time problem, which are most related to our point of view are in order. In [33], for $U = \mathbb{R}^m$ and the target a subset of \mathbb{R}^n , under a quite strong controllability hypothesis, the authors show that the minimum time over space-time controls coincides with the infimum over regular controls and is the unique viscosity solution of a suitable boundary value problem among continuous functions. In [29], dropping the controllability and the v -dependence of the data, the last characterization is obtained among l.s.c. functions. Recently, in [19] the authors investigate an impulsive minimum time problem for a compact control set $U \subset \mathbb{R}^m$. They prove the existence of an optimal control and, following a level set approach, they characterize the associated capture basin. However, their impulsive optimization problem differs from our, since they take the infimum of the time just in the subclass of space-time controls associated to rectilinear graph completions of u (see Subsection 2.2). Moreover, they disregard the explicit dependence of the minimum time function on the variation bound K , which plays here an essential role (see Example 5.1).

1.1. Notation and preliminaries. Let $E \subset \mathbb{R}^N$ be a nonempty subset. For any $r > 0$, $B_N(E, r) := \{z \in \mathbb{R}^N : d(z, E) < r\}$, where $d(z, E)$ denotes the distance of z from the set E . For any interval I and function $f : I \rightarrow E$, $\text{Var}_I(f)$ denotes the (total) variation of f on I . When the domain of f is clear, we simply write $\text{Var}(f)$. Let $\text{AC}(I, E)$, $\text{BV}(I, E)$ denote the set of AC (absolutely continuous) and BV (with bounded variation) functions $f : I \rightarrow E$, respectively. The set $L^1(I, E)$ is the usual quotient space with respect to the Lebesgue measure. When no confusion on either the domain or the codomain may arise, we will sometimes omit one or both of them. We set $\mathbb{R}_+ := [0, +\infty[$. For any $f : E \rightarrow \mathbb{R}^M$ we call *modulus (of continuity) of f*

any increasing, continuous function $\omega : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that $\omega(0) = 0$, $\omega(r) > 0$ for every $r > 0$ and $|f(x_1) - f(x_2)| \leq \omega(|x_1 - x_2|)$ for all $x_1, x_2 \in E$.

Let us recall some basic concepts of non smooth analysis and viability theory, which may be found e.g. in [11], [2].

The *contingent cone* $T_E(z)$ to E at $z \in E$ is defined by

$$w \in T_E(z) \iff \liminf_{h \rightarrow 0^+} \frac{d(z + hw, E)}{h} = 0.$$

The *polar cone* T^- to a subset $T \subset \mathbb{R}^N$ is

$$T^- := \{p \in \mathbb{R}^N : \forall w \in T, p \cdot w \leq 0\}.$$

The *paratingent cone* P_E^G to E relative to a subset $G \subset E$ at $z \in G$ is defined by

$$w \in P_E^G(z) \iff \exists h_n \rightarrow 0^+, w_n \rightarrow w, z_n \rightarrow z (z_n \in G) \text{ s.t. } z_n + h_n w_n \in E.$$

Let E be a closed set. The *limiting normal cone* $N_E(z)$ of E at $z \in E$ is

$$N_E(z) := \{p : \exists z_i \rightarrow z, z_i \in E, p_i \rightarrow p \text{ s.t. } \limsup_{\hat{z} \rightarrow z_i, \hat{z} \in E} \frac{p_i \cdot (\hat{z} - z_i)}{|\hat{z} - z_i|} \leq 0 \forall i\}.$$

If E is closed and convex, then the contingent cone is convex and coincides with the Clarke tangent cone; precisely, $T_E(z)$ is the closed cone spanned by $E - z$. In this case $T_E(z)$ is simply called the *tangent cone* to E at z . Moreover, the polar cone to the tangent cone to E is called the *normal cone* to E at z . It coincides with the limiting normal cone and is still denoted by $N_E(z) = T_E(z)^-$.

E is locally compact if for every $z \in E$ there exists $B_N(z, r)$, $r > 0$, such that the set $E \cap \overline{B_N(z, r)}$ is closed. A locally compact set E is a *viability domain* of a set-valued map $\hat{F} : \mathbb{R}^N \rightarrow \mathbb{R}^N$ if, for every $z \in E$,

$$\hat{F}(z) \cap T_E(z) \neq \emptyset.$$

E is called instead an *invariance domain* [a *backward invariance domain*] of \hat{F} , if, for every $z \in E$,

$$\hat{F}(z) \subset T_E(z) \quad [-\hat{F}(z) \subset T_E(z)].$$

If $\hat{F}(\cdot) = F(\cdot, A)$ for some map $F : \mathbb{R}^N \times A \rightarrow \mathbb{R}^N$, we also say that E is either a *viability domain* or an *invariance domain* [a *backward invariance domain*], respectively, for the control system $z'(s) = F(z(s), a(s))$.

Let E be closed. Let $W : E \rightarrow \mathbb{R}$ be a bounded, l.s.c. function and let $M > 0$ be the supnorm of W . Extend W to \mathbb{R}^N by setting $W(z) = M + 1$ for any $z \notin E$. The *subdifferential* $D^-W(z)$ of W at $z \in E$ is defined by

$$D^-W(z) := \left\{ p \in \mathbb{R}^N : \liminf_{\hat{z} \rightarrow z} \frac{W(\hat{z}) - W(z) - p \cdot (\hat{z} - z)}{|\hat{z} - z|} \geq 0 \right\}.$$

It is well known that

$$p \in D^-W(z) \iff (p, -1) \in [T_{Epi(W)}(z, W(z))]^-,$$

where the *epigraph* of W over E is $Epi(W) := \{(z, r) : z \in E, r \geq W(z)\}$.

Let $\mathcal{H} : E \times \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R}$ be a continuous map. W is said to verify, for any $z \in E$,

$$\mathcal{H}(z, W, DW) \geq 0 \text{ in the viscosity sense} \iff \mathcal{H}(z, W(z), p) \geq 0 \quad \forall p \in D^-W(z).$$

2. The impulsive minimum time problem. In this section we revisit space-time solutions, graph completion solutions and limit solutions in relation with the minimum time problem and prove that the corresponding minimum time functions are the same (see Theorems 2.5, 2.7). In particular, in Theorem 2.8 we prove that graph completion solutions and limit solutions coincide.

Throughout the whole paper we assume that

(H0) $U \subset \mathbb{R}^m$ is a closed, convex subset, which is not a singleton; the control vector fields f, g_1, \dots, g_m , are continuous, locally Lipschitz in (x, u) uniformly w.r.t. $v \in V$, and there is some $M > 0$ such that

$$|f(x, u, v)|, |g_1(x, u, v)|, \dots, |g_m(x, u, v)| \leq M(1 + |(x, u)|) \quad \forall (x, u, v) \in \mathbb{R}^n \times U \times V.$$

Let $(\bar{x}_0, \bar{u}_0, K) \in \mathbb{R}^n \times U \times \mathbb{R}_+$. We define the set of *regular controls* as

$$\text{AC}(\bar{u}_0, K) := \{u \in \text{AC}(\mathbb{R}_+, U) : u(0) = \bar{u}_0, \text{Var}(u) \leq K\}. \quad (5)$$

For any $(u, v) \in \text{AC}(\bar{u}_0, K) \times \mathcal{M}$ ³ there is a unique Carathéodory solution x to

$$\begin{cases} \dot{x}(t) = f(x(t), u(t), v(t)) + \sum_{i=1}^m g_i(x(t), u(t), v(t)) \dot{u}_i(t), & t \geq 0, \\ x(0) = \bar{x}_0. \end{cases} \quad (6)$$

We use the notation $x[\bar{x}_0, \bar{u}_0, u, v] := x$ and call x a *regular solution* and (x, u, v) a *regular trajectory-control pair* of (3), (4).

The *minimum time over regular controls* is given by

$$T_{ac}(\bar{x}_0, \bar{u}_0, K) := \inf_{(u, v) \in \text{AC}(\bar{u}_0, K) \times \mathcal{M}} t(\bar{x}_0, \bar{u}_0, u, v) \quad \forall (\bar{x}_0, \bar{u}_0, K) \in \mathbb{R}^n \times U \times \mathbb{R}_+,$$

where

$$t(\bar{x}_0, \bar{u}_0, u, v) := \inf\{t \geq 0 : (x[\bar{x}_0, \bar{u}_0, u, v](t), u(t)) \in \mathcal{S}\}.$$

Hence $T_{ac} = 0$ on $\mathcal{S} \times \mathbb{R}_+$ and we set $T_{ac} = +\infty$ if the set in the above minimization is empty. The next subsections are devoted to extend the minimum time problem to the case of impulsive controls and trajectories.

2.1. Space-time controls and solutions. To begin with, let us consider an auxiliary control system, the so-called space-time system. For $L > 0$ and $\bar{u}_0 \in U$, let $\text{Lip}(\bar{u}_0, L)$ denote the subset of L -Lipschitz maps

$$(\varphi_0, \varphi) : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \times U,$$

such that

$$(\varphi_0, \varphi)(0) = (0, \bar{u}_0), \quad \varphi_0'(s) \geq 0, \quad 0 < \varphi_0'(s) + |\varphi'(s)| \leq L \quad \text{for a.e. } s \in \mathbb{R}_+.$$

We use the apex " ' " to denote differentiation with respect to the parameter s in order to underline that it does not coincide, in general, with the time variable, t .

Definition 2.1 (Space-time control and solution). Let $(\bar{x}_0, \bar{u}_0) \in \mathbb{R}^n \times U$. We call an element $((\varphi_0, \varphi), \psi) \in \text{Lip}(\bar{u}_0, L) \times \mathcal{M}$ for some $L > 0$, a *space-time control* and call space-time control system, the system

$$\begin{cases} \xi'(s) = f(\xi(s), \varphi(s), \psi(s))\varphi_0'(s) + \sum_{i=1}^m g_i(\xi(s), \varphi(s), \psi(s))\varphi_i'(s) \\ \xi(0) = \bar{x}_0. \end{cases} \quad (7)$$

We will write $\xi[\bar{x}_0, \bar{u}_0, \varphi_0, \varphi, \psi]$ to denote the solution of (7).

³Here and in the sequel we use, for brevity, \mathcal{M} in place of $\mathcal{M}(\mathbb{R}_+, V)$.

For brevity, in (7) we omit the equation $t'(s) = \varphi'_0(s)$, that actually justifies the term ‘space-time control system’. The set of regular trajectory-control pairs to (3),(4) is in one-to-one correspondence with the subset of space-time trajectory-control pairs $(\xi, \varphi_0, \varphi, \psi)$ with $\varphi'_0 > 0$ a.e.. Precisely, by a standard application of the chain rule together with the well known fact that the inverse of an increasing absolutely continuous function $\varphi_0 : \mathbb{R}_+ \rightarrow \mathbb{R}$ is absolutely continuous if and only if $\varphi'_0(s) > 0$ for almost all $s \geq 0$, we have what follows.

Proposition 1. (i) *Given $(u, v) \in \text{AC}(\bar{u}_0, K) \times \mathcal{M}$ for some $K > 0$ and $x := x[\bar{x}_0, \bar{u}_0, u, v]$, set*⁴

$$\begin{aligned} \sigma(t) &:= \int_0^t (1 + |\dot{u}(\tau)|) d\tau \quad \forall t \in \mathbb{R}_+, \\ \varphi_0 &:= \sigma^{-1}, \quad \varphi := u \circ \varphi_0, \quad \psi := v \circ \varphi_0, \quad \xi := \xi[\bar{x}_0, \bar{u}_0, \varphi_0, \varphi, \psi]. \end{aligned} \quad (8)$$

Then $((\varphi_0, \varphi), \psi) \in \text{Lip}(\bar{u}_0, 1) \times \mathcal{M}$, $(\xi, \varphi, \psi) \circ \sigma = (x, u, v)$, and u and φ have the same total variation. We will call such space-time trajectory the arc-length graph-parametrization of (x, u, v) .

(ii) *Vice-versa, given $((\varphi_0, \varphi), \psi) \in \text{Lip}(\bar{u}_0, L) \times \mathcal{M}$ with $\text{Var}(\varphi) \leq K$ for some $L, K > 0$ and*

$$\varphi'_0(s) > 0 \text{ for almost all } s \geq 0,$$

let us set $\xi := \xi[\bar{x}_0, \bar{u}_0, \varphi_0, \varphi, \psi]$ and

$$(u, v) := (\varphi, \psi) \circ \varphi_0^{-1}, \quad x := x[\bar{x}_0, \bar{u}_0, u, v].$$

Then $(u, v) \in \text{AC}(\bar{u}_0, K) \times \mathcal{M}$, $(x, u, v) \circ \varphi_0 = (\xi, \varphi, \psi)$ and the total variations of u and φ coincide.

Considering space-time controls where φ'_0 is zero on some intervals, allows to define discontinuous solutions for the original control system.

Remark 1. The space-time system is *rate-independent*. Precisely, as shown in [26, Sect.3], if $((\varphi_0, \varphi), \psi), ((\tilde{\varphi}_0, \tilde{\varphi}), \tilde{\psi})$ verify $((\varphi_0, \varphi), \psi) = ((\tilde{\varphi}_0, \tilde{\varphi}), \tilde{\psi}) \circ \tilde{s}$ for some Lipschitz continuous, strictly increasing, surjective map $\tilde{s} : \mathbb{R}_+ \rightarrow \mathbb{R}_+$, then one has $\xi = \tilde{\xi} \circ \tilde{s}$, if ξ and $\tilde{\xi}$ denote the solutions to (7) corresponding to $((\varphi_0, \varphi), \psi)$ and $((\tilde{\varphi}_0, \tilde{\varphi}), \tilde{\psi})$, respectively. It is thus not restrictive to consider in the sequel only (φ_0, φ) verifying

$$\varphi'_0(s) + |\varphi'(s)| = 1 \text{ a.e..}$$

For any $(\bar{u}_0, K) \in U \times \mathbb{R}_+$, we define the set of *feasible space-time controls* as

$$\Gamma(\bar{u}_0, K) := \{(\varphi_0, \varphi) \in \text{Lip}(\bar{u}_0, 1) : \varphi'_0(s) + |\varphi'(s)| = 1 \text{ a.e., } \text{Var}(\varphi) \leq K\}. \quad (9)$$

Notice that when $(\varphi_0, \varphi) \in \Gamma(\bar{u}_0, K)$, the parameter s coincides with the *arc-length parameter* of the curve (φ_0, φ) (with respect to the norm $\varphi'_0(s) + |\varphi'(s)|$) and we have the identity

$$s = \varphi_0(s) + \text{Var}_{[0,s]}[\varphi] \quad \forall s \geq 0. \quad (10)$$

As a consequence,

$$\lim_{s \rightarrow +\infty} \varphi_0(s) = +\infty. \quad (11)$$

⁴ Since every L^1 equivalence class contains Borel measurable representatives, here and in the sequel we tacitly assume that the maps v and ψ are Borel measurable on compact intervals, when necessary.

Let $(\bar{x}_0, \bar{u}_0, K) \in \mathbb{R}^n \times U \times \mathbb{R}_+$. Given $((\varphi_0, \varphi), \psi) \in \Gamma(\bar{u}_0, K) \times \mathcal{M}$, we introduce the *pseudo exit-time* (from $(\mathbb{R}^n \times U) \setminus \mathcal{S}$)

$$S = S_{(\bar{x}_0, \bar{u}_0)}((\varphi_0, \varphi), \psi) := \inf\{s \geq 0 : (\xi[\bar{x}_0, \bar{u}_0, \varphi_0, \varphi, \psi](s), \varphi(s)) \in \mathcal{S}\}, \quad (12)$$

where S is 0 when $(\bar{x}_0, \bar{u}_0) \in \mathcal{S}$ and we define $S = +\infty$ if the above set is empty. The *minimum time function over space-time controls* is defined by

$$T_{st}(\bar{x}_0, \bar{u}_0, K) := \inf_{((\varphi_0, \varphi), \psi) \in \Gamma(\bar{u}_0, K) \times \mathcal{M}} \varphi_0(S_{(\bar{x}_0, \bar{u}_0)}((\varphi_0, \varphi), \psi)), \quad (13)$$

where, in view of (11), we set $\varphi_0(+\infty) := +\infty$.

2.2. Graph completions and graph completion solutions. From a space-time trajectory-control pair we can obtain a notion of generalized solution to (3), (4) associated to a control (u, v) in $BV \times \mathcal{M}$. Precisely, given $(\bar{u}_0, K) \in U \times \mathbb{R}_+$ let us define the set of *feasible controls* u as

$$BV(\bar{u}_0, K) := \{u \in BV(\mathbb{R}_+, U) : u(0) = \bar{u}_0, \text{Var}(u) \leq K\}. \quad (14)$$

Definition 2.2 (Graph completion). Let $(u, v) \in BV(\bar{u}_0, K) \times \mathcal{M}$. We call a space-time control $((\varphi_0, \varphi), \psi) \in \Gamma(K, \bar{u}_0) \times \mathcal{M}$ a *graph completion* of (u, v) if for all $t \geq 0$ there is $s \geq 0$ such that $((\varphi_0, \varphi), \psi)(s) = (t, u(t), v(t))$.

This definition gives rise naturally to a *set-valued* notion of solution for the original system (3), (4), by considering

$$x_{set} : \mathbb{R}_+ \rightsquigarrow \mathbb{R}^n, \quad x_{set}(t) := \xi \circ \varphi_0^{-1}(t) \quad \forall t \geq 0.$$

In the literature, controls $u \in BV$ which are right or left continuous are often considered and, accordingly, a univalued, right or left continuous selection of x_{set} is chosen. Recently, for general controls $u \in BV$, [3] proposed a univalued notion of solution to (3), (4) associated to a pointwise selection of the (set-valued) inverse $(\varphi_0, \varphi)^{-1}$. This selection is called a *clock*.

Definition 2.3 (Clock). Given a graph completion $((\varphi_0, \varphi), \psi) \in \Gamma(K, \bar{u}_0) \times \mathcal{M}$ of a control $(u, v) \in BV(\bar{u}_0, K) \times \mathcal{M}$, we call a *clock* any strictly increasing function $\sigma : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that

$$(\varphi_0, \varphi)(\sigma(t)) = (t, u(t)) \quad \text{for every } t \geq 0, \text{ and } \sigma(0) = 0.$$

Definition 2.4 (Graph completion solution). Given a control $(u, v) \in BV(\bar{u}_0, K) \times \mathcal{M}$, let $((\varphi_0, \varphi), \psi) \in \Gamma(K, \bar{u}_0) \times \mathcal{M}$ be one of its graph completions and let σ be a clock. Set $\xi := \xi[\bar{x}_0, \bar{u}_0, \varphi_0, \varphi, \psi]$. We define a *graph completion solution* (shortly, g.c. solution) to (3), (4) associated to $((\varphi_0, \varphi), \psi)$ and σ , the map

$$x(t) := \xi \circ \sigma(t) \quad \forall t \geq 0.$$

$\Sigma_{gc}(\bar{x}_0, \bar{u}_0, K, u, v)$ denotes the set of g.c. solutions associated to (u, v) .

Since U is convex, the simplest graph completion of u can be obtained by bridging each of its jumps with a straight segment (see e.g. [6, Lemma 10.1]).

Notice that if $\text{Var}(u) = K$ this is the only possible graph completion of u belonging to $\Gamma(\bar{u}_0, K)$, while when $\text{Var}(u) < K$ also non rectilinear completions with variation not greater than K can be considered. In fact, in this case graph completions allow for jumps of the trajectory even at times t where u is continuous. Indeed at these instants, owing to the non-triviality of the Lie algebra generated by $\{(\mathbf{e}_1, g_1), \dots, (\mathbf{e}_m, g_m)\}$, a loop of u might determine a discontinuity in x .

Let $(\bar{x}_0, \bar{u}_0, K) \in \mathbb{R}^n \times U \times \mathbb{R}_+$. Given $(u, v) \in BV(\bar{u}_0, K) \times \mathcal{M}$, for any g.c. solution $x \in \Sigma_{gc}(\bar{x}_0, \bar{u}_0, K, u, v)$ we set

$$t(x, u, v) := \inf\{t \geq 0 : (x, u)(t) \in \mathcal{S}\} (\leq +\infty)$$

and define

$$T_{gc}(\bar{x}_0, \bar{u}_0, K) := \inf_{(u, v) \in BV(\bar{u}_0, K) \times \mathcal{M}, x \in \Sigma_{gc}(\bar{x}_0, \bar{u}_0, K, u, v)} t(x, u, v).$$

Notice that $T_{gc}(\bar{x}_0, \bar{u}_0, K) = 0$ when $(\bar{x}_0, \bar{u}_0) \in \mathcal{S}$, while we define it to be equal to $+\infty$ if no trajectory-control pairs reach the target.

Theorem 2.5. *Assume (H0). Then $T_{st} = T_{gc}$.*

Proof. The inequality $T_{st} \leq T_{gc}$ follows directly from the definitions, since any g.c. solution-control pair (x, u, v) is a selection associated to a space-time trajectory-control pair $(S, \xi, \varphi_0, \varphi, \psi)$. It is thus enough to show that $T_{gc} \leq T_{st}$. Suppose, on the contrary, that there exist some $(\bar{x}_0, \bar{u}_0, K) \in ((\mathbb{R}^n \times U) \setminus \mathcal{S}) \times \mathbb{R}_+$ and $\varepsilon > 0$ such that

$$T_{st}(\bar{x}_0, \bar{u}_0, K) < +\infty \quad \text{and} \quad T_{st}(\bar{x}_0, \bar{u}_0, K) \leq T_{gc}(\bar{x}_0, \bar{u}_0, K) - 2\varepsilon.$$

Let $(\xi, \varphi_0, \varphi, \psi)$ be an ε -optimal space-time trajectory-control pair, so that it verifies $(\xi(S), \varphi(S)) \in \mathcal{S}$ and

$$\varphi_0(S) \leq T_{st}(\bar{x}_0, \bar{u}_0, K) + \varepsilon \leq T_{gc}(\bar{x}_0, \bar{u}_0, K) - \varepsilon \quad (15)$$

(here $S = S_{(\bar{x}_0, \bar{u}_0)}((\varphi_0, \varphi), \psi)$). Now any increasing selection $\sigma(t) \in \varphi_0^{-1}(t)$, $t \geq 0$ satisfying

$$\sigma(0) = 0, \quad \sigma(\varphi_0(S)) = S,$$

is a clock and $(x, u, v) := (\xi, \varphi, \psi) \circ \sigma$ is a g.c. solution-control pair associated to $(\xi, \varphi_0, \varphi, \psi)$ and σ , with exit-time $T := t(x, u, v) = \varphi_0(S)$, since $(x, u)(T) = (\xi, \varphi)(S)$. Therefore $T_{gc}(\bar{x}_0, \bar{u}_0, K) \leq \varphi_0(S)$ and by (15) this leads to the contradiction $0 < -\varepsilon$, which allows us to conclude that $T_{st} = T_{gc}$. \square

Remark 2. Several concepts of generalized control and solution considered in the literature rely on the graph completion approach, whose roots can be found already in the pioneer works [34], [35], [38], [39], and can be proven to lead to the minimum time function T_{st} , arguing similarly as above. This is true, in particular, for the generalized controls and solutions introduced by the russian school (see e.g. [18], [16], [12] concerning commutative systems, [21]–[25] for the general case and the references therein), the impulsive control ϑ of [1], [17], and the Fréchet generalized trajectories defined by [15]. The minimum time defined in [19] instead, is in general strictly greater than T_{st} , since the authors take the infimum only over the subset of rectilinear graph completions.

In all the above references one considers controls u with bounded variation. Very recently, in [30]–[32] the graph completion technique has been extended to the case of controls u with possibly unbounded variation.

The following example shows how the existence of an optimal control is verified, in general, only in the enlarged class of discontinuous controls u and g.c. solutions, defined as above. In particular, no optimal control exists in the subset of rectilinear graph completions of u .

Example 2.1. Consider the impulsive control system in \mathbb{R}^2

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \dot{u}_1 + \begin{pmatrix} 0 \\ x_1 \end{pmatrix} \dot{u}_2 = g_1(x) \dot{u}_1 + g_2(x) \dot{u}_2$$

with control set $U := [0, 1] \times [0, 1]$, target $\mathcal{S} = \{(1, 1), (1, 1)\}$, bound on the variation $K = 2$ and initial condition

$$(x_1, x_2)(0) = (0, 0), \quad (u_1, u_2)(0) = (0, 0).$$

In this case, the minimum time to reach the target over the subset of regular (that is, absolutely continuous) controls and trajectories is equal to 0. Indeed, the following sequence (u_h, x_h) , where, for each h ,

$$u_h(t) := \begin{cases} (0, 0) & \text{if } t \in [0, 1/h] \\ (h(t - \frac{1}{h}), 0) & \text{if } t \in [1/h, 2/h] \\ (1, h(t - \frac{2}{h})) & \text{if } t \in [2/h, 3/h] \\ (1, 1) & \text{if } t > 3/h. \end{cases} \quad (16)$$

and $x_h := x[0, 0, 0, 0, u_h]$ given by

$$x_h(t) = \begin{cases} (0, 0) & \text{if } t \in [0, 1/h] \\ (h(t - \frac{1}{h}), 0) & \text{if } t \in [1/h, 2/h] \\ (1, h(t - \frac{2}{h})) & \text{if } t \in [2/h, 3/h] \\ (1, 1) & \text{if } t > 3/h, \end{cases} \quad (17)$$

is a feasible minimizing sequence, since

$$\text{Var}(u_h) \leq 2 \quad \text{and} \quad (x_h, u_h)(3/h) \in \mathcal{S} \quad \text{for all } h.$$

Clearly, the minimum time $T_{ac}((0, 0), (0, 0), 2) = 0$ over regular controls is just an infimum and a regular optimal control does not exist: an optimal trajectory-control pair (\tilde{x}, \tilde{u}) should be discontinuous at $t = 0$ and jump from $(\tilde{x}, \tilde{u})(0) = ((0, 0), (0, 0))$ to $(\tilde{x}, \tilde{u})(0^+) = ((1, 1), (1, 1))$. This ‘lack of compactness’ is exactly the reason why impulsive controls have been introduced.

Notice that the map $\tilde{x} : \mathbb{R}_+ \rightarrow \mathbb{R}^2$,

$$\tilde{x}(0) = (0, 0), \quad \tilde{x}(t) = (1, 1) \quad \text{for all } t > 0 \quad (18)$$

is an optimal g.c. solution corresponding to the control $\tilde{u} : \mathbb{R}_+ \rightarrow [0, 1] \times [0, 1]$,

$$\tilde{u}(0) = (0, 0), \quad \tilde{u}(t) = (1, 1) \quad \text{for all } t > 0, \quad (19)$$

associated to the space-time control and trajectory

$$\begin{cases} (\varphi_0, \varphi) := (0, (s, 0))\chi_{[0,1]} + (0, (1, s-1))\chi_{[1,2]} + (s-2, (1, 1))\chi_{[2,+\infty[}, \\ \xi[0, 0, 0, 0, \varphi_0, \varphi] = (s, 0)\chi_{[0,1]} + (1, s-1)\chi_{[1,2]} + (0, (1, 1))\chi_{[2,+\infty[} \end{cases} \quad (20)$$

and to the clock

$$\sigma(0) := 0, \quad \sigma(t) := t + 2 \quad \text{for all } t > 0.$$

Let us remark that if we consider rectilinear graph completions only, no optimal g.c. solution exists for this minimum time problem. Precisely, the optimal control \tilde{u} must verify $\tilde{u}(0) = (0, 0)$ and $\tilde{u}(0^+) = (1, 1)$, but the rectilinear completion is

$$(\varphi_0, \varphi)(s) = (0, s(1/\sqrt{2}, 1/\sqrt{2})) \quad s \in [0, \sqrt{2}]$$

so that the corresponding space-time trajectory $\xi := \xi[0, 0, 0, 0, \varphi_0, \varphi]$ is given by

$$\xi(s) = (s/\sqrt{2}, s^2/4) \quad s \in [0, \sqrt{2}].$$

Now, the (unique) associated g.c. solution verifies $\hat{x}(0^+) = \xi(\sqrt{2}) = (1, 1/2)$. Thus $(\hat{x}, \hat{u})(0^+) \notin \mathcal{S}$ and the target cannot be reached in time 0.

The point here is that different completions of the graph of u lead actually to different g.c. solutions x , since the vector fields g_1, g_2 do not commute:

$$[g_1, g_2](x) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

2.3. Limit solutions. The next definition is an adaptation of the notion of *extended BV simple limit solution* (shortly *E-BVS limit solution*), introduced in [31], to the case of free final time and impulsive controls u with a prescribed bound K on the variation.⁵

Definition 2.6 (Limit solution). Let $(\bar{x}_0, \bar{u}_0, K) \in \mathbb{R}^n \times U \times \mathbb{R}_+$ and $(u, v) \in BV(\bar{u}_0, K) \times \mathcal{M}$. We say that a map $x \in BV(\mathbb{R}_+, \mathbb{R}^n)$ is a limit solution (shortly, ℓ . solution) of the Cauchy problem (3), (4) if there are $\psi \in \mathcal{M}$ and a sequence (x_h, u_h, v_h) , where $(u_h, v_h) \in AC(\bar{u}_0, K) \times \mathcal{M}$, $x_h := x[\bar{x}_0, \bar{u}_0, u_h, v_h]$, such that for any $T > 0$:

(i) for every $t \in [0, T]$,

$$|(x_h, u_h)(t) - (x, u)(t)| + \|(x_h, u_h, v_h) - (x, u, v)\|_{L^1([0, T])} \rightarrow 0 \text{ as } k \rightarrow +\infty;$$

(ii) setting $\sigma_k(t) := t + Var_{[0, t]}(u_k)$, $V_k := Var_{\mathbb{R}_+}(u_k) \leq K$, one has

$$\|(v_k \circ (\sigma_k)^{-1} - \psi) \chi_{[0, T+V_k]}\|_{L^1([0, T+K])} \rightarrow 0 \text{ as } k \rightarrow +\infty.$$

We use $\Sigma_\ell(\bar{x}_0, \bar{u}_0, K, u, v)$ to denote the set of ℓ . solutions associated to (u, v) .

Remark 3. Condition (ii) in Definition 2.6 takes into account the interplay between u and v which is due to the presence of the ordinary control v in the non-drift terms g_1, \dots, g_m . As shown in [31], when these latter maps do not depend on v , Definition 2.6 can be equivalently stated without hypothesis (ii). More precisely, in this case x is an ℓ . solution if and only if it is the pointwise limit of a sequence of regular trajectories $x_h = x[\bar{x}_0, \bar{u}_0, u_h, v]$ with u_h as above and v fixed. This was in fact the original notion of BVS limit solution, given in [3], [4] when g_1, \dots, g_m do not depend on v .

Let $(\bar{x}_0, \bar{u}_0, K) \in \mathbb{R}^n \times U \times \mathbb{R}_+$. For any ℓ . solution $x \in \Sigma_\ell(\bar{x}_0, \bar{u}_0, K, u, v)$ associated to $(u, v) \in BV(\bar{u}_0, K) \times \mathcal{M}$, we define the *exit-time*

$$t(x, u, v) := \inf\{t \geq 0 : (x, u)(t) \in \mathcal{S}\} (\leq +\infty)$$

and introduce the minimum time function

$$T_\ell(\bar{x}_0, \bar{u}_0, K) := \inf_{(u, v) \in BV(\bar{u}_0, K) \times \mathcal{M}, x \in \Sigma_\ell(\bar{x}_0, \bar{u}_0, K, u, v)} t(x, u, v).$$

As in the case of g.c. solutions, we have $T_\ell(\bar{x}_0, \bar{u}_0, K) = 0$ when $(\bar{x}_0, \bar{u}_0) \in \mathcal{S}$ and we set $T_\ell(\bar{x}_0, \bar{u}_0, K) := +\infty$ if no trajectory-control pairs reaching the target exist.

Theorem 2.7. *Assume (H0). Then $T_\ell = T_{gc}$.*

⁵In fact, the general concept of limit solution has been developed for U compact, but it is easy to see that the boundedness of U is unnecessary, if we consider only controls u with a-priori bounded variation.

This result is a straightforward consequence of the following equivalence.

Theorem 2.8. *Assume (H0) and let $(\bar{x}_0, \bar{u}_0, K) \in \mathbb{R}^n \times U \times \mathbb{R}_+$, $(u, v) \in BV(\bar{u}_0, K) \times \mathcal{M}$. A map x is an ℓ .solution to (3), (4) associated to (u, v) if and only if it is a g.c. solution to (3), (4) associated to the same control.*

Proof. Preliminarily, let us observe that the proof of [31, Thm. 5.2] of the identity on $[0, T]$ between E-BVS limit solutions and graph completion solutions, does not require to increase the variations of the involved controls. Precisely, if x is an E-BVS limit solution to (3), (4) on $[0, T]$ defined by an approximating regular sequence (x_h, u_h, v_h) verifying for every h $\text{Var}_{[0, T]}(u_h) \leq K$, then it is a graph completion solution associated to a feasible space-time control $((\varphi_0, \varphi), \psi)$ on $[0, S]$ and to a clock $\sigma : [0, T] \rightarrow [0, S]$, for some $S \geq T$. In particular, in the proof of [31, Thm. 5.2], (φ_0, φ) is defined as the uniform limit of the arc-length graph-parametrization of the u_h . Hence by the variation's bound K on the u_h , Proposition 1 implies that $\text{Var}_{[0, S]}(\varphi) \leq K$. Conversely, let x be a graph completion solution to (3), (4) on $[0, T]$ associated to a feasible space-time control $((\varphi_0, \varphi), \psi)$ defined on $[0, S]$ with $\text{Var}_{[0, S]}(\varphi) \leq K$, and to a clock $\sigma : [0, T] \rightarrow [0, S]$. Then, again by the same theorem, x is an E-BVS limit solution to (3), (4) on $[0, T]$ and its proof yields that the regular approximating sequence (x_h, u_h, v_h) defining x verifies $\text{Var}_{[0, T]}(u_h) = \text{Var}_{[0, S]}(\varphi) \leq K$ for every h .

By the above arguments we derive that if x is an ℓ .solution to (3), (4) corresponding to (u, v) , for any $T > 0$ the map x is a g.c. solution to (3), (4) on $[0, T]$.

Conversely, if x is a g.c. solution to (3), (4), for any $T > 0$ x is an ℓ .solution to (3), (4) on $[0, T]$. The arbitrariness of $T > 0$ allows us to conclude that the assertions of the theorem are proved. \square

In view of Theorems 2.5, 2.7 and 2.8, from now on let us call *impulsive solution* any x which is an ℓ .solution/g.c.solution to (3), (4) and let us simply use T to denote the unique minimum time function in impulsive control. Moreover, we set

$$\mathcal{R} := \{(\bar{x}_0, \bar{u}_0, K) \in \mathbb{R}^n \times U \times \mathbb{R}_+ : T(\bar{x}_0, \bar{u}_0, K) < +\infty\}. \quad (21)$$

3. Existence of time-optimal controls. Let us introduce the following hypothesis:

(H1) For every $(x, u) \in (\mathbb{R}^n \times U) \setminus \mathcal{S}$, the set

$$F(x, u) := \left\{ \left(f(x, u, v)w_0 + \sum_{i=1}^m g_i(x, u, v) w_i, w_0, w \right) : \right. \\ \left. v \in V, (w_0, w) \in \mathbb{R}_+ \times \mathbb{R}^m, w_0 + \|w\| \leq 1 \right\} \quad (22)$$

is convex.

Remark 4. If the vector fields f and g_i , $i = 1, \dots, m$, do not depend on $v \in V$, then the set $F(x, u)$ is trivially convex, for all (x, u) .

Theorem 3.1. *Assume (H0), (H1). Then for any $(\bar{x}_0, \bar{u}_0, K) \in \mathcal{R} \setminus (\mathcal{S} \times \mathbb{R}_+)$ there exists an optimal space-time control $((\varphi_0, \varphi), \psi) \in \Gamma(\bar{u}_0, K) \times \mathcal{M}$.*

Proof. Let $((\varphi_{0h}, \varphi_h), \psi_h) \subset \Gamma(\bar{u}_0, K) \times \mathcal{M}$ be a minimizing sequence and set

$$\xi_h := \xi[\bar{x}_0, \bar{u}_0, \varphi_{0h}, \varphi_h, \psi_h], \quad S_h := S_{(\bar{x}_0, \bar{u}_0)}(\varphi_{0h}, \varphi_h, \psi_h).$$

Hence we have

$$\lim_{h \rightarrow +\infty} \varphi_{0h}(S_h) = T(\bar{x}_0, \bar{u}_0, K) \quad (23)$$

and, for each $h \in \mathbb{N}$,

$$\int_0^{S_h} |\varphi'_h(s)| ds \leq K, \quad (\xi_h, \varphi_h)(S_h) \in \mathcal{S}. \quad (24)$$

Moreover, for h large enough, S_h verifies $S_h \leq \bar{S}$ for $\bar{S} := T(\bar{x}_0, \bar{u}_0, K) + 1 + K$. Let us set $(\varphi_{0h}, \varphi_h, \xi_h)(s) := (\varphi_{0h}, \varphi_h, \xi_h)(S_h)$ for all $s \in [S_h, \bar{S}]$. Since the sequence (S_h) is bounded and the sequence $((\varphi_{0h}, \varphi_h), \xi_h)$ is equi-bounded and equi-Lipschitzian in $[0, \bar{S}]$, compactness and Ascoli-Arzelà's Theorem, respectively, imply that there exist some subsequences (we do not relabel), a Lipschitz continuous functions $(\varphi_0, \varphi, \xi) : [0, \bar{S}] \rightarrow \mathbb{R} \times U \times \mathbb{R}^n$ and $S \leq \bar{S}$, such that

$$\lim_{h \rightarrow +\infty} S_h = S, \quad \lim_{h \rightarrow +\infty} \sup_{s \in [0, \bar{S}]} |(\varphi_{0h}, \varphi_h, \xi_h)(s) - (\varphi_0, \varphi, \xi)(s)| = 0.$$

Notice that $S > 0$, since, for all h , $S_h \geq \frac{d((\bar{x}_0, \bar{u}_0), \mathcal{S})}{L} > 0$ if $L > 0$ is the common Lipschitz constant of the $(\varphi_{0h}, \varphi_h, \xi_h)$.

By passing to the limit in (23) and (24) as $h \rightarrow +\infty$, we derive

$$\varphi_0(S) = T(\bar{x}_0, \bar{u}_0, K) \quad (25)$$

and

$$\int_0^S |\varphi'(s)| ds \leq K, \quad (\xi, \varphi)(S) \in \mathcal{S}. \quad (26)$$

Owing to (H1), Filippov's Theorem applied to the control system

$$\begin{cases} \xi'(s) = f(\xi(s), \varphi(s), \psi(s))w_0(s) + \sum_{i=1}^m g_i(\xi(s), \varphi(s), \psi(s))w_i(s) \\ \varphi'_0(s) = w_0(s) \\ \varphi'(s) = w(s) \\ (\xi, \varphi_0, \varphi)(0) = (\bar{x}_0, 0, \bar{u}_0). \end{cases} \quad (27)$$

yields the existence of a measurable map $(w_0, w, \psi) : [0, S] \rightarrow \mathbb{R}_+ \times \mathbb{R}^m \times V$, such that

$$(w_0, w)(s) = (\varphi'_0, \varphi')(s), \quad w_0(s) + |w(s)| \leq 1 \quad \text{for almost all } s \in [0, S]$$

and $\xi = \xi[\bar{x}_0, \bar{u}_0, \varphi_0, \varphi, \psi]$.

In general, the control $((\varphi_0, \varphi), \psi)$ does not belong to $\Gamma(\bar{u}_0, K) \times \mathcal{M}$, since the quantity $\varphi'_0(s) + |\varphi'(s)|$ might be strictly smaller than 1 on a set of positive measure. To recover a feasible space-time control, let us introduce the change of variable

$$\eta(s) := \int_0^s [\varphi'_0(r) + |\varphi'(r)|] dr \quad \forall s \in [0, S], \quad \tilde{S} := \eta(S).$$

Notice that $\tilde{S} > 0$ since, on the contrary, we would have $\varphi'_0(r) + |\varphi'(r)| = 0$ a.e., so that $(\xi, \varphi)(S) = (\xi, \varphi)(0)$, while $(\xi, \varphi)(S) \in \mathcal{S}$ but $(\xi, \varphi)(0) = (\bar{x}_0, \bar{u}_0) \notin \mathcal{S}$. We can now obtain a space-time control in $\Gamma(\bar{u}_0, K) \times \mathcal{M}$ by considering, e.g. the strictly increasing right-inverse $s(\cdot) : [0, \tilde{S}] \rightarrow [0, S]$ of η , and defining

$$((\tilde{\varphi}_0, \tilde{\varphi}), \tilde{\psi}) := ((\varphi_0 \circ s, \varphi \circ s), \psi \circ s), \quad \tilde{\xi} := \xi[\bar{x}_0, \bar{u}_0, \tilde{\varphi}_0, \tilde{\varphi}, \tilde{\psi}].$$

Indeed, $((\varphi_0, \varphi), \psi)$ is constant on any interval $[s_1, s_2]$ where η is constant, so that $(\tilde{\varphi}_0, \tilde{\varphi}, \tilde{\xi}) \circ \eta = (\varphi_0, \varphi, \xi)$ and clearly $\tilde{\varphi}'_0(s) + |\tilde{\varphi}'(s)| = 1$ for almost all $s \in [0, \tilde{S}]$. Hence $((\tilde{\varphi}_0, \tilde{\varphi}), \tilde{\psi}) \in \Gamma(\bar{u}_0, K) \times \mathcal{M}$,

$$\tilde{\varphi}_0(\tilde{S}) = \varphi_0(S) = T(\bar{x}_0, \bar{u}_0, K) \quad \text{and} \quad (\tilde{\xi}, \tilde{\varphi})(\tilde{S}) = (\xi, \varphi)(S) \in \mathcal{S},$$

so proving that $((\tilde{\varphi}_0, \tilde{\varphi}), \tilde{\psi})$ is optimal. \square

In view of the results in the previous section, by Theorem 3.1 we derive the following

Corollary 1. *Assume (H0), (H1). Then for any $(\bar{x}_0, \bar{u}_0, K) \in \mathcal{R} \setminus (\mathcal{S} \times \mathbb{R}_+)$ there exists an optimal impulsive solution x associated to an optimal control $(u, v) \in BV(\bar{u}_0, K) \times \mathcal{M}$.*

4. Lavrentiev-type gap phenomena. This section is devoted to investigate the possible occurrence of a gap between the minimum time over impulsive and regular controls, respectively.

Definition 4.1 (Lavrentiev-type gap). We say that we have a *Lavrentiev-type gap* at $(\bar{x}_0, \bar{u}_0, K) \in \mathbb{R}^n \times U \times \mathbb{R}_+$, if

$$T(\bar{x}_0, \bar{u}_0, K) < T_{ac}(\bar{x}_0, \bar{u}_0, K).$$

The gap may occur for "lack of controllability", as in Example 4.1, or for "shortage of total variation", as in Example 4.2.

Example 4.1. Consider the impulsive control system in \mathbb{R}^3

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \dot{u}_1 + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \dot{u}_2 = f(x) + g_1(x) \dot{u}_1 + g_2(x) \dot{u}_2$$

with control set $U := [0, 1] \times [0, 1]$, target $\mathcal{S} = \{((0, 0, 0), (0, 0))\}$, bound on the variation $K = 2$ and initial conditions

$$(x_1, x_2, x_3)(0) = (1, 0, 0), \quad (u_1, u_2)(0) = (1, 0).$$

The associated space-time control system is

$$\xi' = f(\xi) \varphi_0'(s) + g_1(\xi) \varphi_1'(s) + g_2(\xi) \varphi_2'(s)$$

and it is immediate to see that the space-time control

$$(\varphi_0, \varphi_1, \varphi_2)(s) := (0, 1 - s, 0) \quad \forall s \geq 0 \quad (28)$$

is (feasible and) optimal, with $T((1, 0, 0), (1, 0), 2) = \varphi_0(1) = 0$. On the contrary, the minimum function $T_{ac}((1, 0, 0), (1, 0), 2) = +\infty$, since $x_3(t) > 0 \quad \forall t > 0$ in correspondence of any regular control $u \in AC$. Notice that the control system without drift, that is,

$$\xi' = g_1(\xi) \varphi_1'(s) + g_2(\xi) \varphi_2'(s)$$

is not controllable to the origin.

Example 4.2. Consider the impulsive control system in \mathbb{R}^3

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ x_2 \end{pmatrix} \dot{u}_1 + \begin{pmatrix} 0 \\ 1 \\ -x_1 \end{pmatrix} \dot{u}_2 = f(x) + g_1(x) \dot{u}_1 + g_2(x) \dot{u}_2$$

with control set $U := [0, 1] \times [0, 1]$, target $\mathcal{S} = \{((0, 0, 0), (0, 0))\}$, bound on the total variation $K = 1$ and initial conditions

$$(x_1, x_2, x_3)(0) = (1, 0, 0), \quad (u_1, u_2)(0) = (1, 0).$$

The space-time control in (28) is still optimal and verifies $V(\varphi) = K = 1$, so that $T((1, 0, 0), (1, 0), 1) = 0$. Notice that no feasible controls with total variation less than 1 exist and by the form of the drift one can easily deduce that there are

no feasible space-time controls with $\varphi'_0(s) > 0$ a.e.. Recalling that such controls correspond to regular controls for the original system (see Proposition 1), this proves that $T_{ac}((1, 0, 0), (1, 0), 1) = +\infty$, even if the system without drift is the well known nonholonomic integrator, which is completely controllable to \mathcal{S} .

Observe that $T_{ac}((1, 0, 0), (1, 0), K) = T((1, 0, 0), (1, 0), K) = 0$ for every $K > 1$.

Remark 5. In Definition 4.1 we consider regular controls that satisfy the target constraint exactly. In alternative, one might consider regular controls as approximations of generalized controls and require that they satisfy the endpoint conditions just approximately. Following the latter point of view, one would say that one has a *Laurentiev-type gap* at $(\bar{x}_0, \bar{u}_0, K) \in \mathbb{R}^n \times U \times \mathbb{R}_+$, when

$$T(\bar{x}_0, \bar{u}_0, K) < \lim_{\varepsilon \rightarrow 0^+} \inf_{(u,v) \in \text{AC}(\bar{u}_0, K)} t_\varepsilon(\bar{x}_0, \bar{u}_0, u, v), \quad (29)$$

where $t_\varepsilon(\bar{x}_0, \bar{u}_0, u, v) := \inf\{t \geq 0 : (x[\bar{x}_0, \bar{u}_0, u, v](t), u(t)) \in B_{n+m}(\mathcal{S}, \varepsilon)\}$. By the density result in [26, Prop.4.1], no gap of type (29) may occur.

4.1. No gap sufficient conditions. Let $\text{aff}(U)$ denote the smallest affine set containing U and let $\text{vec}(U)$ be the associated vector space. By basic results in non smooth analysis (see e.g. [36], [2, Sect. 4.2]) one has:

Lemma 4.2. *Let U be a nonempty closed, convex set. Then*

- (i) *its relative interior D (w.r.t. $\text{aff}(U)$) is not empty;*
- (ii) *for any $\bar{u} \in U$, the tangent cone $T_U(\bar{u}) = \cup_{h>0} \frac{U-\bar{u}}{h} \subset \text{vec}(U)$ and, if $\bar{u} \in D$, one has $T_U(\bar{u}) = \text{vec}(U)$;*
- (iii) *as a consequence, any Lipschitz continuous map $\varphi : \mathbb{R}_+ \rightarrow U$ verifies $\varphi'(s) \in \text{vec}(U)$ for almost all $s \geq 0$ and D turns out to be both an invariance and a backward invariance domain for the differential inclusion*

$$\varphi'(s) \in \text{vec}(U) \cap \overline{B_m(0, 1)}. \quad (30)$$

In particular, for any $\bar{u} \in D$ such that $\overline{B(\bar{u}, \varepsilon)} \cap U \subset D$ for some $\varepsilon > 0$, every 1-Lipschitz continuous solution φ of (30) such that $\varphi(0) = \bar{u}$ verifies $\varphi(s) \in D$ for all $s \in [0, \varepsilon]$.

Let us set

$$S_m^+(U) := \{(w_0, w) \in \mathbb{R}_+ \times \text{vec}(U) : w_0 + |w| = 1\} \quad (31)$$

and introduce some controllability notions for the control system

$$\begin{cases} \xi'(s) = f(\xi(s), \varphi(s), \psi(s))w_0(s) + \sum_{i=1}^m g_i(\xi(s), \varphi(s), \psi(s))w_i(s) \\ \varphi'(s) = w(s) \end{cases} \quad (32)$$

with controls $(w_0, w, \psi) \in \mathcal{M}(\mathbb{R}_+, S_m^+(U) \times V)$.

Definition 4.3 (STLC). The control system (32) is said to be small time locally controllable (in short, STLC) to the target set \mathcal{S} if for each $\varepsilon > 0$ there exists $\delta \in]0, \varepsilon]$ such that for any $(\bar{x}, \bar{u}) \in \mathbb{R}^n \times U$ with $d((\bar{x}, \bar{u}), \mathcal{S}) < \delta$ there exists a control $(w_0, w, \psi) \in \mathcal{M}([0, \varepsilon], S_m^+(U) \times V)$ such that the associated solution (ξ, φ) of (32) with $(\xi, \varphi)(0) = (\bar{x}, \bar{u})$, verifies $(\xi(\varepsilon), \varphi(\varepsilon)) \in \mathcal{S}$.

We say that (32) is STLC to \mathcal{S} with non impulsive controls for some $\alpha \in]0, 1[$, if it is STLC to \mathcal{S} and, in addition, the control (w_0, w, ψ) verifies $|w| \leq \alpha$ ⁶. Finally, (32) is null STLC to \mathcal{S} if it is STLC to \mathcal{S} with $w \equiv 0$.

Theorem 4.4. *Let D denote the relative interior of U . Assume (H0) and let $\mathcal{S} \subset \mathbb{R}^n \times D$.*

- (i) *If (32) is null STLC to \mathcal{S} , then $T_{ac} = T$.*
- (ii) *If (32) is STLC to \mathcal{S} with non impulsive controls for some $\alpha \in]0, 1[$, then $T_{ac}(\bar{x}_0, \bar{u}_0, K) = T(\bar{x}_0, \bar{u}_0, K)$ for all $(\bar{x}_0, \bar{u}_0, K) \in \mathbb{R}^n \times U \times \mathbb{R}_+$ such that $K > 0$ and $T(\bar{x}_0, \bar{u}_0, K') = T(\bar{x}_0, \bar{u}_0, K)$ for some $0 \leq K' < K$.* (33)

Proof. Let $(\bar{x}_0, \bar{u}_0, K) \in \mathbb{R}^n \times U \times \mathbb{R}_+$. Clearly, $T(\bar{x}_0, \bar{u}_0, K) \leq T_{ac}(\bar{x}_0, \bar{u}_0, K)$, thus proving that $T(\bar{x}_0, \bar{u}_0, K) = T_{ac}(\bar{x}_0, \bar{u}_0, K)$ is equivalent to showing that, if we suppose

$$T(\bar{x}_0, \bar{u}_0, K) \leq T_{ac}(\bar{x}_0, \bar{u}_0, K) - \eta \quad \text{for some } \eta > 0, \quad (34)$$

we get a contradiction. Without loss of generality, one can assume $T(\bar{x}_0, \bar{u}_0, K) < +\infty$ and $(\bar{x}_0, \bar{u}_0, K) \notin \mathcal{S} \times \mathbb{R}_+$, so that there exists some $\eta/4$ -optimal control $((\varphi_0, \varphi), \psi) \in \Gamma(\bar{u}_0, K) \times \mathcal{M}$ with $S := S_{(\bar{x}_0, \bar{u}_0)}((\varphi_0, \varphi), \psi)$ and associated solution $\xi := \xi[\bar{x}_0, \bar{u}_0, \varphi_0, \varphi, \psi]$, such that $(\xi(S), \varphi(S)) \in \mathcal{S}$ and $T(\bar{x}_0, \bar{u}_0, K) \leq \varphi_0(S) + (\eta/4)$. Therefore (34) yields

$$\varphi_0(S) \leq T_{ac}(\bar{x}_0, \bar{u}_0, K) - \frac{3\eta}{4}. \quad (35)$$

Let us now consider the control sequence $((\varphi_{0_h}, \varphi_h), \psi)$ where, for each h ,

$$\varphi_h(s) = \bar{u}_0 + \frac{h}{h+1} \int_0^s \varphi'(r) dr, \quad \varphi_{0_h}(s) = \int_0^s (1 - |\varphi'_h(r)|) dr \quad \text{for almost all } s \in [0, S] \quad (36)$$

and let $\xi_h := \xi[\bar{x}_0, \bar{u}_0, \varphi_{0_h}, \varphi_h, \psi]$. Notice that

$$\sup_{s \in [0, S]} |(\varphi_{0_h}, \varphi_h)(s) - (\varphi_0, \varphi)(s)| \leq \rho_h \quad \text{for some } \rho_h \rightarrow 0^+ \text{ as } h \rightarrow +\infty$$

and

$$\begin{aligned} \varphi_h(s) &= \frac{1}{h+1} \bar{u}_0 + \frac{h}{h+1} \varphi(s) \in U \quad \forall s \in [0, S], \\ \text{Var}(\varphi_h) &= \text{Var}(\varphi) - \frac{1}{h+1} \int_0^S |\varphi'(s)| ds, \\ \varphi_{0_h}(S) &= \varphi_0(S) + \frac{1}{h+1} \int_0^S |\varphi'(s)| ds. \end{aligned} \quad (37)$$

By the continuity of the input-output map associated to the control system (7) (see [26, Thm. 4.1]) it follows that (for a larger ρ_h , if necessary)

$$\sup_{s \in [0, S]} |\xi_h(s) - \xi(s)| \leq \rho_h.$$

Therefore

$$d((\xi_h(S), \varphi_h(S)), \mathcal{S}) \leq |(\xi_h(S), \varphi_h(S)) - (\xi(S), \varphi(S))| \leq \rho_h. \quad (38)$$

Let us assume that the control system (32) is STLC to \mathcal{S} . Since $\mathcal{S} \subset \mathbb{R}^n \times D$ and has compact boundary, it is possible to choose

$$\varepsilon \in]0, \eta/4[\quad (39)$$

⁶By Proposition 1 one derives that (32) is STLC to \mathcal{S} with non impulsive controls if and only if the control system $(\dot{x}, \dot{u})(t) = (f(x, u, v) + \sum_{i=1}^m g_i(x, u, v) \omega_i, \omega_i)$ is STLC to \mathcal{S} with bounded controls $(\omega, v) \in \text{vec}(U) \times V$, verifying $|\omega| \leq \frac{\alpha}{1-\alpha}$.

verifying

$$B_{n+m}(\mathcal{S}, 3\varepsilon) \cap (\mathbb{R}^n \times U) \subset \mathbb{R}^n \times D. \quad (40)$$

Let $\delta \in]0, \varepsilon[$ be as in Definition 4.3. Then by (38) for h large enough, the value $(\bar{x}_h, \bar{u}_h) := (\xi_h(S), \varphi_h(S))$ satisfies

$$d((\bar{x}_h, \bar{u}_h), \mathcal{S}) < \delta, \quad B_m(\bar{u}_h, 2\varepsilon) \cap U \subset D \quad (41)$$

and by (37) we can also assume that

$$\varphi_{0_h}(S) \leq \varphi_0(S) + \frac{\eta}{4}. \quad (42)$$

CASE (i). Let (32) be null STLC to \mathcal{S} . Then for each h , there exists a control $(1, 0, \tilde{\psi}_h) \in \mathcal{M}([0, \varepsilon], S_m^+(U) \times V)$ such that the solution $(\tilde{\xi}_h, \tilde{\varphi})$ of

$$\begin{cases} \tilde{\xi}_h'(s) = f(\tilde{\xi}_h(s), \tilde{\varphi}_h(s), \tilde{\psi}_h(s)) \\ \tilde{\varphi}_h'(s) = 0 \\ (\tilde{\xi}_h, \tilde{\varphi}_h)(0) = (\bar{x}_h, \bar{u}_h) \end{cases}$$

verifies $(\tilde{\xi}_h(\varepsilon), \tilde{\varphi}_h(\varepsilon)) \in \mathcal{S}$. Hence, setting $\hat{S} := S + \varepsilon$, the control $((\hat{\varphi}_{0_h}, \hat{\varphi}_h), \hat{\psi}_h)$ defined, for any $s \in [0, \hat{S}]$, by

$$\begin{aligned} & ((\hat{\varphi}_{0_h}, \hat{\varphi}_h), \hat{\psi}_h)(s) := \\ & ((\varphi_{0_h}, \varphi_h), \psi)(s)\chi_{[0, S]}(s) + ((\varphi_{0_h}(S) + (s - S), \bar{u}_h), \tilde{\psi}_h(s - S))\chi_{]S, \hat{S}]}(s), \end{aligned}$$

belongs to $\Gamma(\bar{u}_0, K) \times \mathcal{M}$. Now, the associated solution $\hat{\xi}_h := \xi[\bar{x}_0, \bar{u}_0, \hat{\varphi}_{0_h}, \hat{\varphi}_h, \hat{\psi}_h]$ verifies $(\hat{\xi}_h(\hat{S}), \hat{\varphi}_h(\hat{S})) \in \mathcal{S}$ and by (42) and (35), we obtain that

$$\varphi_{0_h}(\hat{S}) = \varphi_{0_h}(S) + \varepsilon \leq \varphi_0(S) + \frac{\eta}{2} \leq T_{ac}(\bar{x}_0, \bar{u}_0, K) - \frac{\eta}{4}. \quad (43)$$

CASE (ii). Let (32) be STLC to \mathcal{S} with non impulsive controls for some $0 < \alpha < 1$ and let $T(\bar{x}_0, \bar{u}_0, K') = T(\bar{x}_0, \bar{u}_0, K)$ for some $0 \leq K' < K$. Assume, as it is not restrictive, that

$$\varepsilon \leq K - K' \quad (44)$$

and, owing to (33), choose the $\eta/4$ -optimal control $((\varphi_0, \varphi), \psi) \in \Gamma(\bar{u}_0, K) \times \mathcal{M}$, S and the associated solution ξ defined as above, verifying (all conditions (36)–(42) and) in addition,

$$Var(\varphi) \leq K' < K,$$

so that, by (37),

$$Var(\varphi_h) = Var(\varphi) - \frac{1}{h+1} \int_0^S |\varphi'(s)| ds \leq K' < K. \quad (45)$$

By the assumptions, for each h , there exists a control $(\tilde{w}_{0_h}, \tilde{w}_h, \tilde{\psi}_h) \in \mathcal{M}([0, \varepsilon], S_m^+(U) \times V)$ with

$$w_{0_h} \geq 1 - \alpha > 0, \quad (46)$$

whose associated solution $(\tilde{\xi}_h, \tilde{\varphi})$ of (32) with $(\tilde{\xi}_h, \tilde{\varphi}_h)(0) = (\bar{x}_h, \bar{u}_h)$, verifies $(\tilde{\xi}_h(\varepsilon), \tilde{\varphi}_h(\varepsilon)) \in \mathcal{S}$. Moreover by (39)–(41) and Lemma 4.2, (iii), it follows that

$$\tilde{\varphi}_h(s) \in D \quad \forall s \in [0, \varepsilon], \quad \int_0^\varepsilon |\tilde{w}_h(s)| ds \leq K - K',$$

so that, by (45), setting $\hat{S} := S + \varepsilon$, the control $((\hat{\varphi}_{0_h}, \hat{\varphi}_h), \hat{\psi}_h)$ defined by

$$\begin{aligned} ((\hat{\varphi}_{0_h}, \hat{\varphi}_h), \hat{\psi}_h)(s) &:= ((\varphi_{0_h}, \varphi_h), \psi)(s)\chi_{[0, S]}(s) + \\ &((\varphi_{0_h}(S) + \int_S^{\hat{S}} \tilde{w}_{0_h}(r - S) dr, \tilde{\varphi}_h(s - S)), \tilde{\psi}_h(s - S))\chi_{]S, \hat{S}]}(s) \quad \forall s \in [0, \hat{S}], \end{aligned}$$

belongs to $\Gamma(\bar{u}_0, K) \times \mathcal{M}$. Now the associated solution $\hat{\xi}_h := \xi[\bar{x}_0, \bar{u}_0, \hat{\varphi}_{0_h}, \hat{\varphi}_h, \hat{\psi}_h]$ verifies $(\hat{\xi}_h(\hat{S}), \hat{\varphi}_h(\hat{S})) \in \mathcal{S}$ and by (39), (42) and (35), we have (43).

Notice that, in both cases (i) and (ii), for each h , $\hat{\varphi}'_{0_h}(s) > 0$ for almost all $s \in [0, \hat{S}]$. Hence by Proposition 1 it follows that $(\hat{\xi}_h, \hat{\varphi}_{0_h}, \hat{\varphi}_h, \hat{\psi}_h)$ is the arc-length graph parametrization of a (admissible) regular trajectory-control pair (x_h, u_h, v_h) for the original system (3), (4), with $(u_h, v_h) \in \text{AC}(\bar{u}_0, K) \times \mathcal{M}$ and exit-time $t(\bar{x}_0, \bar{u}_0, u_h, v_h) \leq \hat{\varphi}_{0_h}(\hat{S})$. Thus (43) implies the contradiction

$$T_{ac}(\bar{x}_0, \bar{u}_0, K) \leq t(\bar{x}_0, \bar{u}_0, u_h, v_h) \leq \hat{\varphi}_{0_h}(\hat{S}) \leq T_{ac}(\bar{x}_0, \bar{u}_0, K) - \frac{\eta}{4},$$

and the proof of the theorem is concluded. \square

There exists in the literature a variety of explicit controllability conditions which imply STLC with non impulsive controls or null STLC of (32) to \mathcal{S} (see e.g. [20] and the references therein for some recent results and an overview on classical conditions). In the next corollary we consider just the simplest, first order conditions. Here $N_{\mathcal{S}}(x, u)$ denotes the limiting normal cone to \mathcal{S} at (x, u) (see the Notation).

Proposition 2. *Assume (H0) and let $\mathcal{S} \subset \mathbb{R}^n \times D$.*

(i) *If $\forall (x, u) \in \partial\mathcal{S}$ one has*

$$\min_{v \in V} p \cdot f(x, u, v) < 0 \quad \forall p \in N_{\mathcal{S}}(x, u) \cap \partial B_n(0, 1), \quad (47)$$

then $T_{ac} = T$.

(ii) *If for every $(x, u) \in \partial\mathcal{S}$ one has*

$$\min_{(w_0, w, v) \in S_m^+(U) \times V} p \cdot (f(x, u, v)w_0 + \sum_{i=1}^m g_i(x, u)w_i) < 0 \quad \forall p \in N_{\mathcal{S}}(x, u) \cap \partial B_n(0, 1), \quad (48)$$

then $T_{ac}(\bar{x}_0, \bar{u}_0, K) = T(\bar{x}_0, \bar{u}_0, K)$ for all $(\bar{x}_0, \bar{u}_0, K) \in \mathbb{R}^n \times U \times \mathbb{R}_+$ such that (33) holds true for some $K' \in [0, K[$.

Proof. Let us prove statement (ii). Since $\partial\mathcal{S}$ is compact, (48) yields that for some $\eta > 0$, $\forall (x, u) \in \partial\mathcal{S}$ and $\forall p \in N_{\mathcal{S}}(x, u) \cap \partial B_n(0, 1)$, one has

$$\min_{(w_0, w, v) \in S_m^+(U) \times V} p \cdot (f(x, u, v)w_0 + \sum_{i=1}^m g_i(x, u)w_i) \leq -\eta. \quad (49)$$

Indeed, if on the contrary there exist two sequences $(x_h, u_h) \subset \partial\mathcal{S}$ and $(p_h) \subset N_{\mathcal{S}}(x_h, u_h) \cap \partial B_n(0, 1)$ such that, for each h ,

$$p_h \cdot (f(x_h, u_h, v)w_0 + \sum_{i=1}^m g_i(x_h, u_h)w_i) \geq -\frac{1}{h} \quad \forall (w_0, w, v) \in S_m^+(U) \times V, \quad (50)$$

then there exist subsequences, not relabeled, verifying $\lim_{h \rightarrow +\infty} (x_h, u_h) = (\bar{x}, \bar{u})$ and $\lim_{h \rightarrow +\infty} p_h = \bar{p}$, for some $(\bar{x}, \bar{u}) \in \mathcal{S}$, $\bar{p} \in \partial B_n(0, 1)$. By [?, Prop. 4.2.6] we derive that the set-valued map $(x, u) \rightsquigarrow N_{\mathcal{S}}(x, u)$ has a closed graph, so that $\bar{p} \in N_{\mathcal{S}}(\bar{x}, \bar{u}) \cap$

$\partial B_n(0, 1)$. Hence taking the limit in (50) we would have a contradiction to (48). Set

$$M := \max\{|f(x, u, v)|, |g_1(x, u, v)|, \dots, |g_m(x, u, v)| : (x, u, v) \in \partial\mathcal{S} \times V\}.$$

By a straightforward calculation (49) implies that, choosing $\alpha \geq 1 - \left(\frac{\eta}{2M(m+1)} \wedge \frac{1}{2}\right)$, one has

$$\min_{(w_0, w, v) \in S_m^+(U) \times V, |w| \leq \alpha} p \cdot (f(x, u, v)w_0 + \sum_{i=1}^m g_i(x, u)w_i) \leq -\frac{\eta}{2}.$$

$\forall (x, u) \in \partial\mathcal{S}$ and $\forall p \in N_{\mathcal{S}}(x, u) \cap \partial B_n(0, 1)$. This is the well known Petrov's condition for a general target, which by [10, Thm. 8.2.3] implies that (32) is STLC to \mathcal{S} with non impulsive controls for α as above. Now statement (ii) follows by Theorem 4.4, (ii).

We omit the proof of statement (i), since it is analogous and actually simpler. \square

Remark 6. A condition similar to (i) of Proposition 2 has been already introduced in [33]. Incidentally, arguing as in [33], it is not difficult to prove that such condition, actually, the null STLC to \mathcal{S} , yields the continuity of T in \mathcal{R} . Finer regularity properties of T with the origin as target can be found in [28].

STLC of (32) to \mathcal{S} is far to be a necessary condition to have $T_{ac} = T$. For instance, T_{ac} always coincides with T in the case of control systems without drift.

Proposition 3. *Assume (H0) and $f \equiv 0$ in the control system (3). Then $T_{ac} = T = 0$ in \mathcal{R} and $T_{ac} = T = +\infty$ otherwise.*

Proof. This result is consequence of the more general fact that, when $f \equiv 0$, given $(\bar{x}_0, \bar{u}_0, K)$, for any pair $((\varphi_0, \varphi), \psi), ((\tilde{\varphi}_0, \varphi), \psi) \in \text{Lip}(\bar{u}_0, K) \times \mathcal{M}$, one has $\xi[\bar{x}_0, \bar{u}_0, \varphi_0, \varphi, \psi] \equiv \xi[\bar{x}_0, \bar{u}_0, \tilde{\varphi}_0, \varphi, \psi]$. Thus, if for any $((\varphi_0, \varphi), \psi) \in \Gamma(\bar{u}_0, K) \times \mathcal{M}$ we set, for each $h \geq 1$, $\varphi_{0_h}(s) := \frac{s}{h}$ for all $s \geq 0$, we obtain a (possibly not feasible) control $((\varphi_{0_h}, \varphi), \psi)$ which, by Proposition 1, is a graph parametrization of a regular trajectory-control pair (x_h, u_h, v_h) given by

$$(x_h, u_h, v_h)(t) := (\xi, \varphi, \psi)(ht) \quad \forall t \geq 0.$$

Now if $(\bar{x}_0, \bar{u}_0, K) \in \mathcal{R}$ and $(\xi(S), \varphi(S)) \in \mathcal{S}$ for some $S > 0$, we get $(x_h, u_h)(S/h) \in \mathcal{S}$. Hence $T_{ac}(\bar{x}_0, \bar{u}_0, K) = T(\bar{x}_0, \bar{u}_0, K) = 0$. \square

5. Boundary value problem. We assume for simplicity the convexity hypothesis (H1), implying that the minimum time function T is lower semicontinuous (in short, l.s.c.), and characterize T as unique l.s.c. solution of a HJB equation, verifying suitable boundary conditions. As usual, if (H1) does not hold all the results in this section remain true replacing T with the value function T^r , obtained taking the infimum in (54) over relaxed trajectories, i.e. trajectories whose velocities evolve in the convex hull of the original velocity set (see e.g. [6, Subsect. 3.9]).

The purpose of the following example is to demonstrate that T (and hence V below) depends explicitly on K . As a consequence, the dynamic programming principle and the associated HJ equation exhibit such a dependence too.

Example 5.1. Consider the unidimensional impulsive control system

$$\dot{x} = x\dot{u} - 1, \quad (x(0), u(0)) = (\bar{x}_0, \bar{u}_0)$$

with control set $U := [-1, 1]$ and target $\mathcal{S} = \{0\} \times [-1, 1]$. Let $(\bar{x}_0, \bar{u}_0, K) \in]0, +\infty[\times [-1, 1] \times \mathbb{R}_+$. One can easily derive that, setting $\bar{S} := K \wedge (\bar{u}_0 + 1)$ and $S := \bar{S} + \bar{x}_0 e^{-\bar{S}}$, the space-time trajectory-control pair

$$\begin{aligned} \xi(s) &= \bar{x}_0 e^{-s} \chi_{[0, \bar{S}]}(s) + (\bar{x}_0 e^{-\bar{S}} - (s - \bar{S})) \chi_{] \bar{S}, S]}(s), \\ (\varphi_0, \varphi)(s) &= (0, \bar{u}_0 - s) \chi_{[0, \bar{S}]}(s) + (s - \bar{S}, \bar{u}_0 - \bar{S}) \chi_{] \bar{S}, S]}(s) \end{aligned}$$

is optimal and the impulsive minimum time function is defined by

$$T(\bar{x}_0, \bar{u}_0, K) = \begin{cases} \bar{x}_0 e^{-K} & \text{if } 0 \leq K \leq \bar{u}_0 + 1, \\ \bar{x}_0 e^{-(\bar{u}_0 + 1)} & \text{if } K > \bar{u}_0 + 1. \end{cases}$$

Along the optimal trajectory this function T verifies the following Dynamic Programming Principle:

$$T(\bar{x}_0, \bar{u}_0, K) = \varphi_0(s) + T(\xi(s), \varphi(s), k(s)) \quad \forall s \in]0, S],$$

where

$$k(s) := K - \int_0^s |\varphi'(r)| dr$$

represents the "variation remaining at s ". Indeed the dependence on the variable k is crucial, since if we suppose that T is constant along an optimal trajectory without taking into account the decrease of the variation, that is,

$$T(\bar{x}_0, \bar{u}_0, K) = \varphi_0(s) + T(\xi(s), \varphi(s), K) \quad \forall s \in]0, S], \quad (51)$$

we get a contradiction. For instance, at any $(\bar{x}_0, \bar{u}_0, K)$ with $\bar{u}_0 > -1$ and $0 < K < (\bar{u}_0 + 1)/2$, choosing $s = \bar{S} = K$, by (51) we would have

$$\bar{x}_0 e^{-K} = T(\bar{x}_0, \bar{u}_0, K) = \varphi_0(K) + T(\xi(K), \varphi(K), K) = T(\bar{x}_0 e^{-K}, \bar{u}_0 - K, K) = \bar{x}_0 e^{-2K}.$$

5.1. Boundary value problem and uniqueness. Let us define the following Kruzkov-type transform of T ,

$$V(\bar{x}_0, \bar{u}_0, K) := 1 - e^{-T(\bar{x}_0, \bar{u}_0, K)} \quad \forall (\bar{x}_0, \bar{u}_0, K) \in \mathbb{R}^n \times U \times \mathbb{R}_+,$$

and the associated Hamiltonian

$$\begin{aligned} H(x, u, \mathcal{V}, p_x, p_u, p_k) &:= \sup_{(w_0, w, v) \in S_m^+(U) \times V} \{w_0 \mathcal{V} - w_0 + \\ &- p_x \cdot (f(x, u, v)w_0 + \sum_{i=1}^m g_i(x, u, v) w_i) - p_u \cdot w + p_k |w|\}. \end{aligned} \quad (52)$$

Notice that the minimization in the definition of H is taken for (w_0, w) in the subset

$$S_m^+(U) = \{(w_0, w) \in \mathbb{R}_+ \times \text{vec}(U) : w_0 + |w| = 1\}.$$

This fact will play a crucial role in the proofs.

In the sequel we characterize the map V instead of the minimum time function T , since V is a (l.s.c.) bounded map satisfying the boundary value problem (53) below on the whole space $\mathbb{R}^n \times U \times \mathbb{R}_+$, while T is unbounded and defined just on the (unknown) set \mathcal{R} (see (21)). Clearly, when V is known, one derives both T and \mathcal{R} by

$$\begin{cases} T(\bar{x}_0, \bar{u}_0, K) = -[\log(1 - V(\bar{x}_0, \bar{u}_0, K))], \\ \mathcal{R} = \{(\bar{x}_0, \bar{u}_0, K) : V(\bar{x}_0, \bar{u}_0, K) < 1\}. \end{cases}$$

Let us introduce a "transversality condition" on the target set \mathcal{S} , involving the relative interior D of U :

(H2) For any $(\bar{x}, \bar{u}) \in \mathcal{S}$ and $\varepsilon > 0$, $\mathcal{S} \cap B_{n+m}((\bar{x}, \bar{u}), \varepsilon) \cap (\mathbb{R}^n \times D) \neq \emptyset$.

For instance, $\mathcal{S} \subset \mathbb{R}^n \times D$ and $\mathcal{S} = \hat{\mathcal{S}} \times U$ for some closed set $\hat{\mathcal{S}} \subset \mathbb{R}^n$ satisfy (H2). In particular, hypothesis (H2) guarantees that, if $(\bar{x}, \bar{u}) \in \mathcal{S}$, then either $\bar{u} \in D$ or $\bar{u} \in U \setminus D$ and there is $(x_h, u_h) \subset \mathcal{S}$ converging to (\bar{x}, \bar{u}) , with $u_h \in D$ for all h .

The main results of this section are Theorems 5.1, 5.2 below, which allow us to characterize V . Their proofs are postponed to Subsection 5.2.

Theorem 5.1 (Boundary Value Problem). *Assume (H0), (H1), (H2). Then the value function V verifies the boundary value problem*

$$\begin{cases} H(x, u, \mathcal{V}, \mathcal{V}_x, \mathcal{V}_u, \mathcal{V}_k) \geq 0 & \text{in } ((\mathbb{R}^n \times U) \setminus \mathcal{S}) \times \mathbb{R}_+ \\ -H(x, u, \mathcal{V}, \mathcal{V}_x, \mathcal{V}_u, \mathcal{V}_k) \geq 0 & \text{in } \mathbb{R}^n \times D \times]0, +\infty[\\ \mathcal{V}(x, u, k) = 0 & \text{in } \mathcal{S} \times \mathbb{R}_+, \\ \mathcal{V}(x, u, k) = \liminf_{(y, \nu, \varkappa) \rightarrow (x, u, k), \nu \in D, \varkappa > 0} \mathcal{V}(y, \nu, \varkappa) & \forall (x, u, k) \in \mathbb{R}^n \times U \times \mathbb{R}_+ \end{cases} \quad (53)$$

in the viscosity sense.

Remark 7. As shown in [27, Thm. 4.2], the minimization in the definition of H could be taken over the subset of $(w_0, w, v) \in S_m^+(U) \times V$ where either $w_0 = 0$ or $w = 0$. Hence in Theorem 5.1 we could replace H with

$$\max \{ \mathcal{H}_1(x, u, \mathcal{V}, p_x), \mathcal{H}_2(x, u, p_x, p_u, p_k) \},$$

where

$$\begin{aligned} \mathcal{H}_1(x, u, \mathcal{V}, p_x) &:= \mathcal{V} + \sup_{v \in V} \{-1 - p_x \cdot f(x, u, v)\}, \\ \mathcal{H}_2(x, u, p_x, p_u, p_k) &:= \\ &\sup_{w \in \text{vec}(U), |w|=1, v \in V} \{-p_x \cdot \sum_{i=1}^m g_i(x, u, v) w_i - p_u \cdot w + p_k\}. \end{aligned}$$

In this way, we obtain a quasi-variational inequality.

Theorem 5.2 (Uniqueness). *Assume (H0), (H1), (H2). Then V is the unique bounded, l.s.c. viscosity solution to (53).*

5.2. Proofs of Theorems 5.1 and 5.2. For any $(\bar{x}_0, \bar{u}_0, K) \in \mathbb{R}^n \times U \times \mathbb{R}_+$, we preliminarily introduce an equivalent formulation of the minimum time problem, as a free-time Mayer problem with bounded controls and subject to state and endpoint constraints:

$$\inf_{S > 0, (w_0, w, \psi) \in \mathcal{M}([0, S], S_m^+(U) \times V)} \varphi_0(S), \quad (54)$$

with $(\xi, \varphi_0, \varphi, k)$ solution of

$$\begin{cases} \xi'(s) = f(\xi(s), \varphi(s), \psi(s)) w_0(s) + \sum_{i=1}^m g_i(\xi(s), \varphi(s), \psi(s)) w_i(s) \\ \varphi_0'(s) = w_0(s) \\ \varphi'(s) = w(s) \\ k'(s) = -|w(s)|, \end{cases} \quad (55)$$

and verifying the initial condition

$$(\xi, \varphi_0, \varphi, k)(0) = (\bar{x}_0, 0, \bar{u}_0, K) \quad (56)$$

and the state and end-point constraints

$$k(s) \geq 0, \quad \varphi(s) \in U \quad \text{for } s \in [0, S], \quad (\xi(S), \varphi(S)) \in \mathcal{S}. \quad (57)$$

Therefore, if we define the set of *admissible controls*

$$\begin{aligned} \Lambda(\bar{x}_0, \bar{u}_0, K) := \{ & (S, w_0, w, \psi) : S > 0, \quad (w_0, w, \psi) \in \mathcal{M}([0, S], S_m^+(U) \times V), \\ & \text{s.t. } (\xi, \varphi_0, \varphi, k) \text{ solution to (55), (56) satisfies (57)} \}, \end{aligned} \quad (58)$$

we get

$$T(\bar{x}_0, \bar{u}_0, K) = \inf_{(w_0, w, \psi) \in \Lambda(\bar{x}_0, \bar{u}_0, K)} \varphi_0(S).$$

We need the following preliminary result.

Lemma 5.3. *Let $U \subset \mathbb{R}^m$ be a closed, convex set, which is not a singleton. Let D denote its relative interior and set $M := U \setminus D$. Then M is closed and nonempty and for any $\bar{u} \in M$ there exists $w \in \text{vec}(U)$ such that $w \notin P_U^M(\bar{u})$ and $|w| = 1$. Moreover, for any compact set Q there exists some $\varepsilon > 0$ such that*

$$\begin{aligned} \forall u \in U \cap B_m(M, \varepsilon) \cap Q \quad \exists w_u \in \text{vec}(U), \quad |w_u| = 1, \quad \text{verifying} \\ \left(\tilde{u} +]0, \varepsilon[B_m(-w_u, \varepsilon) \right) \cap U \cap Q \subset D \quad \forall \tilde{u} \in U \cap B_m(u, \varepsilon) \cap Q. \end{aligned} \quad (59)$$

Proof. Let $\bar{u} \in M$. By Lemma 4.2, D is nonempty. In the sequel, for any $u \in U$, $\text{Int}(T_U(u))$ denotes the relative interior (w.r.t. $\text{vec}(U)$) of the tangent set $T_U(u)$. By [2, Prop. 4.2.3] we derive that $\text{Int}(T_U(u)) = \cup_{h>0} \frac{D-u}{h}$ and the set-valued map $U \ni u \rightsquigarrow \text{Int}(T_U(u))$ has open graph.

Since U is not a singleton, there exists $\hat{u} \in D$, $\hat{u} \neq \bar{u}$, such that $[\hat{u}, \bar{u}[\subset D$. Moreover, if we set

$$h := |\hat{u} - \bar{u}|, \quad \hat{w} := \frac{\hat{u} - \bar{u}}{h} \neq 0,$$

\hat{w} belongs to $\text{Int}(T_U(\bar{u}))$. We claim that $w := -\hat{w}$ is the requested vector in $\text{vec}(U)$ satisfying $w \notin P_U^M(\bar{u})$. Indeed, if we suppose, on the contrary, that $w \in P_U^M(\bar{u})$, there are $h_n \rightarrow 0^+$, $w_n \rightarrow w$ ($w_n \neq 0$), $u_n \rightarrow \bar{u}$ ($u_n \in M$) such

$$u_n + h_n w_n \in U. \quad (60)$$

Notice that (60) yields that $w_n \in \text{vec}(U)$ for each n . Since $-w = \hat{w} \in \text{Int}(T_U(\bar{u}))$, by (iv) above it follows that $-w_n \in \text{Int}(T_U(u_n))$ for n large enough, so that, for some $\tilde{h}_n > 0$, one has

$$u_n - \tilde{h}_n w_n \in D. \quad (61)$$

At this point, the segment $[u_n + h_n w_n, u_n - \tilde{h}_n w_n]$ should be entirely contained in the convex set U . But this is not possible, since (61) implies that the open segment $]u_n + h_n w_n, u_n - \tilde{h}_n w_n[$ lays in D , while $u_n \in M$. This proves that $w \notin P_U^M(\bar{u})$.

As shown in [13, Prop. 3], it follows that for every $u \in M$ there exist $w_u \in \text{vec}(U)$, $|w_u| = 1$ and $\varepsilon_u > 0$ such that

$$\left(\tilde{u} +]0, \varepsilon_u[B_m(-w_u, \varepsilon_u) \right) \cap U \subset D \quad \forall \tilde{u} \in U \cap B_m(u, \varepsilon_u).$$

Then, since the set $U \cap Q$ is compact, there exists some $\varepsilon \in]0, 1[$ such that (59) holds. \square

Proof of Theorem 5.1. Thanks to the formulation (54) of the minimum time problem, the proof that V verifies the first supersolution condition in (5.1) derives in a standard way from the following Dynamic Programming Principle:

(DPP) for any $(\bar{x}_0, \bar{u}_0, K) \in ((\mathbb{R}^n \times U) \setminus \mathcal{S}) \times \mathbb{R}_+$ and $s > 0$, one has

$$V(\bar{x}_0, \bar{u}_0, K) = \inf_{(S, w_0, w, \psi) \in \Lambda(\bar{x}_0, \bar{u}_0, K)} \left\{ \int_0^{s \wedge S} e^{-\varphi_0(r)} w_0(r) dr + e^{-\varphi_0(s)} V(\xi(s), \varphi(s), k(s)) \chi_{[0, S]}(s) \right\}.$$

Moreover, since the boundary datum $h(x, u, k) = 0$ verifies the so-called "compatibility condition":

$$-H(x, u, h, h_x, h_u, h_k) = 0 \quad \text{in } \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R},$$

the value function V satisfies the following Backward Dynamic Programming Principle:

(BDPP) for any $(\bar{x}_0, \bar{u}_0, K) \in \mathbb{R}^n \times D \times]0, +\infty[$, $s \in]-\infty, 0[$, and for every $(w_0, w, \psi) \in \mathcal{M}[-\infty, 0], S_m^+(U) \times V$, one has⁷

$$V(\bar{x}_0, \bar{u}_0, K) \geq e^{-\varphi_0(s)} V(\xi(s), \varphi(s), k(s)) - \int_s^0 e^{-\varphi_0(r)} w_0(r) dr.$$

At this point, the choice of the minimization set in the definition of the Hamiltonian implies that the epigraph of V over $\mathbb{R}^n \times D \times]0, +\infty[$ is a backward invariance domain for the control system⁸

$$\begin{cases} \xi'(s) = f(\xi(s), \varphi(s), \psi(s))w_0(s) + \sum_{i=1}^m g_i(\xi(s), \varphi(s), \psi(s))w_i(s) \\ \varphi'(s) = w(s) \\ k'(s) = -|w(s)|, \\ r'(s) = w_0(s)r(s) - w_0(s) \end{cases} \quad (62)$$

with $(w_0, w, v) \in S_m^+(U) \times V$, arguing as in [13, Prop. 6]. Hence we can derive that V verifies the second supersolution condition in (53) by [13, Thm. 11].

The third relation in (53) is trivially satisfied. The last statement of the theorem is equivalent to show that, for every $(\bar{x}_0, \bar{u}_0, K) \in \mathbb{R}^n \times U \times \mathbb{R}_+$, there exists a sequence $(\bar{x}_{0h}, \bar{u}_{0h}, K_h) \subset \mathbb{R}^n \times D \times]0, +\infty[$ such that $\lim_h (\bar{x}_{0h}, \bar{u}_{0h}, K_h) = (\bar{x}_0, \bar{u}_0, K)$ and

$$\lim_{h \rightarrow +\infty} V(\bar{x}_{0h}, \bar{u}_{0h}, K_h) = V(\bar{x}_0, \bar{u}_0, K). \quad (63)$$

If $(\bar{x}_0, \bar{u}_0, K) \notin \mathcal{R}$, then $V(\bar{x}_0, \bar{u}_0, K) = 1$ and since V is l.s.c., this guarantees that $V = 1$ in some neighborhood of $(\bar{x}_0, \bar{u}_0, K)$ in $\mathbb{R}^n \times U \times \mathbb{R}_+$, so that (63) is clearly true.

Let $(\bar{x}_0, \bar{u}_0, K) \in \mathcal{R} \setminus (\mathcal{S} \times \mathbb{R}_+)$. Since the epigraph of V over $\mathbb{R}^n \times D \times]0, +\infty[$ is a backward invariance domain for (62) and V is l.s.c., it is enough to consider either $\bar{u}_0 \in M$ or $K = 0$. If $\bar{u}_0 \in D$ and $K = 0$, the result follows from the monotonicity of $k \rightarrow V(\cdot, \cdot, k)$. If $\bar{u}_0 \in M$, by Lemma 5.3 we easily derive that there exist $\bar{w} \in \text{vec}(U)$, $\bar{w} \notin P_U^M(\bar{u}_0)$, $|\bar{w}| = 1$ and $\varepsilon > 0$ such that the function φ solution to $\varphi'(s) = \bar{w}$, $\varphi(0) = \bar{u}_0$, belongs to D for all $s \in [-\varepsilon, 0]$. At this point, for $(\bar{w}_0, \bar{w}, \bar{v})$ with $\bar{w}_0 = 0$

⁷Here we consider solutions to (55) also for $s < 0$.

⁸In system (62) we consider also the variable φ_0 to be consistent with the control system (55), even though the function V does not depend on φ_0 , since the original problem is time-invariant.

and $\bar{v} \in V$ arbitrary, let us define the space-time trajectory-control pair $(\xi, \varphi_0, \varphi, k)$, verifying, for almost all $s \in [-\varepsilon, 0]$,

$$\begin{cases} \xi'(s) = \sum_{i=1}^m g_i(\xi(s), \varphi(s), \bar{v}) \bar{w}_i \\ \varphi_0'(s) = 0 \\ \varphi'(s) = \bar{w} \\ k'(s) = -1 \\ (\xi, \varphi_0, \varphi, k)(0) = (\bar{x}_0, 0, \bar{u}_0, K). \end{cases}$$

Let (s_h) be a sequence converging to 0, with $-\varepsilon < s_h < 0$ and introduce for any h the translation $(\xi_h, \varphi_{0h}, \varphi_h, k_h)(s) := (\xi, \varphi_0, \varphi, k)(s + s_h)$. If we set $(\bar{x}_{0h}, \bar{u}_{0h}, K_h) := (\xi_h(0), \varphi_h(0), K - s_h)$, we have $\lim_{h \rightarrow +\infty} (\bar{x}_{0h}, \bar{u}_{0h}, K_h) = (\bar{x}_0, \bar{u}_0, K)$, $\bar{u}_{0h} \in D$, $K_h > 0$ and by the Dynamic Programming Principle we obtain that

$$V(\bar{x}_{0h}, \bar{u}_{0h}, K_h) \leq \int_0^{-s_h} e^{-\varphi_{0h}(r)} \bar{w}_0(r) dr + e^{-\varphi_{0h}(-s_h)} V(\bar{x}_0, \bar{u}_0, K) = V(\bar{x}_0, \bar{u}_0, K).$$

Hence we can conclude that (63) holds, owing to the lower semicontinuity of V . Let $(\bar{x}_0, \bar{u}_0, K) \in \mathcal{S} \times \mathbb{R}_+$. In this case $V(\bar{x}_0, \bar{u}_0, K) = 0$ by definition and the existence of a sequence as in (63), actually, with $V(\bar{x}_{0h}, \bar{u}_{0h}, K_h) = 0$ for all h , is ensured by hypothesis (H2). \square

In order to characterize V as unique solution to (53), for any $(\bar{x}_0, \bar{u}_0, K) \in \mathbb{R}^n \times D \times]0, +\infty[$, let us introduce the following subset of admissible controls

$$\begin{aligned} \Lambda^0(\bar{x}_0, \bar{u}_0, K) &:= \{(S, w_0, w, \psi) \in \Lambda(\bar{x}_0, \bar{u}_0, K) : (\xi, \varphi_0, \varphi, k) \\ &\text{solution to (55), (56) verifies } (\varphi(s), k(s)) \in D \times]0, +\infty[\forall s \in [0, S]\}, \end{aligned} \quad (64)$$

and define

$$T^0(\bar{x}_0, \bar{u}_0, K) := \inf_{(S, w_0, w, \psi) \in \Lambda^0(\bar{x}_0, \bar{u}_0, K)} \varphi_0(S)$$

and

$$V^0(\bar{x}_0, \bar{u}_0, K) := 1 - e^{-T^0(\bar{x}_0, \bar{u}_0, K)}.$$

Let V_*^0 denote the l.s.c. envelope of V^0 , namely,

$$V_*^0(\bar{x}_0, \bar{u}_0, K) := \liminf_{(y, \nu, \varkappa) \rightarrow (\bar{x}_0, \bar{u}_0, K), \nu \in D, \varkappa > 0} V^0(y, \nu, \varkappa) \quad \forall (\bar{x}_0, \bar{u}_0, K) \in \mathbb{R}^n \times U \times \mathbb{R}_+.$$

The proof of Theorem 5.2 is based on the following proposition, which gives an approximation of admissible trajectories $(\xi, \varphi_0, \varphi, k)(\cdot)$ with possibly $\varphi(s) \in M$ or $k(s) = 0$ for some s , by means of *internal* admissible trajectories, that is, admissible trajectories satisfying $(\varphi, k)(s) \in D \times]0, +\infty[$ for all s .

Proposition 4. *Assume (H0), (H2). Given $(\bar{x}_0, \bar{u}_0, K) \in \mathbb{R}^n \times U \times \mathbb{R}_+$, assume that $(S, w_0, w, \psi) \in \Lambda(\bar{x}_0, \bar{u}_0, K)$ and let $(\xi, \varphi_0, \varphi, k)$ be the associated solution to (55), (56). Then there exists a sequence $(\bar{x}_{0h}, \bar{u}_{0h}, K_h) \subset \mathbb{R}^n \times D \times]0, +\infty[$ verifying*

- (i) $\lim_{h \rightarrow +\infty} (\bar{x}_{0h}, \bar{u}_{0h}, K_h) = (\bar{x}_0, \bar{u}_0, K)$;
- (ii) for each h , there is some $(S, w_{0h}, w_h, \psi) \in \Lambda^0(\bar{x}_{0h}, \bar{u}_{0h}, K_h)$ such that the associated solution $(\xi_h, \varphi_{0h}, \varphi_h, k_h)$ of (55) satisfying $(\xi_h, \varphi_{0h}, \varphi_h, k_h)(0) =$

$(\bar{x}_{0_h}, 0, \bar{u}_{0_h}, K_h)$, verifies

$$\lim_{h \rightarrow +\infty} \left[\sup_{s \in [0, S]} |(\xi_h, \varphi_{0_h}, \varphi_h, k_h)(s) - (\xi, \varphi_0, \varphi, k)(s)| \right] = 0.$$

We postpone the proof of Proposition 4 at the end of this section.

As a consequence, we have:

Corollary 2. *Assume (H0), (H1), (H2). Then $V \equiv V_*^0$.*

Proof. The relations $V \leq V_*^0$ everywhere and $V = V^0 \equiv 1$ outside \mathcal{R} , are trivially satisfied. If $(\bar{x}_0, \bar{u}_0, K) \in \mathcal{S} \times \mathbb{R}_+$, we have $V(\bar{x}_0, \bar{u}_0, K) = V_*^0(\bar{x}_0, \bar{u}_0, K) = 0$, since by (H2) there exists a sequence $(\bar{x}_h, \bar{u}_h) \subset \mathcal{S}$ such that $\bar{u}_h \in D$ for each h and $\lim_{h \rightarrow +\infty} (\bar{x}_h, \bar{u}_h) = (\bar{x}_0, \bar{u}_0)$. Therefore for any K one has $V^0(\bar{x}_h, \bar{u}_h, K) = 0$ and

$$V_*^0(\bar{x}_0, \bar{u}_0, K) \leq \liminf_{h \rightarrow +\infty} V^0(\bar{x}_h, \bar{u}_h, K) = 0.$$

If $(\bar{x}_0, \bar{u}_0, K) \in \mathcal{R} \setminus (\mathcal{S} \times \mathbb{R}_+)$, by Theorem 3.1, there exists an optimal space-time control $((\varphi_0, \varphi), \psi) \in \Gamma(\bar{u}_0, K) \times \mathcal{M}$. Hence the input $(S, w_0, w, \psi) := (S_{(\bar{x}_0, \bar{u}_0)}((\varphi_0, \varphi), \psi), \varphi'_0, \varphi', \psi) \in \Lambda(\bar{x}_0, \bar{u}_0, K)$ and the associated solution $(S, \xi, \varphi_0, \varphi, k)$ to (55), (56), verify

$$T(\bar{x}_0, \bar{u}_0, K) = \varphi_0(S) \implies V(\bar{x}_0, \bar{u}_0, K) = 1 - e^{-\varphi_0(S)}.$$

Now Proposition 4 implies that there are a sequence $(\bar{x}_{0_h}, \bar{u}_{0_h}, K_h) \subset \mathbb{R}^n \times D \times]0, +\infty[$ converging to $(\bar{x}_0, \bar{u}_0, K)$ and some admissible controls $(S, w_{0_h}, w_h, \psi) \in \Lambda^0(\bar{x}_{0_h}, \bar{u}_{0_h}, K_h)$, such that the associated solutions $(\xi_h, \varphi_{0_h}, \varphi_h, k_h)$ to (55) satisfying $(\xi_h, \varphi_{0_h}, \varphi_h, k_h)(0) = (\bar{x}_{0_h}, \bar{u}_{0_h}, K_h)$, verify, in particular,

$$\lim_{h \rightarrow +\infty} \varphi_{0_h}(S) = \varphi_0(S).$$

Hence

$$V_*^0(\bar{x}_0, \bar{u}_0, K) \leq \liminf_{h \rightarrow +\infty} V^0(\bar{x}_{0_h}, \bar{u}_{0_h}, K_h) \leq \lim_{h \rightarrow +\infty} (1 - e^{-\varphi_{0_h}(S)}) = 1 - e^{-\varphi_0(S)} = V(\bar{x}_0, \bar{u}_0, K).$$

This concludes the proof of the equality $V = V_*^0$. \square

Proof of Theorem 5.2. We split the proof in two parts: first we show that any bounded, l.s.c. viscosity solution W of (53) verifies $W \geq V$. Then we prove the inequality $W \leq V_*^0$, which, owing to Corollary 2, implies that $W \leq V$.

Step 1. Let $W : \mathbb{R}^n \times U \times \mathbb{R}_+ \rightarrow \mathbb{R}$ be a bounded, l.s.c. function satisfying

$$\begin{cases} H(x, u, W, W_x, W_u, W_k) \geq 0 & \text{in } ((\mathbb{R}^n \times U) \setminus \mathcal{S}) \times \mathbb{R}_+, \\ W(x, u, k) \geq 0 & \text{in } \mathcal{S} \times \mathbb{R}_+ \end{cases} \quad (65)$$

in the viscosity sense. The inequality $W \geq V$ on $\mathbb{R}^n \times U \times \mathbb{R}_+$ follows straightforwardly from [37, Thm. 4.5, (i)]. This shows that the value function V is the minimal bounded, l.s.c. viscosity supersolution of (65).

Step 2. Let $W : \mathbb{R}^n \times U \times \mathbb{R}_+ \rightarrow \mathbb{R}$ be a bounded, l.s.c. function satisfying

$$\begin{cases} -H(x, u, W, W_x, W_u, W_k) \geq 0 & \text{in } \mathbb{R}^n \times D \times]0, +\infty[\\ W(x, u, k) = 0 & \text{in } \mathcal{S} \times \mathbb{R}_+, \\ W(x, u, k) = \liminf_{(y, \nu, \varkappa) \rightarrow (x, u, k), \nu \in D, \varkappa > 0} W(y, \nu, \varkappa) & \forall (x, u, k) \in \mathbb{R}^n \times U \times \mathbb{R}_+ \end{cases} \quad (66)$$

in the viscosity sense. To prove the inequality $W \leq V_*^0$ in $\mathbb{R}^n \times U \times \mathbb{R}_+$, by the lower semicontinuity of W and the last condition in (66) (verified by both W and V_*^0), it suffices to show that

$$W \leq V^0 \quad \text{in } \mathbb{R}^n \times D \times]0, +\infty[.$$

By [13, Thm. 11] we can derive that

$$\text{Epi}(W) := \{(x, u, k, r) : (x, u, k) \in \mathbb{R}^n \times D \times]0, +\infty[, r \geq W(x, u, k)\}$$

is a backward invariance domain for the control system (62). Let $(\bar{x}_0, \bar{u}_0, K) \in \mathbb{R}^n \times D \times]0, +\infty[$ and consider any control $(w_0, w, \psi) \in \mathcal{M}(\mathbb{R}_+, S_m^+(U) \times V)$ such that the associated solution (ξ, φ, k, r) to (62) verifying $(\xi, \varphi, k)(0) = (\bar{x}_0, \bar{u}_0, K)$ and

$$r(S) = W(\xi(S), \varphi(S), k(S)) \quad \text{for an arbitrary } S > 0$$

satisfies $(\varphi(s), k(s)) \in D \times]0, +\infty[$ for all $s \geq 0$. Thus, setting $\varphi_0(s) := \int_0^s w_0(r) dr$ for any $s \geq 0$, we get

$$r(0) = 1 - e^{-\varphi_0(S)} + e^{-\varphi_0(S)} W(\xi(S), \varphi(S), k(S)) \geq W(\bar{x}_0, \bar{u}_0, K) \quad (67)$$

Now, if $(\bar{x}_0, \bar{u}_0, K) \in \mathcal{R}^0 := \{(x, u, k) : T^0(x, u, k) < +\infty\}$, for any $(S, w_0, w, \psi) \in \Lambda^0(\bar{x}_0, \bar{u}_0, K) \neq \emptyset$, one has

$$r(S) = W(\xi(S), \varphi(S), k(S)) = 0$$

and by (67) this implies that

$$W(\bar{x}_0, \bar{u}_0, K) \leq 1 - e^{-\varphi_0(S)}.$$

By the definition of V^0 , this yields that $W \leq V^0$ in \mathcal{R}^0 . If $(\bar{x}_0, \bar{u}_0, K) \notin \mathcal{R}^0$, by the arbitrariness of $S > 0$ and by (11), saying that $\lim_{s \rightarrow +\infty} \varphi_0(s) = +\infty$, (67) implies that

$$W(\bar{x}_0, \bar{u}_0, K) \leq 1 = V^0(\bar{x}_0, \bar{u}_0, K).$$

By Step 1 and Step 2 the proof of the uniqueness result is concluded. \square

Incidentally, the inequality $W \geq V$ proved in Step 1 above could have been alternatively obtained adapting the arguments of [13, Prop. 7]. Moreover, when the set U has nonempty interior, we could derive the inequality $W \leq V^0$ in Step 2, by [37, Thm. 4.5, (iii)].

Proof of Proposition 4. Let $(\bar{x}_0, \bar{u}_0, K) \in \mathbb{R}^n \times U \times \mathbb{R}_+$ verify $\Lambda(\bar{x}_0, \bar{u}_0, K) \neq \emptyset$, fix $(S, w_0, w, \psi) \in \Lambda(\bar{x}_0, \bar{u}_0, K)$ and let $(\xi, \varphi_0, \varphi, k)$ be the associated solution to (55), (56). We note at the outset that, by standard estimates, there exists some $\hat{M} > 0$ such that any solution $(\hat{\xi}, \hat{\varphi}_0, \hat{\varphi}, \hat{k})$ to (55) in $[0, S]$ associated to a control in $\mathcal{M}([0, S], S_m^+(U) \times V)$ and with $(\hat{\xi}, \hat{\varphi}_0, \hat{\varphi}, \hat{k})(S) \in B_{n+m+1}((\xi, \varphi_0, \varphi, k)(S), 1)$, verifies

$$\sup_{s \in [0, S]} |(\hat{\xi}, \hat{\varphi}_0, \hat{\varphi}, \hat{k})(s)| \leq \hat{M}.$$

Choosing $Q := \overline{B_m(0, \bar{M})}$, by Lemma 5.3, we derive that (59) holds for some $\varepsilon \in]0, 1[$. So, if $\hat{u} \in U \cap Q$ and $0 < r < \varepsilon$,

$$|\hat{u} - (\tilde{u} - rw_u)| < r\varepsilon \implies \hat{u} \in D. \quad (68)$$

Since (S, w_0, w, ψ) is admissible, one has

$$(\bar{x}, \bar{u}) := (\xi(S), \varphi(S)) \in \mathcal{S}$$

and by hypothesis (H2) there exists a sequence

$$(\bar{x}_h, \bar{u}_h) \subset \mathcal{S}, \quad \bar{u}_h \in D \text{ for each } h, \quad \lim_{h \rightarrow +\infty} (\bar{x}_h, \bar{u}_h) = (\bar{x}, \bar{u}).$$

Let us choose $\Delta\sigma > 0$ and an integer N such that

$$\Delta\sigma < \frac{\varepsilon}{4}, \quad N\Delta\sigma > S \quad (69)$$

Set

$$C_2 := \frac{4}{\varepsilon}, \quad C_1 := (1 + 2C_2)^{N+2} (> 1), \quad d := \frac{\varepsilon^2}{8} \quad (70)$$

Without loss of generality, let us assume that, for each h ,

$$|\bar{u}_h - \bar{u}| < \frac{d}{C_1} \wedge \frac{\Delta\sigma}{C_2}. \quad (71)$$

Step 1. Let us begin constructing an approximation of φ laying in D through a recursive procedure. The proof of this step is an akin adaptation of the proof of [13, Prop. 9]. For each $h \in \mathbb{N}$, we are going to define a control $w_h : [0, S] \rightarrow \text{vec}(U) \cap \overline{B_m(0, 1)}$ and a corresponding solution $\varphi_h : [0, S] \rightarrow D$, which satisfy:

$$\begin{aligned} \varphi_h(S) &= \bar{u}_h, \quad \varphi_h(s) \in D \quad \forall s \in [0, S]; \\ |\varphi_h(s) - \varphi(s)| &\leq C_1 |\bar{u}_h - \bar{u}| \quad \forall s \in [0, S]; \\ [0, S] &\text{ is divided in subintervals } [s_{i+1}^h, s_i^h], \quad i = 0, \dots, n_h \leq N. \text{ Moreover,} \quad (72) \\ w_h(s) &= w(s + \varepsilon_i^h) \quad \forall s \in [s_{i+1}^h, s_i^h - \varepsilon_i^h], \quad i = 0, \dots, n_h - 1, \\ \text{where } \varepsilon_i^h &= C_2 |\varphi_h(s_i^h) - \varphi(s_i^h)|. \end{aligned}$$

Here, we mean that, determined w_h, φ_h on the interval $[s_i^h, s_0^h]$, we choose a $s_{i+1}^h \leq (s_i^h - \Delta\sigma) \vee 0$ as illustrated below and extend w_h, φ_h to $[s_{i+1}^h, s_i^h]$. The construction is based on the assumption that, at any step, the final condition $s' := s_i^h, u' := \varphi_h(s_i^h)$, satisfies

$$u' \in D, \quad |\varphi(s') - u'| < d, \quad u' \in B_m(M, \varepsilon) \cap Q. \quad (73)$$

At the end of Step 1 we will show that (73) holds true.

Set $\varepsilon' := C_2 |\varphi(s') - u'|$ and define the control $\tilde{w} : [0, s'] \rightarrow \text{vec}(U) \cap \overline{B_m(0, 1)}$ by

$$\tilde{w}(s) := \begin{cases} w' & s \in [s' - \varepsilon', s'], \\ w(s + \varepsilon') & s \in [0, s' - \varepsilon'], \end{cases}$$

where $w' := w_{u'}$ verifies (59). Then the solution $\tilde{\varphi}$ of

$$\begin{cases} \tilde{\varphi}'(s) = \tilde{w}(s) & s \in [0, s'], \\ \tilde{\varphi}(s') = u', \end{cases} \quad (74)$$

satisfies $\tilde{\varphi}(s) \in D$ for all $s \in [s' - \Delta\sigma, s']$. Indeed,

$$\tilde{\varphi}(s) = \begin{cases} u' - (s' - s)w' & s \in [s' - \varepsilon', s'], \\ u' - \varepsilon'w' - \varphi(s') + \varphi(s + \varepsilon') & s \in [s' - \Delta\sigma, s' - \varepsilon'] \end{cases} \quad (75)$$

and by (69), (71) together with (59), it follows that $\tilde{\varphi}(s) \in D$ for all $s \in [s' - \varepsilon', s']$. Moreover, observing that

$$\varepsilon' < C_2 d = \frac{4}{\varepsilon} \frac{\varepsilon^2}{8} = \frac{\varepsilon}{2} < \varepsilon, \quad \Delta\sigma + d < \frac{\varepsilon}{4} + \frac{\varepsilon^2}{8} < \varepsilon,$$

for any $s \in [s' - \Delta\sigma, s' - \varepsilon']$, one has

$$\begin{aligned} |\tilde{\varphi}(s) - (\varphi(s + \varepsilon') - \varepsilon'w')| &= |u' - \varphi(s')| = \frac{\varepsilon'}{C_2} = \frac{\varepsilon'\varepsilon}{4} < \varepsilon'\varepsilon, \\ |\varphi(s + \varepsilon') - u'| &\leq |\varphi(s + \varepsilon') - \varphi(s')| + |\varphi(s') - u'| < \Delta\sigma + d < \varepsilon, \end{aligned}$$

and by (68) one derives that $\varphi_h^-(s) \in D$, as long as $\varphi_h^-(s) \in U$. Let us show that this is actually true for all $s \in [s' - \Delta\sigma, s' - \varepsilon']$. Indeed, let

$$\tilde{s} := \inf\{s \in [0, s'] : \tilde{\varphi}(s) \in D\}.$$

Then $\tilde{\varphi}(\tilde{s}) \in U$ and, if we assume that $\tilde{s} > s' - \Delta\sigma$, this implies that $\tilde{\varphi}(\tilde{\sigma}) \in D$. Therefore, by the invariance theorem we have that there is $\delta > 0$ such that $\tilde{\varphi}(s) \in D$ for $s \in [\tilde{s} - \delta, \tilde{s}]$, in contradiction with the definition of \tilde{s} . Thus $\tilde{s} \leq s' - \Delta\sigma$. Moreover, $\tilde{\varphi}(\tilde{s}) \in M$ if $\tilde{s} > 0$, again by the invariance theorem. So, one can choose $s'' \in [0, s']$ such that

$$s'' \leq s' - \Delta\sigma, \quad \tilde{\varphi}(s) \in D \quad \forall s \in [s'', s'], \quad s'' = 0 \quad \text{or} \quad \tilde{\varphi}(s'') \in B_m(M, \varepsilon).$$

To repeat the same argument in the subsequent step, (73) has to be satisfied for s' and u' replaced by s'' and $\tilde{\varphi}(s'')$, respectively. Then it remains to prove that

$$|\tilde{\varphi}(s'') - \varphi(s'')| < d. \quad (76)$$

Fix $h \in \mathbb{N}$ and let us prove this inequality directly for the pair $(\tilde{w}_h, \tilde{\varphi}_h)$ of (72), constructed as described above. By (71) we have

$$|\tilde{\varphi}_h(s_0^h) - \varphi(s_0^h)| = |\tilde{\varphi}_h(S) - \varphi(S)| = |\bar{u}_h - \bar{u}| < d,$$

and by (75) one obtains that, for any $i = 1, \dots, n_h$,

$$\begin{aligned} |\tilde{\varphi}_h(s_i^h) - \varphi(s_i^h)| &= |\tilde{\varphi}_h(s_{i-1}^h) - \varepsilon_{i-1}^h w_{\tilde{\varphi}_h(s_{i-1}^h)} - \varphi(s_{i-1}^h) + \varphi(s_i^h + \varepsilon_{i-1}^h) - \varphi(s_i^h)| \\ &\leq |\tilde{\varphi}_h(s_{i-1}^h) - \varphi(s_{i-1}^h)| + \varepsilon_{i-1}^h + |\varphi(s_i^h + \varepsilon_{i-1}^h) - \varphi(s_i^h)| \\ &\leq (1 + 2C_2)|\tilde{\varphi}_h(s_{i-1}^h) - \varphi(s_{i-1}^h)| \end{aligned}$$

recalling that φ is 1-Lipschitz continuous and $\varepsilon_{i-1}^h = C_2|\tilde{\varphi}_h(s_{i-1}^h) - \varphi(s_{i-1}^h)|$.

We can now prove by induction the first of the following relation, while the second follows from (70), (71):

$$|\tilde{\varphi}_h(s_i^h) - \varphi(s_i^h)| \leq (1 + 2C_2)^N |\bar{u}_h - \bar{u}| < d. \quad (77)$$

At this point, using again the explicit expression (75) it is easy to conclude the proof by showing that $|\varphi_h(s) - \varphi(s)| \leq C_1|\bar{u}_h - \bar{u}| \quad \forall s \in [0, S]$.

Step 2. Set $\bar{k} := k(S)$ and let (\bar{k}_h) be an arbitrary sequence with $\bar{k}_h > 0$ for each h and $\lim_{h \rightarrow +\infty} \bar{k}_h = \bar{k}$. Fixed $h \in \mathbb{N}$, let (φ_{0_h}, k_h) be the solution of

$$\begin{cases} \varphi'_{0_h}(s) = w_h(s), & \varphi_{0_h}(0) = 0, \\ k'_h(s) = -|w_h(s)|, & k_h(S) = \bar{k}_h, \end{cases}$$

where w_h is the same as in Step 1. For any $s \in [0, S]$,

$$\begin{aligned} |\varphi_{0_h}(s) - \varphi_0(s)| &= \left| \int_0^s (1 - |w_h(\sigma)|) d\sigma - \int_0^s (1 - |w(\sigma)|) d\sigma \right| = \\ &= \left| \sum_{i=0}^{n_h-1} \int_{s_{i+1}^h}^{s_{i+1}^h + \varepsilon_i^h} (1 - |w(\sigma)|) d\sigma \right| \leq \sum_{i=0}^{n_h-1} \varepsilon_i^h \leq N C_2 (1 + 2C_2)^N |\bar{u}_h - \bar{u}| \end{aligned}$$

and the last inequality follows from (77) and the definition of the ε_i^h 's. Hence, in particular, $\lim_{h \rightarrow +\infty} \varphi_{0_h}(S) = \varphi_0(S)$. Similarly, for any $s \in [0, S]$,

$$\begin{aligned} |k_h(s) - k(s)| &= \left| \bar{k}_h + \int_s^S |w_h(\sigma)| d\sigma - \bar{k} - \int_s^S |w(\sigma)| d\sigma \right| \leq \\ &= |\bar{k}_h - \bar{k}| + \sum_{i=0}^{n_h-1} \varepsilon_i^h \leq |\bar{k}_h - \bar{k}| + N C_2 (1 + 2C_2)^N |\bar{u}_h - \bar{u}|. \end{aligned}$$

Moreover, $k_h(s) \geq \bar{k}_h > 0$ for each h and $\lim_{h \rightarrow +\infty} k_h(0) = K$.

Step 3. For any h , let ξ_h denote the solution of

$$\begin{cases} \xi_h'(s) = f(\xi_h(s), \varphi_h(s), \psi(s)) \varphi_{0_h}'(s) + \sum_{i=1}^m g_i(\xi_h(s), \varphi_h(s), \psi(s)) \varphi_{i_h}'(s) & s \in [0, S], \\ \xi_h(S) = \bar{x}_h. \end{cases}$$

Since $\bar{x}_h \rightarrow \bar{x}$ and $\sup_{s \in [0, S]} |(\varphi_{0_h, h}, \varphi_h)(s) - (\varphi_0, \varphi)(s)| \rightarrow 0$ as $h \rightarrow +\infty$, by the continuity of the input-output map

$$(\tilde{x}_0, \tilde{u}_0, \tilde{\varphi}_0, \tilde{\varphi}, \tilde{\psi}) \mapsto \xi[\tilde{x}_0, \tilde{u}_0, \tilde{\varphi}_0, \tilde{\varphi}, \tilde{\psi}]$$

proven in [26, Thm. 4.1], we derive that

$$\lim_{h \rightarrow +\infty} \sup_{s \in [0, S]} |\xi_h(s) - \xi(s)| = 0.$$

At this point we have shown that, choosing $(\bar{x}_{0_h}, \bar{u}_{0_h}, K_h) := (\xi_h, \varphi_h, k_h)(0)$, statement (i) of the proposition holds true. Moreover, $(\xi_h, \varphi_{0_h}, \varphi_h, k_h)$ is the solution of (55) satisfying $(\xi_h, \varphi_{0_h}, \varphi_h, k_h)(0) = (\bar{x}_{0_h}, 0, \bar{u}_{0_h}, K_h)$ associated to the control (S, w_{0_h}, w_h, ψ) with $w_{0_h} := 1 - |w_h|$ a.e.. Therefore (S, w_{0_h}, w_h, ψ) belongs to $\Lambda^0(\bar{x}_{0_h}, \bar{u}_{0_h}, K_h)$ and the proof of Proposition 4 is concluded. \square

Acknowledgments. The author thanks Caterina Sartori for useful discussions and for carefully reading the paper and the anonymous referees for suggestions and bibliographical remarks, which improved the presentation.

REFERENCES

- [1] A. Arutyunov, D. Karamzin and F.L. Pereira, On a generalization of the impulsive control concept: controlling system jumps. *Discrete Contin. Dyn. Syst.*, **29** (2011), no. 2, 403–415.
- [2] J.P. Aubin, and H. Frankowska, *Set-valued analysis*. Reprint of the 1990 edition. Modern Birkhuser Classics. Birkhuser Boston, Inc., Boston, MA, 2009.
- [3] S. Aronna and F. Rampazzo, L^1 limit solutions for control systems. *J. Differential Equations*, **258** (2015), no. 3, 954–979.
- [4] M.S. Aronna, M. Motta and F. Rampazzo, Infimum gaps for limit solutions. *Set-Valued Var. Anal.* **23** (2015), no. 1, 3–22.
- [5] D. Azimov and R. Bishop New trends in astrodynamics and applications: optimal trajectories for space guidance. *Ann. New York Acad. Sci.* **1065** (1), (2005), 189–209. doi: 10.1196/annals.1370.002
- [6] A. Bressan and B. Piccoli, *Introduction to the mathematical theory of control*. AIMS Series on Applied Mathematics, 2. American Institute of Mathematical Sciences (AIMS), Springfield, MO, 2007.
- [7] A. Bressan and F. Rampazzo, On differential systems with vector-valued impulsive controls. *Boll. Un. Mat. Ital. B* (7) 2 (1988), no. 3, 641–656.

- [8] P. Bousquet, C. Mariconda and G. Treu, On the Lavrentiev phenomenon for multiple integral scalar variational problems. *J. Funct. Anal.* **266** (2014), no. 9, 5921–5954.
- [9] A. Catllá, D. Schaeffer, T. Witelski, E. Monson and A. Lin, On spiking models for synaptic activity and impulsive differential equations, *SIAM Rev.* **50** (2008) (3), 553–569.
- [10] P. Cannarsa and C. Sinestrari Semiconcave functions, Hamilton-Jacobi equations, and optimal control. *Progress in Nonlinear Differential Equations and their Applications*, **58**. Birkhuser Boston, Inc., Boston, MA, 2004.
- [11] F. H. Clarke, Y. S. LedyaeV, R. J. Stern and P. R. Wolenski, *Nonsmooth Analysis and Control Theory*, Graduate Texts in Mathematics vol. 178, Springer-Verlag, New York, 1998.
- [12] V. A. Dykhta, Impulse-trajectory extension of degenerated optimal control problems. The Lyapunov functions method and applications, *IMACS Ann. Comput. Appl. Math.*, **8**, Baltzer, Basel, (1990) 10309.
- [13] H. Frankowska and S. Plaskacz, Semicontinuous solutions of Hamilton-Jacobi-Bellman equations with degenerate state constraints, *J. Math. Anal. Appl.* **251** (2000), no. 2, 818–838.
- [14] P. Gajardo, H. Ramirez C. and A. Rapaport, Minimal time sequential batch reactors with bounded and impulse controls for one or more species, *SIAM J. Control Optim.* **47** (2008) (6) 2827–2856.
- [15] M. Guerra, A. Sarychev, Fréchet generalized trajectories and minimizers for variational problems of low coercivity, *J. Dyn. Control Syst.* **21** (2015), no. 3, p. 351–377.
- [16] V. I. Gurman, Optimal processes of singular control. *Automat. Remote Control* **26** (1965) 783–792.
- [17] D.Y. Karamzin; V.A. de Oliveira, F.L. Pereira and G.N. Silva, On the properness of an impulsive control extension of dynamic optimization problems, *ESAIM Control Optim. Calc. Var.* **21** (2015), no. 3, 857–875.
- [18] V.F. Krotov, Global methods in optimal control theory. *Monographs and Textbooks in Pure and Applied Mathematics*, **195**. Marcel Dekker, Inc., New York, 1996.
- [19] K. Kumisch, Z. Rao, Minimal time problem with impulsive controls. *Appl. Math. Optim.* **75** (2017), no. 1, 75–97.
- [20] T.T. Le Thuy and A. Marigonda, Small-time local attainability for a class of control systems with state constraints. *ESAIM Control Optim. Calc. Var.* **23** (2017), no. 3, 1003–1021.
- [21] B. M. Miller, Optimization of dynamical systems with generalized control. (Russian) *Avtomat. i Telemekh.* 1989, no. 6, 23–34; *translation in Automat. Remote Control* **50** (1989), no. 6, part 1, 733–742.
- [22] B. M. Miller, The method of discontinuous time substitution in problems of the optimal control of impulse and discrete-continuous systems. (Russian) *Avtomat. i Telemekh.* 1993, no. 12, 3–32; *translation in Automat. Remote Control* **54** (1993), no. 12, part 1, 1727–1750
- [23] B. M. Miller, The generalized solutions of nonlinear optimization problems with impulse control. *SIAM J. Control Optim.* **34** (1996), no. 4, 1420–1440.
- [24] B. M. Miller and E. Ya. Rubinovich, *Impulsive control in continuous and discrete-continuous systems*. Kluwer Academic/Plenum Publishers, New York, 2003.
- [25] B. M. Miller and E. Ya. Rubinovich, Discontinuous solutions in the optimal control problems and their representation by singular space-time transformations. Translation of *Avtomat. i Telemekh.* 2013, no. 12, 56–103. *Autom. Remote Control* **74** (2013), no. 12, 1969–2006.
- [26] M. Motta and F. Rampazzo, Space-time trajectories of nonlinear systems driven by ordinary and impulsive controls. *Differential Integral Equations* **8** (1995), no. 2, 269–288
- [27] M. Motta and F. Rampazzo, Dynamic programming for nonlinear systems driven by ordinary and impulsive controls. *SIAM J. Control Optim.* **34** (1996), no. 1, 199–225.
- [28] M. Motta and C. Sartori, Minimum time with bounded energy, minimum energy with bounded time. *SIAM J. Control Optim.* **42** (2003), no. 3, 789–809.
- [29] M. Motta and C. Sartori, Semicontinuous viscosity solutions to mixed boundary value problems with degenerate convex Hamiltonians, *Nonlinear Anal.* **49** (2002), no. 7, Ser. A: Theory Methods, 905–927.
- [30] M. Motta and C. Sartori, On asymptotic exit-time control problems lacking coercivity. *ESAIM Control Optim. Calc. Var.* **20**, (2014) no. 4, 957–982.
- [31] M. Motta and C. Sartori, On L^1 limit solutions in impulsive control, *Discrete Contin. Dyn. Syst. Ser. S* **11** (2018), no. 6, 1201–1218.
- [32] M. Motta and C. Sartori, Lack of BV bounds for impulsive control systems, *J. Math. Anal. Appl.* **461** (2018), no. 1, 422–450.

- [33] F. Rampazzo and C. Sartori, The minimum time function with unbounded controls. *J. Math. Systems Estim. Control* **8** (1998) no. 2, 1–34.
- [34] A. Razmadzé, Sur les solutions discontinues dans le calcul des variations. (French) *Math. Ann.* **94** (1925), no. 1, 1–52.
- [35] R. Rishel, An extended Pontryagin principle for control systems whose control laws contain measures, *J. Soc. Indust. Appl. Math. Ser. A Control*, **3** (1965) pp. 191–205.
- [36] R. T. Rockafellar, *Convex Analysis*. Princeton, NJ: Princeton University Press. p. 47, 1997 [1970].
- [37] P. Soravia, Optimality principles and representation formulas for viscosity solutions of Hamilton-Jacobi equations. II. Equations of control problems with state constraints. *Differential Integral Equations* **12**, (1999) no. 2, 275–293.
- [38] J. Warga, Relaxed variational problems, *J. Math. Anal. Appl.*, **4**, (1962) 111–128.
- [39] J. Warga, Variational problems with unbounded controls. *J. Soc. Indust. Appl. Math. Ser. A Control* **3** (1965) 424–438.
- [40] A.J. Zaslavski, Nonoccurrence of the Lavrentiev phenomenon for many optimal control problems. *SIAM J. Control Optim.* **45** (2006) no. 3, 1116–1146.