Enhanced nearby and vanishing cycles in dimension one and Fourier transform

ENHANCED NEARBY AND VANISHING CYCLES IN DIMENSION ONE AND FOURIER TRANSFORM

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Abstract. Enhanced ind-sheaves provide a suitable framework for the irregular Riemann-Hilbert correspondence. In this paper, we give some precisions on nearby and vanishing cycles for enhanced perverse objects in dimension one. As an application, we give a topological proof of the following fact. Let $\mathcal{M}$ be a holonomic algebraic $\mathcal{D}$-module on the affine line, and denote by ${}^{\mathsf{L}}\mathcal{M}$ its Fourier-Laplace transform. For a point a on the affine line, denote by ℓ_a the corresponding linear function on the dual affine line. Then, the vanishing cycles of $\mathcal{M}$ at a are isomorphic to the graded component of degree ℓ_a of the Stokes filtration of ${}^{\mathsf{L}}\mathcal{M}$ at infinity.

Contents

1. Introduction

1.1. Let X be a smooth complex curve. Recall that a perverse sheaf F on X admits a quiver description in terms of its nearby and vanishing cycles $\Psi_a(F)$ and $\Phi_a(F)$, where $a \in X$ ranges over the singularities of F. Let S_aX and S^*_aX be the circles of tangent and cotangent directions at a,

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respectively. Using the canonical and variation maps $\Psi_a(F) \underset{v}{\overset{c}{\rightleftarrows}} \Phi_a(F)$, one may upgrade the vector spaces $\Psi_a(F)$ and $\Phi_a(F)$ to local systems on S_aX and S^*_aX, with monodromies $1 - vc$ and $1 - cv$, respectively. Then, one has

$$\Psi_a(F) \simeq \nu^{\mathsf{sph}}_{\{a\}}(F), \quad \Phi_a(F) \simeq \mu^{\mathsf{sph}}_{\{a\}}(F)[1], \tag{1.1}$$

where $\nu^{\mathsf{sph}}_{\{a\}}$ and $\mu^{\mathsf{sph}}_{\{a\}}$ denote the traces on S_aX and S^*_aX of Sato's specialization $\nu_{\{a\}}$ and microlocalization $\mu_{\{a\}}$ functor, respectively.

1.2. Let X^{rb}_a be the real oriented blow-up of X with center a, and consider the natural embeddings

$$S_aX \xhookrightarrow{i} X^{\mathsf{rb}}_a \xhookleftarrow{j} X \setminus \{a\} \xhookrightarrow{j_a} X. \tag{1.2}$$

Recall that one has

$$\nu^{\mathsf{sph}}_{\{a\}}(F) \simeq i^{-1}\mathrm{R}j_* j_a^{-1}F,$$

where i^{-1}, $\mathrm{R}j_*$ and j_a^{-1} denote the external operations for sheaves.

1.3. Let $\mathcal{M}$ be a (not necessarily regular) holonomic $\mathcal{D}_X$-module, and let $F := \mathcal{DR}(\mathcal{M})$ be its de Rham complex, which is a perverse sheaf. If $\mathcal{M}$ is regular, the classical Riemann-Hilbert correspondence implies that $\mathcal{M}$ can be reconstructed from the quiver description of F. If $\mathcal{M}$ is irregular, a result of Deligne and Malgrange (see [7]) implies that $\mathcal{M}$ can be reconstructed by further considering the so-called Stokes filtration $\Psi^{\preccurlyeq\bullet}_a(F,\mathcal{M})$[1] of $\Psi_a(F)$, indexed by Puiseux germs, defined as follows. Let (a,θ,f) be a Puiseux germ, that is, a holomorphic function f on a small sector around $\theta \in S_aX$, which admits a Puiseux series expansion at a. For (a,θ,g) another germ, the order relation $g \preccurlyeq_\theta f$ means that $\mathrm{Re}(g-f)$ is bounded from above on a small sector around θ. Then, an element u of the stalk $\big(\Psi^{\preccurlyeq f}_a(F,\mathcal{M})\big)_\theta$ is a section of the de Rham complex of $\mathcal{M}$ in a sectorial neighborhood of θ such that $\mathrm{e}^{-f}u$ has tempered growth at a. It turns out that the graded component $\Psi^f_a(F,\mathcal{M})$ is a locally constant sheaf on S_aX.

1.4. In [2] we established an extension of the classical Riemann-Hilbert correspondence to the irregular case, in the framework of enhanced ind-sheaves, which has the advantage of working in any dimension. More precisely, there is a quasi-commutative diagram

$$\begin{array}{ccc} \mathrm{Mod}_{\mathrm{rh}}(\mathcal{D}_X) & \xrightarrow[\sim]{\mathcal{DR}} & \mathrm{Perv}(\mathbb{C}_X) \\ \downarrow \iota & & \downarrow e\,\iota \\ \mathrm{Mod}_{\mathrm{hol}}(\mathcal{D}_X) & \xrightarrow[\sim]{\mathcal{DR}^{\mathrm{E}}} & \mathrm{E\text{-}Perv}(\mathrm{I}\mathbb{C}_X). \end{array} \tag{1.3}$$

[1]The Stokes filtration depends on $\mathcal{M}$, and not only on F.

Here, ι embeds regular holonomic $\mathcal{D}$-modules into holonomic $\mathcal{D}$-modules which are not necessarily regular, $e\,\iota$ embeds perverse sheaves into enhanced ones (see Definition 3.4 (ii)), and $\mathcal{DR}^{\mathrm{E}}$ is an enhancement[2] of the de Rham functor $\mathcal{DR}$.

1.5. In [4, §6.2] we explained how to describe the Stokes filtration of $\Psi_a(F)$ in terms of the enhanced de Rham complex $\mathcal{DR}^{\mathrm{E}}(\mathcal{M})$. Here, using enhanced specialization and microlocalization from [5], and making a more excplicit use of the sheafification functor discussed in [6], we propose a description of the Stokes filtration which sheds some light on the geometry underlying these constructions. We also discuss a tempered version of the vanishing cycles $\Phi_a(F)$ as follows.

1.6. Let $\mathbf{k}$ be a field, and consider the natural embedding $e\,\iota\colon \mathrm{D}^{\mathrm{b}}(\mathbf{k}_X) \xrightarrow{\iota} \mathrm{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_X) \xrightarrow{e} \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_X)$, of sheaves into ind-sheaves into enhanced ind-sheaves. Recall that $e\,\iota$ has a left quasi-inverse sh called sheafification functor. We say that an enhanced ind-sheaf K is of sheaf type if it lies in the essential image of $e\,\iota$.

1.7. Let $K \in \mathrm{E}\text{-}\mathrm{Perv}(\mathrm{I}\mathbf{k}_X)$, and let $a \in X$ be a singularity of K. The nearby and vanishing cycles of K are defined as follows. Consider the bordered analogue of (1.2)

$$S_aX \overset{i}{\hookrightarrow} X_a^{\mathsf{rb}} \overset{j}{\hookleftarrow} (X \setminus \{a\})_\infty \overset{j_a}{\hookrightarrow} X,$$

where $(X \setminus \{a\})_\infty$ denotes the bordered space $(X \setminus \{a\}, X)$. We set

$$\begin{aligned}
\Psi_a(K) &:= \nu^{\mathsf{sph}}_{\{a\}}\big(\mathsf{sh}(K)\big) \\
&\simeq i^{-1}\mathrm{R}j_* j_a^{-1}\mathsf{sh}(K) \\
&\simeq i^{-1}\mathrm{R}j_*\mathsf{sh}\big(\mathrm{E}j_a^{-1}K\big), \\
\Psi_a^{\preccurlyeq 0}(K) &:= i^{-1}\mathsf{sh}\big(\mathrm{E}j_*\mathrm{E}j_a^{-1}K\big), \\
\mathrm{E}\nu^{\mathsf{sph}}_{\{a\}}(K) &:= \mathrm{E}i^{-1}\mathrm{E}j_*\mathrm{E}j_a^{-1}K, \\
\Psi_a^0(K) &:= \mathsf{sh}\big(\mathrm{E}\nu^{\mathsf{sph}}_{\{a\}}(K)\big).
\end{aligned}$$

Here, $\mathrm{E}i^{-1}$, $\mathrm{E}j_*$ and $\mathrm{E}j_a^{-1}$ denote the external operations for enhanced ind-sheaves, and $\mathrm{E}\nu^{\mathsf{sph}}_{\{a\}}$ is the natural enhancement of Sato's specialization. Further, for (a, θ, f) a Puiseux germ, set

$$\begin{aligned}
\Psi_a^{\preccurlyeq f}(K) &:= \Psi_a^{\preccurlyeq 0}\big(K(f)\big), \\
\Psi_a^{f}(K) &:= \Psi_a^{0}\big(K(f)\big),
\end{aligned}$$

[2]As $\mathcal{DR}^{\mathrm{E}}$ is only briefly mentioned in this paper, we do not recall its definition, referring instead to [2].

where $K(f)$ is the twist of K by an enhanced ind-sheaf which encodes the exponential growth e^f (see Definition 4.1). Finally, set

$$\Phi_a(K) := \mathrm{E}\mu^{\mathsf{sph}}_{\{a\}}\big(\mathsf{sh}(K)\big)[1],$$
$$\Phi^0_a(K) := \mathsf{sh}\big(\mathrm{E}\mu^{\mathsf{sph}}_{\{a\}}(K)\big)[1],$$

where $\mathrm{E}\mu_{\{a\}}$ is the natural enhancement of Sato's microlocalization.

As it turns out, $\mathrm{E}\nu_{\{a\}}(K)$ and $\mathrm{E}\mu_{\{a\}}(K)$ are of sheaf type.

1.8. If $\mathbf{k} = \mathbb{C}$, and $K = \mathcal{DR}^{\mathrm{E}}(\mathcal{M})$ is the enhanced de Rham complex of a holonomic algebraic $\mathcal{D}_X$-module $\mathcal{M}$ then, setting $F := \mathcal{DR}(\mathcal{M}) \simeq \mathsf{sh}(K)$, one has by definition

$$\Psi_a(K) \simeq \Psi_a(F), \quad \Phi_a(K) \simeq \Phi_a(F).$$

Moreover, one has

$$\Psi^{\precsim\bullet}_a(K) \simeq \Psi^{\precsim\bullet}_a(F, \mathcal{M}), \quad \Psi^{\bullet}_a(K) \simeq \Psi^{\bullet}_a(F, \mathcal{M}).$$

On the other hand, $\Phi^0_a(K) \simeq \Phi_a(F)$ only if $\mathcal{M}$ is regular.

1.9. Recall that an exponential factor at a of a holonomic $\mathcal{D}_X$-module $\mathcal{M}$ is a Puiseux germ (a, θ, f) where the Stokes filtration $\Psi^{\precsim\bullet}_a(F, \mathcal{M})$ jumps. Assume for simplicity that the exponential factors of $\mathcal{M}$ are unramified, so that f is a germ of meromorphic function with poles at a. Let $N^{>0}_\theta$ be a set of representatives of the exponential factors of $\mathcal{M}$, modulo bounded functions. We can assume that if $f \in N^{>0}_\theta$ is bounded, then $f = 0$.

For f unbounded, let $\mathcal{E}^f$ be the germ of $\mathcal{D}_X$-module at a associated with the meromorphic connection $d + df$. Set $\mathcal{E}^0 = \mathcal{O}_X$. The Hukuara-Levelt-Turrittin decomposition theorem asserts that

$$\mathcal{M}^{\mathrm{formal}}_a \simeq \bigoplus_{f \in N^{>0}_\theta} \big(\mathcal{L}_f \otimes^{\mathsf{D}} \mathcal{E}^{-f}\big)^{\mathrm{formal}}_a.$$

Here, $\mathcal{M}^{\mathrm{formal}}_a$ is the formal $\mathcal{D}$-module at a associated with $\mathcal{M}$, $\otimes^{\mathsf{D}}$ is the inner product for $\mathcal{D}_X$-modules, and $\mathcal{L}_f$ is a regular holonomic $\mathcal{D}_X$-module.

Set $K = \mathcal{DR}^{\mathrm{E}}(\mathcal{M})$ and $L_f = \mathcal{DR}(\mathcal{L}_f)$. Then, we have

$$\mathrm{E}\nu^{\mathsf{sph}}_{\{a\}}\big(K(f)\big) \simeq e\big(\nu^{\mathsf{sph}}_{\{a\}}(L_f)\big), \quad \mathrm{E}\nu_{\{a\}}(K) \simeq e\big(\nu_{\{a\}}(L_0)\big).$$

1.10. We give an application of the above constructions to the study of the Fourier-Laplace transform in dimension one.

Let $\mathbb{V}$ be a one-dimensional complex vector space, and $\mathbb{V}^*$ the dual vector space. Set $\mathbb{V}_\infty := (\mathbb{V}, \mathbb{P})$, where $\mathbb{P} = \mathbb{V} \cup \{\infty\}$ is the projective compactification, and similarly define $\mathbb{V}^*_\infty$ and $\mathbb{P}^*$.

Let $K \in \text{E-Perv}(\mathrm{I}\mathbf{k}_{\mathbb{V}_\infty})$, and set ${}^{\mathbb{L}}K := {}^{\mathsf{L}}K[1]$. (Here, the shift ensures compatibility with the Riemann-Hilbert correspondence.) Assume that ${}^{\mathbb{L}}K$ is an enhanced perverse ind-sheaf on $\mathbb{V}^*_\infty$. For $a \in \mathbb{P}$, let (a, θ, f) be a Puiseux germ on $\mathbb{P}$ such that f is unbounded and *not linear* (modulo bounded functions). Then its Legendre transform (b, η, g) is a Puiseux germ on $\mathbb{P}^*$ of the same kind. The stationary phase formula states that there is an isomorphism

$$\big(\Psi_b^g({}^{\mathbb{L}}K)\big)_\eta \simeq \big(\Psi_a^f(K)\big)_\theta. \tag{1.4}$$

This is a classical result for holonomic $\mathcal{D}$-modules, and we gave a proof for enhanced ind-sheaves in [4].

1.11. Here we consider the case of linear Puiseux germs, excluded from (1.4), which goes as follows. For $a \in \mathbb{V}$, denote by ℓ_a the corresponding *linear* function on $\mathbb{V}^*$. Consider the natural identifications $S_\infty \mathbb{P}^* \simeq S_a^* \mathbb{V}$ and $S_b^* \mathbb{V}^* \simeq S_\infty \mathbb{P}$, for $b \in \mathbb{V}^*$. Then, there are isomorphisms

$$\Psi_\infty^{\ell_a}({}^{\mathbb{L}}K) \simeq \Phi_a^0(K), \quad \Phi_b^0({}^{\mathbb{L}}K) \simeq r^{-1}\Psi_\infty^{-\ell_b}(K), \tag{1.5}$$

where r is the antipodal map.

Our proof of (1.5) proceeds as follows. The second isomorphism is obtained from the first one by interchanging $\mathbb{V}$ and $\mathbb{V}^*$, and replacing K by ${}^{\mathbb{L}}K$. After translation from a to 0, the first isomorphism reads

$$\Psi_\infty^0({}^{\mathbb{L}}K) \simeq \Phi_0^0(K).$$

By definition, this is implied by the isomorphism

$$\mathrm{E}\nu_{\{\infty\}}^{\mathsf{sph}}({}^{\mathbb{L}}K) \simeq \mathrm{E}\mu_{\{0\}}^{\mathsf{sph}}(K).$$

We prove the above isomorphism using the so-called smash functor of [1, §6], in its enhanced version from [5, §6].

1.12. Concerning related literature, the $\mathcal{D}$-module counterpart of (1.5) is proved in [12] when $\mathbf{k} = \mathbb{C}$ and $K = \mathcal{DR}^{\mathrm{E}}(\mathcal{M})$ is the enhanced de Rham complex of a holonomic algebraic $\mathcal{D}_{\mathbb{V}}$-module $\mathcal{M}$ which is regular at finite distance, and has only linear exponential factors at infinity. Note that, in this case, ${}^{\mathsf{L}}\mathcal{M}$ satisfies the same conditions.

In the framework of enhanced ind-sheaves, a proof of (1.5) is given in [1], in the case where $K = e(F)$, for F a perverse sheaf[3] on $\mathbb{V}_\infty$.

See [14] for a recent thorough treatment of the Fourier-Laplace transform of holonomic algebraic $\mathcal{D}$-modules on the affine line.

1.13. The contents of this paper are as follows.

After recalling some notations in Section 2, we recall in Section 3 the notion of perverse enhanced ind-sheaf on a complex analytic curve. For such a perverse object, we show that its specialization and microlocalization are perverse sheaves in the classical sense.

In Section 4 we discuss nearby and vanishing cycles along the lines presented in §1.7 above.

[3]For $F = \mathcal{DR}(\mathcal{M})$, this means that $\mathcal{M}$ is regular everywhere, including at infinity.

In Section 5 we apply our constructions to the Fourier-Laplace transform in dimension one. In particular, we give a proof of (1.5).

Finally, we present in the Appendix an alternative description of vanishing cycles in terms of blow-up transforms. In this setting, both nearby and vanishing cycles are realized on the circle of normal directions S_aX.

2. Review on enhanced ind-sheaves

We recall here some notions and results, mainly to fix notations, referring to the literature for details. In particular, we refer to [9] for sheaves, to [15] (see also [8, 3]) for enhanced sheaves, to [10] for ind-sheaves, to [2] (see also [11, 3, 6]) for bordered spaces and enhanced ind-sheaves, and to [5] for enhanced specialization and microlocalization.

In this paper, $\mathbf{k}$ denotes a base field.

2.1. Ind-sheaves and bordered spaces.

A good space is a topological space which is Hausdorff, locally compact, countable at infinity, and with finite soft dimension. Let M be a good space.

Denote by $\mathrm{Mod}(\mathbf{k}_M)$ the category of sheaves of $\mathbf{k}$-vector spaces on M, and by $\mathrm{D^b}(\mathbf{k}_M)$ its bounded derived category. For $f\colon M \to N$ a morphism of good spaces, denote by $\otimes$, f^{-1}, $\mathrm{R}f_!$ and $\mathrm{R}\mathcal{H}om$, $\mathrm{R}f_*$, $f^!$ the six operations.

For $S \subset M$ locally closed, we denote by $\mathbf{k}_S$ the extension by zero to M of the constant sheaf on S with stalk $\mathbf{k}$.

A bordered space is a pair $\mathsf{M} = (M, C)$ with M an open subset of a good space C. We set $\mathring{\mathsf{M}} := M$ and $\check{\mathsf{M}} := C$. A morphism $f\colon \mathsf{M} \to \mathsf{N}$ of bordered spaces is a morphism $\mathring{f}\colon \mathring{\mathsf{M}} \to \mathring{\mathsf{N}}$ of good spaces such that the projection $\overline{\Gamma}_{\mathring{f}} \to \check{\mathsf{M}}$ is proper. Here, $\overline{\Gamma}_{\mathring{f}}$ denotes the closure in $\check{\mathsf{M}} \times \check{\mathsf{N}}$ of the graph $\Gamma_{\mathring{f}}$ of $\mathring{f}$. The morphism f is called semi-proper if the projection $\overline{\Gamma}_{\mathring{f}} \to \check{\mathsf{N}}$ is proper.

By definition, a subset Z of M is a subset of $\mathring{\mathsf{M}}$. For $Z \subset \mathsf{M}$ locally closed, we set $Z_\infty = (Z, \overline{Z})$ where $\overline{Z}$ is the closure of Z in $\check{\mathsf{M}}$.

We denote by $\mathrm{D^b}(\mathrm{I}\mathbf{k}_{\mathsf{M}})$ the bounded derived category of ind-sheaves of $\mathbf{k}$-vector spaces on M, and by $\otimes$, f^{-1}, $\mathrm{R}f_{!!}$ and $\mathrm{R}\mathcal{I}hom$, $\mathrm{R}f_*$, $f^!$ the six operations.

We denote by $\iota_{\mathsf{M}}\colon \mathrm{D^b}(\mathbf{k}_{\mathring{\mathsf{M}}}) \to \mathrm{D^b}(\mathrm{I}\mathbf{k}_{\mathsf{M}})$ the natural embedding, by α_{M} its left adjoint, and we set $\mathrm{R}\mathcal{H}om := \alpha_{\mathsf{M}}\mathrm{R}\mathcal{I}hom$. For $F \in \mathrm{D^b}(\mathbf{k}_{\mathring{\mathsf{M}}})$, we often write simply F instead of $\iota_{\mathsf{M}}F$ in order to make notations less heavy.

Recall that ι commutes with $\mathrm{R}\mathring{f}_*$, $\mathring{f}^{-1}$ and $\mathring{f}^!$, but it does not commute in general with $\mathrm{R}\mathring{f}_{!!}$. If f is semi-proper, then

$$\iota_{\mathsf{N}}\,\mathrm{R}\mathring{f}_!\simeq \mathrm{R}f_{!!}\,\iota_{\mathsf{M}}. \tag{2.1}$$

2.2. **Enhanced ind-sheaves.** Denote by $t \in \mathbb{R}$ the coordinate on the affine line, consider the two-point compactification $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$, and set $\mathbb{R}_\infty := (\mathbb{R}, \overline{\mathbb{R}})$. For M a bordered space, consider the projection

$$\pi_{\mathsf{M}} \colon \mathsf{M} \times \mathbb{R}_\infty \to \mathsf{M}.$$

Denote by $\mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathsf{M}}) := \mathrm{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathsf{M}\times\mathbb{R}_\infty})/\pi_{\mathsf{M}}^{-1}\mathrm{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathsf{M}})$ the bounded derived category of enhanced ind-sheaves of $\mathbf{k}$-vector spaces on M. Denote by $\mathrm{Q}_{\mathsf{M}} \colon \mathrm{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathsf{M}\times\mathbb{R}_\infty}) \to \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathsf{M}})$ the quotient functor. Recall that there is a natural splitting $\mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathsf{M}}) \simeq \mathrm{E}^{\mathrm{b}}_+(\mathrm{I}\mathbf{k}_{\mathsf{M}}) \oplus \mathrm{E}^{\mathrm{b}}_-(\mathrm{I}\mathbf{k}_{\mathsf{M}})$.

For $f \colon \mathsf{M} \to \mathsf{N}$ a morphism of bordered spaces, denote by $\overset{+}{\otimes}$, $\mathrm{E}f^{-1}$, $\mathrm{E}f_{!!}$ and $\mathrm{R}\mathcal{I}hom^+$, $\mathrm{E}f_*$, $\mathrm{E}f^!$ the six operations. Denote by $\mathrm{D}^{\mathrm{E}}_{\mathsf{M}}$ the Verdier dual. Denote by $\mathrm{R}\mathcal{H}om^{\mathrm{E}}$ the hom functor taking values in $\mathrm{D}^{\mathrm{b}}(\mathbf{k}_{\mathring{\mathsf{M}}})$.

One sets

$$\mathbf{k}^{\mathrm{E}}_{\mathsf{M}} := \mathrm{Q}_{\mathsf{M}}\big(\text{“}\varinjlim_{c\to+\infty}\text{”}\ \mathbf{k}_{\{t\geqslant c\}}\big),$$

writing for short $\{t \geqslant c\} = \{(x,t) \in \mathring{\mathsf{M}} \times \mathbb{R};\, t \geqslant c\}$.

There are embeddings

$$\epsilon^+_{\mathsf{M}} \colon \mathrm{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathsf{M}}) \rightarrowtail \mathrm{E}^{\mathrm{b}}_+(\mathrm{I}\mathbf{k}_{\mathsf{M}}), \qquad F \mapsto \mathrm{Q}_{\mathsf{M}}\mathbf{k}_{\{t\geqslant 0\}} \otimes \pi_{\mathsf{M}}^{-1}F,$$

$$e_{\mathsf{M}} \colon \mathrm{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathsf{M}}) \rightarrowtail \mathrm{E}^{\mathrm{b}}_+(\mathrm{I}\mathbf{k}_{\mathsf{M}}), \qquad F \mapsto \mathbf{k}^{\mathrm{E}}_{\mathsf{M}} \otimes \pi_{\mathsf{M}}^{-1}F \simeq \mathbf{k}^{\mathrm{E}}_{\mathsf{M}} \overset{+}{\otimes} \epsilon^+_{\mathsf{M}}(F).$$

Recall that e commutes with $\mathrm{R}f_{!!}$, f^{-1} and $f^!$, but it does not commute in general with $\mathrm{R}f_*$.

The functor e_{M} has as a left quasi-inverse the sheafification functor

$$\mathsf{sh}_{\mathsf{M}} \colon \mathrm{E}^{\mathrm{b}}_+(\mathrm{I}\mathbf{k}_{\mathsf{M}}) \to \mathrm{D}^{\mathrm{b}}(\mathbf{k}_{\mathring{\mathsf{M}}}), \quad K \mapsto \mathrm{R}\mathcal{H}om^{\mathrm{E}}(\mathbf{k}^{\mathrm{E}}_{\mathsf{M}}, K).$$

We call $\mathsf{sh}_{\mathsf{M}}(K)$ the sheaf associated with K. We say that $K \in \mathrm{E}^{\mathrm{b}}_+(\mathrm{I}\mathbf{k}_{\mathsf{M}})$ is of sheaf type if it is in the essential image of e_{M}. This is a local property[4] on M. The full subcategory of $\mathrm{E}^{\mathrm{b}}_+(\mathrm{I}\mathbf{k}_{\mathsf{M}})$ consisting of objects of sheaf type is closed by extensions, and equivalent to $\mathrm{D}^{\mathrm{b}}(\mathbf{k}_{\mathring{\mathsf{M}}})$.

For $U \subset \mathsf{M}$ an open subset, and $\varphi \colon U_\infty \to \mathbb{R}_\infty$ a morphism of bordered spaces, we set

$$\mathsf{E}^{\varphi}_{U|\mathsf{M}} := \mathrm{Q}_{\mathsf{M}}\mathbf{k}_{\{t+\varphi(x)\geqslant 0\}}, \quad \mathbb{E}^{\varphi}_{U|\mathsf{M}} := \mathbf{k}^{\mathrm{E}}_{\mathsf{M}} \overset{+}{\otimes} \mathsf{E}^{\varphi}_{U|\mathsf{M}}, \tag{2.2}$$

writing for short $\{t + \varphi(x) \geqslant 0\} = \{(x,t) \in U \times \mathbb{R};\, t + \varphi(x) \geqslant 0\}$.

2.3. **Specialization and microlocalization.** Let N be a smooth manifold, $V \to N$ an $\mathbb{R}$-vector bundle, and $\mathbb{S}V$ its fiberwise sphere compactification given by $\mathbb{S}V := \big((\mathbb{R} \times V) \setminus (\{0\} \times N)\big)/\mathbb{R}^{\times}_{>0}$. Set $V_\infty := (V, \mathbb{S}V)$. Let $V^* \to N$ be the dual bundle.

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[4] A property is local on M if any $x \in \check{\mathsf{M}}$ has an open neighborhood $V \subset \check{\mathsf{M}}$ such that the property holds on $(V \cap \mathring{\mathsf{M}})_\infty$

The enhanced Fourier-Sato transforms

$$\begin{aligned}(*)^\wedge\colon \mathrm{E}^{\mathrm{b}}_+(\mathrm{I}\mathbf{k}_{V_\infty}) &\to \mathrm{E}^{\mathrm{b}}_+(\mathrm{I}\mathbf{k}_{V^*_\infty}) \cap \mathrm{E}^{\mathrm{b}}_{(\mathbb{R}^\times_{>0})_\infty}(\mathrm{I}\mathbf{k}_{V^*_\infty}),\\ {}^{\mathsf{L}}(*)\colon \mathrm{E}^{\mathrm{b}}_+(\mathrm{I}\mathbf{k}_{V_\infty}) &\to \mathrm{E}^{\mathrm{b}}_+(\mathrm{I}\mathbf{k}_{V^*_\infty}),\end{aligned}$$

are the integral transforms with kernel, respectively,

$$\mathsf{F} := \epsilon^+_{(V\times V^*)_\infty}\mathbf{k}_{\{\langle v,w\rangle\leqslant 0\}}, \quad \mathsf{L} := \mathsf{E}^{-\langle v,w\rangle}_{V\times V^*|(V\times V^*)_\infty}$$

Here, $\mathrm{E}^{\mathrm{b}}_{(\mathbb{R}^\times_{>0})_\infty}(\mathrm{I}\mathbf{k}_{V^*_\infty})$ is the full triangulated subcategory of $\mathrm{E}^{\mathrm{b}}_+(\mathrm{I}\mathbf{k}_{V^*_\infty})$ whose objects are conic for the natural action of the group object $(\mathbb{R}^\times_{>0})_\infty$. Note that ${}^{\mathsf{L}}(*)$ sends conic objects to conic objects.

Let M be a smooth manifold, $N \subset M$ a submanifold, and denote by

$$T_N M \xrightarrow{\ \tau\ } N \xleftarrow{\ \varpi\ } T^*_M N$$

the normal and conormal bundles. Consider the normal deformation $p_{\mathsf{nd}}\colon M^{\mathsf{nd}}_N \to M$ with center N, and the associated commutative diagram of bordered spaces

$$\begin{array}{ccccccc} (T_N M)_\infty & \xhookrightarrow{\ i_{\mathsf{nd}}\ } & (M^{\mathsf{nd}}_N)_\infty & \xrightarrow{\ s_{\mathsf{nd}}\ } & \mathbb{R}_\infty & & \\ \downarrow{\scriptstyle\tau} & \square & \downarrow{\scriptstyle p_{\mathsf{nd}}} & \nwarrow{\scriptstyle j_{\mathsf{nd}}} & \square & \nwarrow & \\ N & \xhookrightarrow[\ i_N\ ]{} & M & \xleftarrow[\ p_\Omega\ ]{} & \Omega_\infty & \longrightarrow & (\mathbb{R}_{>0})_\infty \end{array}$$

where $(M^{\mathsf{nd}}_N)_\infty$ is the bordered compactification of p_{nd}, and $\Omega := s^{-1}_{\mathsf{nd}}(\mathbb{R}_{>0})$. Sato's specialization and microlocalization functors have natural enhancements

$$\begin{aligned}\mathrm{E}\nu_N\colon \mathrm{E}^{\mathrm{b}}_+(\mathrm{I}\mathbf{k}_N) &\to \mathrm{E}^{\mathrm{b}}_{(\mathbb{R}^\times_{>0})_\infty}(\mathrm{I}\mathbf{k}_{(T_N M)_\infty}),\\ \mathrm{E}\mu_N\colon \mathrm{E}^{\mathrm{b}}_+(\mathrm{I}\mathbf{k}_N) &\to \mathrm{E}^{\mathrm{b}}_{(\mathbb{R}^\times_{>0})_\infty}(\mathrm{I}\mathbf{k}_{(T_N M)_\infty}),\end{aligned}$$

defined by

$$\begin{aligned}\mathrm{E}\nu_N(K) &:= \mathrm{E}i^{-1}_{\mathsf{nd}}\mathrm{E}j_{\mathsf{nd}*}\mathrm{E}p^{-1}_\Omega K,\\ \mathrm{E}\mu_N(K) &:= {}^{\mathsf{L}}\mathrm{E}\nu_N(K) \simeq \mathrm{E}\nu_N(K)^\wedge.\end{aligned}$$

Denoting by $\dot{T}_N M$ the complement of the zero-section, and setting $S_N M := \dot{T}_N M/\mathbb{R}^\times_{>0}$, consider the natural morphisms

$$(T_N M)_\infty \xleftarrow{\ u\ } (\dot{T}_N M)_\infty \xrightarrow{\ \gamma\ } S_N M\ .$$

We set

$$\mathrm{E}\nu^{\mathsf{sph}}_N := \mathrm{E}\gamma_*\mathrm{E}u^{-1}\mathrm{E}\nu_N,$$

so that $\mathrm{E}u^{-1}\mathrm{E}\nu_N \simeq \mathrm{E}\gamma^{-1}\mathrm{E}\nu^{\mathsf{sph}}_N$. We similarly define $\mathrm{E}\mu^{\mathsf{sph}}_N$.

Consider the real oriented blowup $p_{\mathsf{rb}}\colon M^{\mathsf{rb}}_N \to M$ with center N, and the associated commutative diagram of bordered spaces

$$\begin{array}{ccccc} S_N M & \xhookrightarrow{i_{\mathsf{rb}}} & M^{\mathsf{rb}}_N & \xhookleftarrow{j_{\mathsf{rb}}} & (M \setminus N)_\infty \\ {\scriptstyle\sigma}\downarrow & \square & {\scriptstyle p_{\mathsf{rb}}}\downarrow & \swarrow {\scriptstyle j_N} & \\ N & \xhookrightarrow[i_N]{} & M. & & \end{array} \tag{2.3}$$

One has an associated functor

$$\mathrm{E}\nu^{\mathsf{rb}}_N\colon \mathrm{E}^{\mathrm{b}}_+(\mathrm{I}\mathbf{k}_N) \to \mathrm{E}^{\mathrm{b}}_+(\mathrm{I}\mathbf{k}_{S_N M}), \quad K \mapsto \mathrm{E}i^{-1}_{\mathsf{rb}}\mathrm{E}j_{\mathsf{rb}*}\mathrm{E}j^{-1}_N K.$$

Note that one has

$$\mathrm{E}\nu^{\mathsf{sph}}_N \simeq \mathrm{E}\nu^{\mathsf{rb}}_N. \tag{2.4}$$

2.4. **Constructibility.** Let M be a subanalytic bordered space.

We denote by $\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathbf{k}_{\mathsf{M}})$ the full triangulated subcategory of $\mathrm{D}^{\mathrm{b}}(\mathbf{k}_{\mathring{\mathsf{M}}})$ whose objects F are such that $\mathrm{R}j_{\mathsf{M}!}F$ is $\mathbb{R}$-constructible in $\check{\mathsf{M}}$. Here, $j_{\mathsf{M}}\colon \mathsf{M} \hookrightarrow \check{\mathsf{M}}$ is the natural morphism.

We denote by $\mathrm{E}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathrm{I}\mathbf{k}_{\mathsf{M}})$ the full triangulated subcategory of $\mathrm{E}^{\mathrm{b}}_+(\mathrm{I}\mathbf{k}_{\mathsf{M}})$ whose objects K satisfy the following property. For any open relatively compact subanalytic subset $U \subset \mathsf{M}$ there exists $F \in \mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathbf{k}_{\mathsf{M}\times\mathbb{R}_\infty})$ such that $\pi^{-1}\mathbf{k}_U \otimes K \simeq \mathbf{k}^{\mathrm{E}}_{\mathsf{M}} \overset{+}{\otimes} \mathrm{Q}_{\mathsf{M}} F$.

3. Enhanced perverse ind-sheaves on a curve

In this section we let X be a smooth complex curve.

3.1. **Normal form.** Consider the real oriented blow-up X^{rb}_a of X with center $a \in X$ as in (2.3), and the associated natural morphisms

$$S_a X \xhookrightarrow{i} X^{\mathsf{rb}}_a \xhookleftarrow{j} (X \setminus \{a\})_\infty \xhookrightarrow{j_a} X\,, \tag{3.1}$$

where we write for short $i = i_{\mathsf{rb}}$, $j = j_{\mathsf{rb}}$, and $j_a = j_{\{a\}}$.

A *sectorial neighborhood* of $\theta \in S_a X$ is an open subset $U \subset X \setminus \{a\}$ such that $S_a X \cup j(U)$ is a neighborhood of θ in X^{rb}_a. We write $U \dot{\ni} \theta$ to indicate that U is a sectorial neighborhood of θ. We say that $U \subset X \setminus \{a\}$ is a sectorial neighborhood of $Z \subset S_a X$, and we write $U \dot{\supset} Z$, if U is a sectorial neighborhood of each $\theta \in Z$.

The sheaf $\mathcal{P}_{S_a X}$ of *Puiseux germs* on $S_a X$ is the subsheaf of $i^{-1}j_*j_a^{-1}\mathcal{O}_X$ whose stalk at $\theta \in S_a M$ are holomorphic functions on small sectors $V \dot{\ni} \theta$ admitting a Puiseux expansion at a. We denote by $\overline{\mathcal{P}}_{S_a X}$ the quotient of $\mathcal{P}_{S_a X}$ modulo bounded functions, and we denote by $[f] \in \overline{\mathcal{P}}_{S_a X}$ the equivalence class of $f \in \mathcal{P}_{S_a X}$.

For $f \neq 0$, we set $\mathrm{ord}_a(f) = -n_0/m$ if f has a Puiseux expansion $\sum_{n\geqslant n_0} c_n z_a^{n/m}$ with $c_{n_0} \neq 0$, where $n, n_0 \in \mathbb{Z}$, $m \in \mathbb{Z}_{>0}$, and z_a is a local coordinate at a with $z_a(a) = 0$. We set $\mathrm{ord}_a(0) = -\infty$. Note that f is bounded if and only if $\mathrm{ord}_a(f) \leqslant 0$cmtm It is false. In this context f is a section of $\mathcal{P}_{S_aX}$. Hence for example, $-1/z$ is bounded on a neighborhood of $0+$.

Definition 3.1. One says that $K \in \mathrm{E}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathrm{I}\mathbf{k}_X)$ has *normal form* at $\theta \in S_aX$ if there exist a finite subset $\Phi_\theta \subset \mathcal{P}_{S_aX,\theta}$ and integers $n_\theta(f) \in \mathbb{Z}_{>0}$ for $f \in \Phi_\theta$ such that

$$\pi^{-1}\mathbf{k}_{V_\theta} \otimes K \simeq \bigoplus_{f\in\Phi_\theta} \big(\mathbb{E}^{\mathrm{Re}\, f}_{V_\theta|X}\big)^{n_\theta(f)}[1] \tag{3.2}$$

for some $V_\theta \mathrel{\dot\ni} \theta$. (Recall that $\mathbb{E}^{\mathrm{Re}\, f}_{V_\theta|X}$ was defined in (2.2).) One says that K has normal form at $I \subset S_aX$ if it has normal form at any $\theta \in I$. One says that K has normal form at a if it has normal form at S_aX.

If K has normal form at a connected open subset $I \subset S_aX$, and $f \in \mathcal{P}_{S_aX}(I)$, the number

$$\overline{N}([f]) = \sum_{h\in\Phi_\theta,\ [f]=[h]} n_\theta(h)$$

is finite, does not depend on the choice of $\theta \in I$, and only depends on the class $[f]$ of f. If $\overline{N}([f]) > 0$, one says that $[f]$ is an *exponential factor* of K, and $\overline{N}([f])$ is called its *multiplicity*.

Proposition 3.2. *Let $K \in \mathrm{E}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathrm{I}\mathbf{k}_X)$ have normal form at $I \subset S_aX$. Then $\mathrm{E}\nu^{\mathsf{rb}}_{\{a\}}(K)|_I$ is of sheaf type. More precisely, $\mathrm{E}\nu^{\mathsf{rb}}_{\{a\}}(K)|_I \simeq e(L)$ for $L \in \mathrm{Mod}(\mathbf{k}_I)$ a local system of rank $\overline{N}([0])$.*

Proof. The statement is a local problem on I.

Let $\theta \in I$. Since K has normal form at θ, there is an open neighborhood $I_\theta \ni \theta$ such that (3.2) holds with $V_\theta \mathrel{\dot\supset} I_\theta$. Thus, we can reduce to the case $K \simeq \mathbb{E}^{\mathrm{Re}\, h}_{V_\theta|X}[1]$ for $h \in \mathcal{P}_{S_aX}(I_\theta)$. By definition of $\mathrm{E}\nu^{\mathsf{rb}}_{\{a\}}$, it is then enough to check that

$$\mathrm{E}i^{-1}\mathrm{E}j_*\mathrm{E}j_a^{-1}\mathbb{E}^{\mathrm{Re}\, h}_{V_\theta|X}\Big|_{I_\theta} \simeq \begin{cases} e(\mathbf{k}_{I_\theta}) & \text{if } \mathrm{ord}_a(h) \leqslant 0, \\ 0 & \text{if } \mathrm{ord}_a(h) > 0. \end{cases}$$

The statement is clear if $\mathrm{ord}_a(h) \leqslant 0$. If $\mathrm{ord}_a(h) > 0$, after a change of variable and a ramification we can assume that $h(z) = z_a^{-1}$. Hence, one concludes using Lemma 3.3. □

Lemma 3.3. *Let $M = \mathbb{R}_{\geqslant 0}\times\mathbb{R}$ with coordinates (ρ, s). Set $U = \{\rho > 0\}$, $N = \{\rho = 0\}$, and consider the embeddings $N \xrightarrow{i} M \xleftarrow{j} U_\infty$. Then, one has*

$$\mathsf{E}^{s/\rho}_{U|M} \simeq \mathrm{E}j_*\mathrm{E}j^{-1}\mathsf{E}^{s/\rho}_{U|M}, \tag{3.3}$$

$$\mathrm{E}i^{-1}\mathrm{E}j_*\mathrm{E}j^{-1}\mathsf{E}^{s/\rho}_{U|M} \simeq 0, \tag{3.4}$$

$$\mathsf{sh}_M\big(\mathbb{E}^{s/\rho}_{U|M}\big) \simeq \mathbf{k}_{\{\rho>0\}\cup\{s<0\}}. \tag{3.5}$$

Proof. Note that (3.3) is a consequence of (3.4).

One has

$$\begin{aligned}\mathrm{E}i^{-1}\mathrm{E}j_*\mathrm{E}j^{-1}\mathsf{E}^{s/\rho}_{U|M} &= \mathrm{E}i^{-1}\mathrm{E}j_*\mathrm{Q}_U\mathbf{k}_{\{\rho>0,\ t+s/\rho\geqslant 0\}}\\ &\simeq \mathrm{Q}_N i_{\mathbb{R}}^{-1}\mathrm{R}j_{\mathbb{R}*}\mathbf{k}_{\{\rho>0,\ s+\rho t\geqslant 0\}}\\ &\simeq \mathrm{Q}_N i_{\mathbb{R}}^{-1}\mathbf{k}_{\{\rho\geqslant 0,\ s+\rho t\geqslant 0\}}\\ &\simeq \mathrm{Q}_N\mathbf{k}_{\{s\geqslant 0\}}\simeq 0.\end{aligned}$$

Now let us show (3.5). We shall apply [11, Proposition 6.6.5]. For $c>0$, we have

$$\begin{aligned}\mathrm{R}\pi_{M*}\big(\mathbf{k}_{\{t>-c\}}\otimes\mathsf{E}^{s/\rho}_{U|M}\big) &\simeq \mathrm{R}\pi_{M*}\mathbf{k}_{\{\rho>0,\,t\geqslant -s/\rho,\,t>-c\}}\\ &\simeq \mathrm{R}\pi_{M*}\mathrm{R}\mathcal{H}om\left(\mathbf{k}_{\{\rho\geqslant 0,\,t\rho+s>0,\,t\geqslant -c\}}[1],\mathbf{k}_{M\times\mathbb{R}}[1]\right)\\ &\simeq \mathrm{R}\mathcal{H}om\left(\mathrm{R}\pi_{M!}\mathbf{k}_{\{\rho\geqslant 0,t\rho+s>0,t\geqslant -c\}}[1],\mathbf{k}_M\right)\\ &\simeq \mathrm{R}\mathcal{H}om\left(\mathbf{k}_{\{\rho>0,\,-s/\rho\geqslant -c\}},\mathbf{k}_M\right)\\ &\simeq \mathrm{R}\mathcal{H}om\left(\mathbf{k}_{\{\rho>0,\,s\leqslant c\rho\}},\mathbf{k}_M\right)\\ &\simeq \mathbf{k}_{\{\rho\geqslant 0,\,s<c\rho\}}.\end{aligned}$$

Taking the inductive limit with respect to $c\to+\infty$, we obtain (3.5). ☐

3.2. **Perversity.** Recall that $F\in\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathbf{k}_X)$ is perverse if and only if there exists a discrete subset $\Sigma\subset X$ such that:

(a) $H^n i_a^{-1}F=0$ for any $n>0$ and $a\in\Sigma$;
(b) $H^n i_a^! F=0$ for any $n<0$ and $a\in\Sigma$;
(c) $F|_{X\setminus\Sigma}\simeq L[1]$, for $L\in\mathrm{Mod}(\mathbf{k}_{X\setminus\Sigma})$ a local system of finite rank.

Denote by $\mathrm{Perv}(\mathbf{k}_X)\subset\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathbf{k}_X)$ the full triangulated subcategory of perverse sheaves.

Definition 3.4. (i) We say that $K\in\mathrm{E}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathrm{I}\mathbf{k}_X)$ is $\mathbb{C}$-constructible if there exists a discrete subset $\Sigma\subset X$ such that:

(a) for any $n\in\mathbb{Z}$, $H^n(K)|_{X\setminus\Sigma}\simeq e(L)$ for a local system L on $X\setminus\Sigma$ of finite rank,
(b) for any $n\in\mathbb{Z}$, $H^n(K)$ has normal form at any $a\in\Sigma$.

Denote by $\mathrm{E}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(\mathrm{I}\mathbf{k}_X)\subset\mathrm{E}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathrm{I}\mathbf{k}_X)$ the full triangulated subcategory of $\mathbb{C}$-constructible enhanced ind-sheaves.

(ii) We say that $K\in\mathrm{E}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathrm{I}\mathbf{k}_X)$ is an enhanced perverse ind-sheaf if there exists a discrete subset $\Sigma\subset X$ such that:

(a) $H^n\mathrm{E}i_a^{-1}K=0$ for any $n>0$ and $a\in\Sigma$;
(b) $H^n\mathrm{E}i_a^!K=0$ for any $n<0$ and $a\in\Sigma$;
(c) $K|_{X\setminus\Sigma}\simeq e(L[1])$, for $L\in\mathrm{Mod}(\mathbf{k}_{X\setminus\Sigma})$ a local system of finite rank,
(d) $H^{-1}(K)$ has normal form at any $a\in\Sigma$.

Denote by $\mathrm{E}\text{-}\mathrm{Perv}(\mathrm{I}\mathbf{k}_X)\subset\mathrm{E}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(\mathrm{I}\mathbf{k}_X)$ the full triangulated subcategory of enhanced perverse ind-sheaves.

Lemma 3.5. *The functor* $e\,\iota\colon \mathrm{D}^{\mathrm{b}}(\mathbf{k}_X) \to \mathrm{E}^{\mathrm{b}}_+(\mathrm{I}\mathbf{k}_X)$ *sends* $\mathrm{Perv}(\mathbf{k}_X)$ *to* $\text{E-Perv}(\mathrm{I}\mathbf{k}_X)$*, and the functor* $\mathsf{sh}\colon \mathrm{E}^{\mathrm{b}}_+(\mathrm{I}\mathbf{k}_X) \to \mathrm{D}^{\mathrm{b}}(\mathbf{k}_X)$ *sends* $\text{E-Perv}(\mathrm{I}\mathbf{k}_X)$ *to* $\mathrm{Perv}(\mathbf{k}_X)$*.*

Proof. The first statement is clear from the definitions. The second statement follows using Lemma 3.3. □

Note that $\text{E-Perv}(\mathrm{I}\mathbf{k}_X)$ is an abelian subcategory of the quasi-abelian heart ${}^{1/2}\mathrm{E}^0_{\mathbb{R}\text{-c}}(\mathrm{I}\mathbf{k}_X)$ for the middle perversity t-structure introduced in [3]. Note also that, using [4, Proposition 4.1.2] (see also [13, Proposition 3.28]), one has

Theorem 3.6. *The enhanced de Rham functor induces an equivalence between* $\text{E-Perv}(\mathrm{I}\mathbb{C}_X)$ *and the category of holonomic* $\mathcal{D}_X$*-modules.*

Proposition 3.7. *Let* $K \in \text{E-Perv}(\mathrm{I}\mathbf{k}_X)$ *and* $a \in X$ *a singularity of* K*. Then both* $\mathrm{E}\nu_{\{a\}}(K)$ *and* $\mathrm{E}\mu_{\{a\}}(K)[1]$ *are of sheaf type. Moreover they, as well as their associated sheaves, are perverse with the zero-section as their only singularity.*

Proof. Consider the morphisms

$$X \xleftarrow{i_a} \{a\} \xrightarrow{o} T_aX \xleftarrow{u} (\dot{T}_aX)_\infty \xrightarrow{\gamma} S_aX,$$

and recall the distinguished triangle

$$\mathrm{E}u_{!!}\mathrm{E}u^{-1}\mathrm{E}\nu_{\{a\}}K \to \mathrm{E}\nu_{\{a\}}K \to \mathrm{E}o_*\mathrm{E}o^{-1}\mathrm{E}\nu_{\{a\}}K \xrightarrow{+1},$$

(i) Let us show that $\mathrm{E}\nu_{\{a\}}K$ is of sheaf type. As in Proposition 3.2, there exists a local system $L \in \mathrm{Mod}(\mathbf{k}_{S_aX})$ such that $\mathrm{E}\nu^{\mathsf{sph}}_{\{a\}}(K) \simeq e(L[1])$. Since u is semiproper, one has by (2.1)

$$\mathrm{E}u_{!!}\mathrm{E}u^{-1}\mathrm{E}\nu_{\{a\}}K \simeq \mathrm{E}u_{!!}\mathrm{E}\gamma^{-1}\mathrm{E}\nu^{\mathsf{sph}}_{\{a\}}K \simeq e(\mathrm{R}u_!\,\gamma^{-1}L[1]).$$

Hence $\mathrm{E}u_{!!}\mathrm{E}u^{-1}\mathrm{E}\nu_{\{a\}}K$ is of sheaf type. Since any $\mathbb{R}$-constructible enhanced ind-sheaf on a point is of sheaf type, $\mathrm{E}o^{-1}\mathrm{E}\nu_{\{a\}}K$ is of sheaf type. This implies that $\mathrm{E}\nu_{\{a\}}K$ is of sheaf type by the distinguished triangle (3.6).

(ii) Let us show that $\mathrm{E}\nu_{\{a\}}(K) \in \mathrm{E}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathrm{I}\mathbf{k}_{T_aX})$ is perverse, with $\{a\}$ as its only singularity. Since $\mathrm{E}o^{-1}\mathrm{E}\nu_{\{a\}}(K) \simeq \mathrm{E}i_a^{-1}K$ and $\mathrm{E}o^!\mathrm{E}\nu_{\{a\}}(K) \simeq \mathrm{E}i_a^!K$ by [5, Lemma 4.8], we have

(a) $H^n(\mathrm{E}o^{-1}\mathrm{E}\nu_{\{a\}}(K)) \simeq H^n(\mathrm{E}i_a^{-1}K) \simeq 0$ for $n > 0$,
(b) $H^n(\mathrm{E}o^!\mathrm{E}\nu_{\{a\}}(K)) \simeq H^n(\mathrm{E}i_a^!K) \simeq 0$ for $n < 0$.
(c) $\mathrm{E}u^{-1}\mathrm{E}\nu_{\{a\}}(K) \simeq e(\gamma^{-1}L[1])$.

(iii) It remains to show that $\mathrm{E}\mu_{\{a\}}(K)$ is of sheaf type, and that its associated sheaf is perverse. Setting $F := \mathsf{sh}\big(\mathrm{E}\nu_{\{a\}}(K)\big)$, this follows from

$$\mathrm{E}\mu_{\{a\}}(K) \simeq \mathrm{E}\nu_{\{a\}}(K)^\wedge \simeq \big(e(F)\big)^\wedge \simeq e(F^\wedge),$$

and the fact that the classical Fourier-Sato transform for sheaves preserves the perversity of $\mathbb{R}^{\times}_{>0}$-conic objects, up to shift [1]. □

4. Nearby and vanishing cycles

As we mentioned in the Introduction, nearby cycles for enhanced ind-sheaves were already discussed in [4, §6.2]. However, defining them through enhanced specialization, as we do here, sheds some light on the underlying geometry. Moreover, using enhanced microlocalization, we can here also deal with vanishing cycles. In this section we thus recall and complement some results from loc. cit.

4.1. **Definitions.** Let X be a smooth complex curve, and $a \in X$. Consider the natural morphisms associated with the real blow-up X^{rb}_a of X with center a as in (3.1):

$$S_aX \xhookrightarrow{i} X^{\mathsf{rb}}_a \xhookleftarrow{j} (X \setminus \{a\})_\infty \xhookrightarrow{j_a} X\,.$$

Definition 4.1. Let $K \in \mathrm{E}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathrm{I}\,\mathbf{k}_X)$.

(i) Consider the objects of $\mathrm{D}^{\mathrm{b}}(\mathbf{k}_{S_aX})$

$$\begin{aligned}
\Psi_a(K) &:= \nu^{\mathsf{sph}}_{\{a\}}\big(\mathsf{sh}_X(K)\big) \simeq \nu^{\mathsf{rb}}_{\{a\}}\big(\mathsf{sh}_X(K)\big)\\
&= i^{-1}\mathrm{R}j_*j_a^{-1}\mathsf{sh}_X(K)\\
&\underset{(*)}{\simeq} i^{-1}\mathrm{R}j_*\mathsf{sh}_{(X\setminus\{a\})_\infty}\big(\mathrm{E}j_a^{-1}K\big),\\
\Psi^{\prec 0}_a(K) &:= i^{-1}\mathsf{sh}_{X^{\mathsf{rb}}_a}\big(\mathrm{E}j_*\mathrm{E}j_a^{-1}K\big),\\
\Psi^{0}_a(K) &:= \mathsf{sh}_{S_aX}\big(\mathrm{E}i^{-1}\mathrm{E}j_*\mathrm{E}j_a^{-1}K\big)\\
&= \mathsf{sh}_{S_aX}\big(\mathrm{E}\nu^{\mathsf{rb}}_{\{a\}}(K)\big) \simeq \mathsf{sh}_{S_aX}\big(\mathrm{E}\nu^{\mathsf{sph}}_{\{a\}}(K)\big),
\end{aligned}$$

where $(*)$ follows from [6, Lemma 3.9].

(ii) Let $I \subset S_aX$ be an open subset and $f \in \mathcal{P}_{S_aX}(I)$. For $U \subset X\setminus\{a\}$ an open subset such that $U \mathrel{\dot{\supset}} I$ and f extends on U, set

$$\begin{aligned}
K(f) &:= \mathrm{R}\mathcal{I}hom^+(\mathbb{E}^{\mathrm{Re}\,f}_{U|X}, K) \in \mathrm{E}^{\mathrm{b}}_+(\mathrm{I}\,\mathbf{k}_X),\\
\Psi^{\prec f}_a(K) &:= \Psi^{\prec 0}_a\big(K(f)\big)|_I \in \mathrm{D}^{\mathrm{b}}(\mathbf{k}_I),\\
\Psi^{f}_a(K) &:= \Psi^{0}_a\big(K(f)\big)|_I \in \mathrm{D}^{\mathrm{b}}(\mathbf{k}_I).
\end{aligned}$$

Note that $\Psi^{\prec f}_a(K)$ and $\Psi^{f}_a(K)$ do not depend on the choice of U.

(iii) Consider the object of $\mathrm{D}^{\mathrm{b}}(\mathbf{k}_{S^*_aX})$

$$\begin{aligned}
\Phi^0_a(K) &:= \mu^{\mathsf{sph}}_{\{a\}}\big(\mathsf{sh}_X(K)\big)[1],\\
\Phi^0_a(K) &:= \mathsf{sh}_{S^*_aX}\big(\mathrm{E}\mu^{\mathsf{sph}}_{\{a\}}(K)\big)[1].
\end{aligned}$$

Lemma 4.2. *Let $I \subset S_aX$ be an open subset, $f, g \in \mathcal{P}_{S_aX}(I)$ with $f \precsim_I g$, and $K \in \mathrm{E}^{\mathrm{b}}_+(\mathrm{I}\mathbf{k}_X)$. Then there are natural morphisms in $\mathrm{E}^{\mathrm{b}}_+(\mathrm{I}\mathbf{k}_I)$*

$$\Psi_a(K)|_I \leftarrow \Psi_a^{\precsim g}(K) \leftarrow \Psi_a^{\precsim f}(K) \to \Psi_a^f(K).$$

Proof. It follows from [6, Lemma 3.9]. □

Let $\theta \in S_aX$, $f \in \mathcal{P}_{S_aX,\theta}$, and denote by z_a a local coordinate at a with $z_a(a) = 0$. Assuming $K \in \mathrm{E}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathrm{I}\mathbf{k}_X)$, it follows from [6, Lemma 5.1] that one has

$$\Psi_a^f(K)_\theta \simeq \varinjlim_{\delta,\varepsilon \to 0+,\ V \dot{\ni} \theta} \mathrm{RHom}^{\mathrm{E}}(\mathsf{E}_{V|X}^{\operatorname{Re} f(x) \rhd \operatorname{Re} f(x) - \delta |z_a(x)|^{-\varepsilon}}, K). \tag{4.1}$$

4.2. **The case of perverse objects.** Let us collect in the following lemma some results from [4, §6]. Note that statement (iii) below also follows from Proposition 3.2.

Lemma 4.3. *Let $f, g \in \mathcal{P}_{S_aX}(I)$ with $f \precsim_I g$, and $K \in \mathrm{E}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathrm{I}\mathbf{k}_X)$. Assume that K has normal form at I. Then*

(i) *$\Psi_a(K)|_I$ is concentrated in degree zero and is a local system on I of rank $\sum_{[h] \in \overline{\mathcal{P}}_{S_aX,\theta}} \overline{N}(h)$ at $\theta \in I$,*

(ii) *$\Psi_a^{\precsim f}(K)$ is concentrated in degree zero, and is an $\mathbb{R}$-constructible sheaf on I. Moreover, the morphisms $\Psi_a^{\precsim f}(K) \to \Psi_a^{\precsim g}(K) \to \Psi_a(K)|_I$ are monomorphisms, and $\Psi_a^{\precsim f}(K) \to \Psi_a^f(K)$ is an epimorphism.*

(iii) *$\Psi_a^f(K)$ is concentrated in degree zero, and is a local system of rank $\overline{N}([f])$ on I.*

Recall the notion of a Stokes filtration, e.g. from [4, §6.1].

Proposition 4.4. *Let $K \in \text{E-Perv}(\mathrm{I}\mathbf{k}_X)$. Then*

(i) *$\Psi_a(K)$ is a local system on S_aX with Stokes filtration $\Psi_a^{\precsim \bullet}(K)$, and associated graded components $\Psi_a^\bullet(K)$ which is a local system on S_aX;*

(ii) *$\Phi_a^0(K)$ is a local system on S_a^*X.*

Proof. (i) is a particular case of Lemma 4.3, and (ii) follows from Proposition 3.7. □

Refer to Appendix A for an alternative description of the vanishing cycles $\Phi_a^0(K)$ as a local system on S_aX, via some blow-up transforms.

5. Fourier transform on the affine line

Let K be an enhanced perverse ind-sheaf on the affine line, and assume that so is its shifted enhanced Fourier-Sato transform ${}^{\mathbb{L}}K := {}^{\mathsf{L}}K[1]$. The stationary phase formula provides a relation (see (1.4)) between the graded components of the Stokes filtrations of K and ${}^{\mathbb{L}}K$, for degrees which are not linear (modulo bounded function). We discuss here the case of linear degrees.

5.1. **Linear exponential factors.** Let z be a coordinate on the complex line $\mathbb{V}$, and w the dual coordinate on $\mathbb{V}^*$, so that the pairing $\mathbb{V}\times\mathbb{V}^*\to\mathbb{C}$ is given by $(z,w)\mapsto zw$. The underlying real vector spaces are in duality by the pairing $\langle z,w\rangle=\mathrm{Re}(zw)$. Denoting by $\mathbb{P}=\mathbb{V}\cup\{\infty\}$ the complex projective line with affine chart $\mathbb{V}$, one has $\mathbb{V}_\infty\simeq(\mathbb{V},\mathbb{P})$. Similarly, $\mathbb{V}^*_\infty\simeq(\mathbb{V}^*,\mathbb{P}^*)$, for $\mathbb{P}^*=\mathbb{V}^*\cup\{\infty\}$.

Let $\mathrm{E\text{-}Perv}(\mathrm{I}\mathbf{k}_{\mathbb{V}_\infty})$ be the full triangulated subcategory of $\mathrm{E}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathrm{I}\mathbf{k}_{V_\infty})$ whose objects are of the form $\mathrm{E}j^{-1}K$ for some $K\in\mathrm{E\text{-}Perv}(\mathrm{I}\mathbf{k}_{\mathbb{P}})$. Here, $j\colon\mathbb{V}_\infty\hookrightarrow\mathbb{P}$ is the natural morphism.

Consider the enhanced Fourier-Sato transform

$$\mathrm{E}^{\mathrm{b}}_+(\mathrm{I}\mathbf{k}_{\mathbb{V}_\infty})\to\mathrm{E}^{\mathrm{b}}_+(\mathrm{I}\mathbf{k}_{\mathbb{V}^*_\infty}),\quad K\mapsto{}^{\mathbb{L}}K:={}^{\mathsf{L}}K[1].$$

Here, the shift ensures compatibility with the Riemann-Hilbert correspondence.

Theorem 5.1. *Let* $K\in\mathrm{E\text{-}Perv}(\mathrm{I}\mathbf{k}_{\mathbb{V}_\infty})$. *Assume* ${}^{\mathbb{L}}K\in\mathrm{E\text{-}Perv}(\mathrm{I}\mathbf{k}_{\mathbb{V}^*_\infty})$. *Then:*

(i) *for any* $a\in\mathbb{V}$, *under the canonical identification* $S_\infty\mathbb{P}^*\simeq S^*_a\mathbb{V}$, *there is an isomorphism of local systems*

$$\Psi^{aw}_\infty({}^{\mathbb{L}}K)\simeq\Phi^0_a(K)\,;$$

(ii) *for any* $b\in\mathbb{V}^*$, *under the canonical identification* $S^*_b\mathbb{V}^*\simeq S_\infty\mathbb{P}$, *there is an isomorphism of local systems*

$$\Phi^0_b({}^{\mathbb{L}}K)\simeq r^{-1}\Psi^{-bz}_\infty(K)\,,$$

where r *denotes the antipodal map.*

Remark 5.2. With notations as in §1.4, for $\mathbf{k}=\mathbb{C}$ let $K:=\mathcal{DR}^{\mathrm{E}}(\mathcal{M})$ for $\mathcal{M}$ an algebraic holonomic $\mathcal{D}_{\mathbb{V}}$-module. Then ${}^{\mathbb{L}}K\simeq\mathcal{DR}^{\mathrm{E}}(\mathcal{M}^\wedge)$, where $\mathcal{M}^\wedge$ is the Fourier-Laplace transform of $\mathcal{M}$. Since $\mathcal{M}^\wedge$ is an algebraic holonomic $\mathcal{D}_{\mathbb{V}^*}$-module, ${}^{\mathbb{L}}K\in\mathrm{E\text{-}Perv}(\mathrm{I}\mathbf{k}_{\mathbb{V}^*_\infty})$.

Proof of Theorem 5.1. (ii) follows from (i). In fact, interchanging the roles of $\mathbb{V}$ and $\mathbb{V}^*$, one has

$$\Phi^0_b({}^{\mathbb{L}}K)\simeq\Psi^{bz}_\infty({}^{\mathbb{L}\mathbb{L}}K)\underset{(*)}{\simeq}\Psi^{bz}_\infty(\mathrm{E}r^{-1}K)\simeq r^{-1}\Psi^{-bz}_\infty(K).$$

For $(*)$ refer e.g. to [5, §5.2].

(i) The translation $\tau_a\colon\mathbb{V}\to\mathbb{V}$, $z\mapsto z+a$, induces an identification $S_a\mathbb{V}\simeq S_0\mathbb{V}$. Moreover, one has

$$\begin{aligned}\Phi^0_a(K)&\simeq\Phi^0_0(\mathrm{E}\tau_a^{-1}K),\\ \Psi^{aw}_\infty\big({}^{\mathbb{L}}(\mathrm{E}\tau_a^{-1}K)\big)&\simeq\Psi^{aw}_\infty\big(({}^{\mathbb{L}}K)(-aw)\big)\\ &\simeq\Psi^0_\infty\big({}^{\mathbb{L}}K\big).\end{aligned}$$

Hence, we may assume $a = 0$. It is then enough to check that there is an isomorphism

$$\Psi^0_\infty({}^{\mathbb{L}}K) \simeq \Phi^0_0(K).$$

Since $\mathrm{E}\nu^{\mathsf{sph}}_{\{\infty\}}({}^{\mathbb{L}}K)$ and $\mathrm{E}\mu^{\mathsf{sph}}_{\{0\}}(K)$ are of sheaf type, it is equivalent to prove that there is an isomorphism

$$\mathrm{E}\nu^{\mathsf{sph}}_{\{\infty\}}({}^{\mathbb{L}}K) \simeq \mathrm{E}\mu^{\mathsf{sph}}_{\{0\}}(K).$$

This follows from Lemma 5.3 and (5.2) below. □

5.2. **Smash functor.** We consider here the smash functor of [1], in its enhanced version from [5], and establish a small additional result needed to complete the proof of Theorem 5.1.

The sphere compactification $\mathbb{S}\mathbb{V} := \big((\mathbb{R}_u \times \mathbb{V}) \setminus \{(0,0)\}\big)/\mathbb{R}^\times_{>0}$ of $\mathbb{V}$ decomposes as $\mathbb{S}\mathbb{V} = \mathbb{V}^+ \sqcup H \sqcup \mathbb{V}^-$, corresponding to $u > 0$, $u = 0$ or $u < 0$. Let us identify $\mathbb{V} = \mathbb{V}^+$. Note that H is a real hypersurface of $\mathbb{S}\mathbb{V}$. One has a natural identification $\dot{\mathbb{V}} := \mathbb{V} \setminus \{0\} = T^+_H\mathbb{S}\mathbb{V}$, where $T^+_H\mathbb{S}\mathbb{V} \subset \dot{T}_H\mathbb{S}\mathbb{V}$ denotes the normal directions pointing to $\mathbb{V} = \mathbb{V}^+$. With these identification, $\mathrm{E}\nu_H$ induces a functor

$$\mathrm{E}\nu_{H|\dot{\mathbb{V}}} \colon \mathrm{E}^{\mathrm{b}}_+(\mathrm{I}\mathbf{k}_{\dot{\mathbb{V}}_\infty}) \to \mathrm{E}^{\mathrm{b}}_{(\mathbb{R}^\times_{>0})_\infty}(\mathrm{I}\mathbf{k}_{\dot{\mathbb{V}}_\infty})$$

which can be considered a "specialization at ∞".

The enhanced smash functor

$$\mathrm{E}\sigma_{\mathbb{V}} \colon \mathrm{E}^{\mathrm{b}}_+(\mathrm{I}\mathbf{k}_{\mathbb{V}_\infty}) \to \mathrm{E}^{\mathrm{b}}_{(\mathbb{R}^\times_{>0})_\infty}(\mathrm{I}\mathbf{k}_{\mathbb{V}_\infty}),$$

for which we refer to [5, §6], provides an extension of $\mathrm{E}\nu_{H|\dot{\mathbb{V}}}$ from $\dot{\mathbb{V}}_\infty$ to $\mathbb{V}$. In fact, $\mathrm{E}\sigma_{\mathbb{V}}$ induces a functor

$$\mathrm{E}\sigma_{\mathbb{V}|\dot{\mathbb{V}}} \colon \mathrm{E}^{\mathrm{b}}_+(\mathrm{I}\mathbf{k}_{\dot{\mathbb{V}}_\infty}) \to \mathrm{E}^{\mathrm{b}}_{(\mathbb{R}^\times_{>0})_\infty}(\mathrm{I}\mathbf{k}_{\dot{\mathbb{V}}_\infty}),$$

and one has

$$\mathrm{E}\sigma_{\mathbb{V}|\dot{\mathbb{V}}} \simeq \mathrm{E}\nu_{H|\dot{\mathbb{V}}}. \tag{5.1}$$

Recall also that, by [5, Proposition 6.6], for $K \in \mathrm{E}^{\mathrm{b}}_+(\mathrm{I}\mathbf{k}_{\mathbb{V}_\infty})$ one has

$$\mathrm{E}\mu_{\{0\}}(K) \simeq \mathrm{E}\sigma_{\mathbb{V}^*}({}^{\mathsf{L}}K). \tag{5.2}$$

Lemma 5.3. *Let $K \in \mathrm{E}^{\mathrm{b}}_+(\mathrm{I}\mathbf{k}_{\mathbb{V}_\infty})$. Then, with the natural identification $S_\infty\mathbb{P} \simeq \dot{\mathbb{V}}/\mathbb{R}^\times_{>0}$, one has*

$$\mathrm{E}\nu^{\mathsf{sph}}_{\{\infty\}}(K) \simeq \mathrm{E}\sigma^{\mathsf{sph}}_{\mathbb{V}}(K).$$

Proof. With $H, \mathbb{V}^\pm \subset \mathbb{S}\mathbb{V}$ defined as above, one has $\mathbb{S}\mathbb{V}^{\mathsf{rb}}_H = (\mathbb{S}\mathbb{V}^{\mathsf{rb}}_H)^+ \sqcup (\mathbb{S}\mathbb{V}^{\mathsf{rb}}_H)^+$ and $S_H\mathbb{S}\mathbb{V} = (S_H\mathbb{S}\mathbb{V})^+ \sqcup (S_H\mathbb{S}\mathbb{V})^-$. Moreover, the identification $\mathbb{V} \simeq \mathbb{V}^+$ implies identifications $\mathbb{P}^{\mathsf{rb}}_\infty \simeq (\mathbb{S}\mathbb{V}^{\mathsf{rb}}_H)^+$ and $S_\infty\mathbb{P} \simeq (S_H\mathbb{S}\mathbb{V})^+$.

With these identifications, and using (2.4) and (5.1), it is enough to prove the isomorphism

$$\mathrm{E}\nu^{\mathsf{rb}}_{\{\infty\}}(K) \simeq \mathrm{E}\nu^{\mathsf{rb}}_H(K)|_{S^+_H\mathbb{S}\mathbb{V}}.$$

This follows by considering the commutative diagram

$$\begin{array}{ccccccc} \mathbb{V}_\infty & \hookleftarrow & \dot{\mathbb{V}}_\infty & \hookrightarrow & \mathbb{P}^{\mathsf{rb}}_\infty & \hookleftarrow & S_\infty\mathbb{P} \\ \wr\,| & & \wr\,| & \square & \wr\,| & \square & \wr\,| \\ \mathbb{V}^+_\infty & \hookleftarrow & \dot{\mathbb{V}}^+_\infty & \hookrightarrow & \big(\mathbb{S}\mathbb{V}^{\mathsf{rb}}_H\big)^+ & \hookleftarrow & \big(S_H\mathbb{S}\mathbb{V}\big)^+. \end{array}$$

□

Appendix A. Vanishing cycles by blow-up transform

A.1. **Blow-up transforms.** Let M be a real analytic manifold, and $N \subset M$ a smooth submanifold. As in (2.3) consider the real oriented blowup M^{rb}_N of M with center N, and the associated commutative diagram of bordered spaces

$$\begin{array}{ccccc} S_NM & \overset{i}{\hookrightarrow} & M^{\mathsf{rb}}_N & \overset{j}{\hookleftarrow} & (M\setminus N)_\infty \\ \sigma\downarrow & \square & p\downarrow & \swarrow_{j_N} & \\ N & \underset{i_N}{\hookrightarrow} & M, & & \end{array}$$

where we write for short $i = i_{\mathsf{rb}}$, $j = j_{\mathsf{rb}}$ and $p = p_{\mathsf{rb}}$.

Definition A.1. For $K \in \mathrm{E}^{\mathrm{b}}_+(\mathrm{I}\mathbf{k}_M)$, consider the objects of $\mathrm{E}^{\mathrm{b}}_+(\mathrm{I}\mathbf{k}_{S_NM})$

$$\mathrm{E}\lambda^{\mathsf{rb}}_N(K) := \mathrm{E}i^!\mathrm{E}p^{-1}K[1],$$
$$\mathrm{E}\widetilde{\lambda}^{\mathsf{rb}}_N(K) := \mathrm{E}i^{-1}\mathrm{E}p^!K.$$

We denote by λ^{rb}_N and $\widetilde{\lambda}^{\mathsf{rb}}_N$ the analogous functors for sheaves.

Note that one has

$$e\circ\lambda^{\mathsf{rb}}_N \simeq \mathrm{E}\lambda^{\mathsf{rb}}_N\circ e, \quad e\circ\widetilde{\lambda}^{\mathsf{rb}}_N \simeq \mathrm{E}\widetilde{\lambda}^{\mathsf{rb}}_N\circ e,$$

and similarly for e replaced by ϵ, ϵ^+ or ϵ^-.

Remark A.2. Note that $\mathrm{E}\lambda^{\mathsf{rb}}_N \not\simeq \mathrm{E}\widetilde{\lambda}^{\mathsf{rb}}_N$ in general, as shown by the following example. (See however Proposition A.8.) For $M = \mathbb{R}_x$ and $N = \{0\}$, one has $M^{\mathsf{rb}}_N \simeq \{x \leqslant 0\} \sqcup \{x \geqslant 0\}$. Restricted to the left component, the maps i and p are the embeddings $\{0\} \xrightarrow{i} \{x \leqslant 0\} \xrightarrow{p} \mathbb{R}$. Then, for $F = \mathbf{k}_{\{x>0\}}$, one has

$$\lambda^{\mathsf{rb}}_N(F) \simeq i^!p^{-1}\mathbf{k}_{\{x>0\}}[1] \simeq i^!\left(\mathbf{k}_{\{x>0\}}|_{\{x\leqslant 0\}}[1]\right) \simeq 0,$$
$$\widetilde{\lambda}^{\mathsf{rb}}_N(F) \simeq i^{-1}p^!\mathbf{k}_{\{x>0\}} \simeq \left(\mathrm{R}\varGamma_{\{x\leqslant 0\}}\mathbf{k}_{\{x>0\}}\right)_0 \simeq \mathbf{k}[-1].$$

Lemma A.3. *For $K \in \mathrm{E}^{\mathrm{b}}_+(\mathrm{I}\mathbf{k}_M)$, there are distinguished triangles*

(i) $$\mathrm{E}\sigma^{-1}\mathrm{E}i_N^{-1}K \to \mathrm{E}\nu^{\mathsf{rb}}_N(K) \xrightarrow{c} \mathrm{E}\lambda^{\mathsf{rb}}_N(K) \xrightarrow{+1},$$

(ii) $$\mathrm{E}\sigma^!\mathrm{E}i_N^!K \to \mathrm{E}\widetilde{\lambda}^{\mathsf{rb}}_N(K) \xrightarrow{v} \mathrm{E}\nu^{\mathsf{rb}}_N(K) \xrightarrow{+1}.$$

Proof. (i) For $L \in \mathrm{E}^{\mathrm{b}}_{+}(\mathrm{I}\mathbf{k}_{M^{\mathsf{rb}}_N})$, there is a distinguished triangle

$$\mathrm{E}j_{!!}\mathrm{E}j^{-1}L \to L \to \mathrm{E}i_{!!}\mathrm{E}i^{-1}L \xrightarrow{+1} .$$

When $L = \mathrm{E}p^{-1}K$, the above distinguished triangle reads

$$\mathrm{E}j_{!!}\mathrm{E}j^{!}_{N}K \to \mathrm{E}p^{-1}K \to \mathrm{E}i_{!!}\mathrm{E}\sigma^{-1}\mathrm{E}i_N^{-1}K \xrightarrow{+1} .$$

By applying $\mathrm{E}i^{!}$ we get (i).

(ii) Consider the distinguished triangle

$$\mathrm{E}i_{*}\mathrm{E}i^{!}L \to L \to \mathrm{E}j_{*}\mathrm{E}j^{!}L \xrightarrow{+1} .$$

When $L = \mathrm{E}p^{!}K$, the above distinguished triangle reads

$$\mathrm{E}i_{*}\mathrm{E}\sigma^{!}\mathrm{E}i^{!}_{N}K \to \mathrm{E}p^{!}K \to \mathrm{E}j_{*}\mathrm{E}j_N^{-1} \xrightarrow{+1} .$$

One concludes by applying $\mathrm{E}i^{-1}$. □

The following result is clear from the definitions and [5, Lemma 4.7].

Lemma A.4. *For $K \in \mathrm{E}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathrm{I}\mathbf{k}_M)$, one has*

$$\mathrm{D}^{\mathrm{E}}\mathrm{E}\nu^{\mathsf{rb}}_N(K) \simeq \mathrm{E}\nu^{\mathsf{rb}}_N(\mathrm{D}^{\mathrm{E}}K), \quad \mathrm{D}^{\mathrm{E}}\mathrm{E}\widetilde{\lambda}^{\mathsf{rb}}_N(K) \simeq \mathrm{E}\lambda^{\mathsf{rb}}_N(\mathrm{D}^{\mathrm{E}}K)[-1].$$

Lemma A.5. *For $K \in \mathrm{E}^{\mathrm{b}}_{+}(\mathrm{I}\mathbf{k}_M)$ one has*

(i) $$\mathrm{E}\lambda^{\mathsf{rb}}_N(K) \simeq \mathrm{E}\lambda^{\mathsf{rb}}_N(\mathrm{E}\nu_N(K)),$$

(ii) $$\mathrm{E}\widetilde{\lambda}^{\mathsf{rb}}_N(K) \simeq \mathrm{E}\widetilde{\lambda}^{\mathsf{rb}}_N(\mathrm{E}\nu_N(K)),$$

with the identification $S_N M \simeq S_N(T_N M)$.

Proof. Since the proofs are similar, we will only discuss (i).

(a) We will construct in part (b) below a natural morphism

$$\mathrm{E}\lambda^{\mathsf{rb}}_N(\mathrm{E}\nu_N(K)) \to \mathrm{E}\lambda^{\mathsf{rb}}_N(K). \tag{A.1}$$

By Lemma A.3 (i), it enters the commutative diagram

$$\begin{array}{ccccc}
\mathrm{E}\sigma^{-1}\mathrm{E}o^{-1}\mathrm{E}\nu_N(K) & \longrightarrow & \mathrm{E}\nu^{\mathsf{rb}}_N(\mathrm{E}\nu_N(K)) & \longrightarrow & \mathrm{E}\lambda^{\mathsf{rb}}_N(\mathrm{E}\nu_N(K)) \xrightarrow{+1} \\
\downarrow\wr & & \downarrow\wr & & \downarrow \\
\mathrm{E}\sigma^{-1}\mathrm{E}i_N^{-1}K & \longrightarrow & \mathrm{E}\nu^{\mathsf{rb}}_N(K) & \longrightarrow & \mathrm{E}\lambda^{\mathsf{rb}}_N(K) \xrightarrow{+1} .
\end{array}$$

Here, the first vertical isomorphism is due to [5, Lemma 4.8], and the second vertical isomorphism follows from [5, Lemma 4.10]. Hence, also the third vertical arrow is an isomorphism, and the statement follows.

(b) In order to obtain (A.1), we are going to connect the relevant spaces in a commutative diagram.

We refer to [5, §§2.3, 2.4] for details on the real oriented blow-up M^{rb}_N, the projective blow-up M^{pb}_N, the normal deformation M^{nd}_N, and the open embedding $M^{\mathsf{nd}}_N \subset \big(M \times \mathbb{R}\big)^{\mathsf{pb}}_{N\times\{0\}}$.

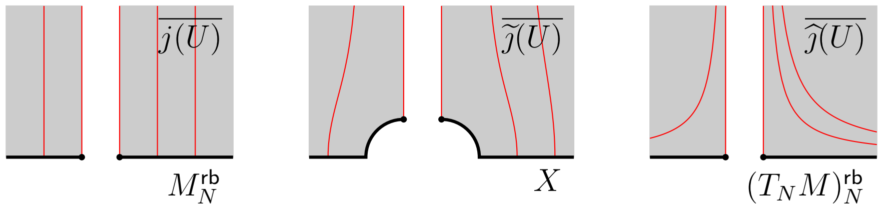


FIGURE 1. The sets $\overline{j(U)}$, $\overline{\widetilde{\jmath}(U)}$ and $\overline{\widehat{\jmath}(U)}$ pictured in the case $M = \mathbb{R}$ and $N = \{0\}$. The red lines are fibers of the projection $p_U : U \to \Omega \to M$.

With the natural identification $M \times \mathbb{R}_{>0} \simeq \Omega \subset M_N^{\mathsf{nd}}$, consider the open embeddings (see Figure 1)

$$\begin{array}{ccl} & & (M \times \mathbb{R})^{\mathsf{rb}}_{N\times\mathbb{R}} \\ & \overset{j}{\nearrow} & \\ U := \Omega^{\mathsf{rb}}_{N\times\mathbb{R}_{>0}} & \xrightarrow{\widehat{\jmath}} & \left(M_N^{\mathsf{nd}}\right)^{\mathsf{rb}}_{\overline{N\times\mathbb{R}_{\neq 0}}} \\ & \underset{\widetilde{\jmath}}{\searrow} & \\ & & \left((M\times\mathbb{R})^{\mathsf{rb}}_{N\times\{0\}}\right)^{\mathsf{rb}}_{\overline{N\times\mathbb{R}_{\neq 0}}}. \end{array}$$

Here, $\widetilde{\jmath}$ is induced by the natural embedding $\overline{\Omega} \subset (M \times \mathbb{R})^{\mathsf{rb}}_{N\times\{0\}}$, compatible with the open embedding $M_N^{\mathsf{nd}} \subset (M \times \mathbb{R})^{\mathsf{pb}}_{N\times\{0\}}$ of [5, §2.4]. More precisely, there is a commutative diagram

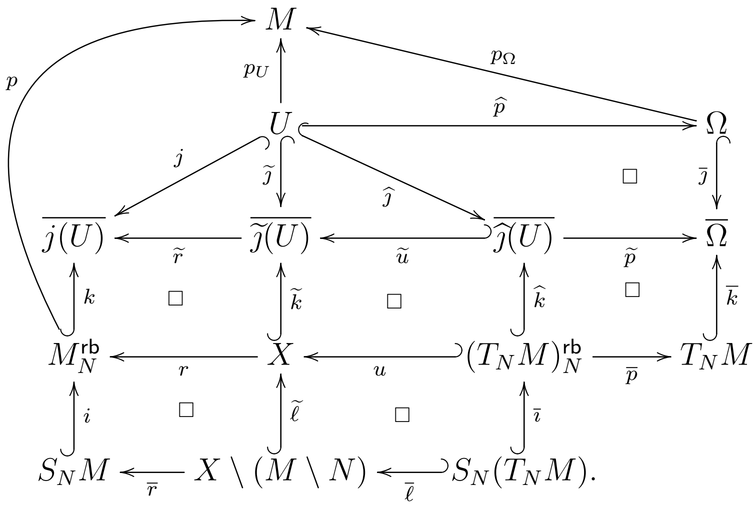
$$\begin{array}{ccc} \overline{\Omega} & \hookrightarrow & (M \times \mathbb{R})^{\mathsf{rb}}_{N\times\{0\}} \\ \downarrow & & \downarrow q \\ M_N^{\mathsf{nd}} & \hookrightarrow & (M \times \mathbb{R})^{\mathsf{pb}}_{N\times\{0\}}, \end{array}$$

where q is the natural projection from the real oriented blow-up to the projective blow-up.

Let X be the closed subset of $\left((M \times \mathbb{R})^{\mathsf{rb}}_{N\times\{0\}}\right)^{\mathsf{rb}}_{\overline{N\times\mathbb{R}_{\neq 0}}}$ given by

$$\begin{aligned} X &:= \overline{\widetilde{\jmath}(U)} \setminus \widetilde{\jmath}(U) \\ &= (M \setminus N) \sqcup S_N M \sqcup \dot{T}_N M \sqcup S_N(T_N M). \end{aligned}$$

Consider the commutative diagram with cartesian squares, where $\widetilde{u}$ and u are open embeddings,



Note that $\overline{r} \circ \overline{\ell}$ gives the identification $S_N(T_NM) \simeq S_NM$. Hence, by definition, (A.1) is written as

$$\mathrm{E}\overline{r}_*\mathrm{E}\overline{\ell}_*\mathrm{E}\overline{\imath}^!\mathrm{E}\overline{p}^{-1}\mathrm{E}\nu_N(K)[1] \to \mathrm{E}i^!\mathrm{E}p^{-1}K[1]. \tag{A.2}$$

On one hand, there is a chain of morphisms

$$\begin{aligned}\mathrm{E}\overline{\imath}^!\mathrm{E}\overline{p}^{-1}\mathrm{E}\nu_N(K) &\simeq \mathrm{E}\overline{\imath}^!\mathrm{E}\overline{p}^{-1}\mathrm{E}\overline{k}^{-1}\mathrm{E}\overline{\jmath}_*\mathrm{E}p_\Omega^{-1}K \simeq \mathrm{E}\overline{\imath}^!\mathrm{E}\widehat{k}^{-1}\mathrm{E}\widetilde{p}^{-1}\mathrm{E}\overline{\jmath}_*\mathrm{E}p_\Omega^{-1}K \\ &\underset{(1)}{\to} \mathrm{E}\overline{\imath}^!\mathrm{E}\widehat{k}^{-1}\mathrm{E}\widehat{\jmath}_*\mathrm{E}\widehat{p}^{-1}\mathrm{E}p_\Omega^{-1}K \simeq \mathrm{E}\overline{\imath}^!\mathrm{E}\widehat{k}^{-1}\mathrm{E}\widehat{\jmath}_*\mathrm{E}p_U^{-1}K \\ &\simeq \mathrm{E}\overline{\imath}^!\mathrm{E}\widehat{k}^{-1}\mathrm{E}\widetilde{u}^{-1}\mathrm{E}\widetilde{\jmath}_*\mathrm{E}p_U^{-1}K \simeq \mathrm{E}\overline{\imath}^!\mathrm{E}u^{-1}\mathrm{E}\widetilde{k}^{-1}\mathrm{E}\widetilde{\jmath}_*\mathrm{E}p_U^{-1}K \\ &\underset{(2)}{\simeq} \mathrm{E}\overline{\ell}^!\mathrm{E}\widetilde{\ell}^!\mathrm{E}\widetilde{k}^{-1}\mathrm{E}\widetilde{\jmath}_*\mathrm{E}p_U^{-1}K = \mathrm{E}\overline{\ell}^!L,\end{aligned}$$

where we set

$$L := \mathrm{E}\widetilde{\ell}^!\mathrm{E}\widetilde{k}^{-1}\mathrm{E}\widetilde{\jmath}_*\mathrm{E}p_U^{-1}K.$$

Here, (1) follows by adjunction from the isomorphism $\mathrm{E}\widehat{\jmath}^{-1}\mathrm{E}\widetilde{p}^{-1}\mathrm{E}\overline{\jmath}_* \simeq \mathrm{E}\widehat{p}^{-1}$, and (2) uses the fact that $\mathrm{E}u^{-1} \simeq \mathrm{E}u^!$.

Hence, there is a morphism

$$\mathrm{E}\overline{r}_*\mathrm{E}\overline{\ell}_*\mathrm{E}\overline{\imath}^!\mathrm{E}\overline{p}^{-1}\mathrm{E}\nu_N(K) \to \mathrm{E}\overline{r}_*\mathrm{E}\overline{\ell}_*\mathrm{E}\overline{\ell}^!L. \tag{A.3}$$

On the other hand, there is a chain of isomorphisms

$$\begin{aligned}\mathrm{E}i^!\mathrm{E}p^{-1}K &\underset{(3)}{\simeq} \mathrm{E}i^!\mathrm{E}k^{-1}\mathrm{E}j_*\mathrm{E}p_U^{-1}K \simeq \mathrm{E}i^!\mathrm{E}k^{-1}\mathrm{E}\widetilde{r}_*\mathrm{E}\widetilde{\jmath}_*\mathrm{E}p_U^{-1}K \\ &\underset{(4)}{\simeq} \mathrm{E}i^!\mathrm{E}r_*\mathrm{E}\widetilde{k}^{-1}\mathrm{E}\widetilde{\jmath}_*\mathrm{E}p_U^{-1}K \simeq \mathrm{E}\overline{r}_*\mathrm{E}\widetilde{\ell}^!\mathrm{E}\widetilde{k}^{-1}\mathrm{E}\widetilde{\jmath}_*\mathrm{E}p_U^{-1}K \\ &\simeq \mathrm{E}\overline{r}_*L.\end{aligned}$$

Here, (3) easily follows using the identification $(M \times \mathbb{R})^{\mathsf{rb}}_{N\times\mathbb{R}} \simeq (M^{\mathsf{rb}}_N) \times \mathbb{R}$, and (4) uses the fact that $\widetilde{r}$ and r are proper.

Hence, the natural morphism $\mathrm{E}\overline{\ell}_*\mathrm{E}\overline{\ell}^!L \to L$, combined with (A.3), induces (A.2). ☐

A.2. **The case of vector bundles.** Let $\tau\colon V \to N$ be a vector bundle, and $o\colon N \to V$ its zero section. Let $\dot{V} = V \setminus o(N)$, and consider the quotient $\gamma\colon \dot{V} \to S_N V$ by the $\mathbb{R}^{\times}_{>0}$-action.

Consider the projections

$$V \xleftarrow{p_1} V \times_N \dot{V} \xrightarrow{p_2} \dot{V}\ .$$

For $K \in \mathrm{E}^{\mathrm{b}}_{+}(\mathrm{I}\mathbf{k}_V)$ and $C \in \mathrm{E}^{\mathrm{b}}_{+}(\mathrm{I}\mathbf{k}_{V\times_N \dot{V}})$, we set

$$\begin{aligned}\Phi_C(K) &:= \mathrm{E}p_{2!!}\big(C \overset{+}{\otimes} \mathrm{E}p_1^{-1}K\big),\\ \Psi_C(K) &:= \mathrm{E}p_{2*}\mathrm{R}\mathcal{I}hom^{+}(C, \mathrm{E}p_1^{!}K).\end{aligned}$$

Lemma A.6. *Let $K \in \mathrm{E}^{\mathrm{b}}_{(\mathbb{R}^{\times}_{>0})_\infty}(\mathrm{I}\mathbf{k}_V)$. With the identifications $N \simeq o(N) \subset V$ and $T_N V \simeq V$, one has*

$$\begin{aligned}\mathrm{E}\gamma^{-1}\mathrm{E}\lambda^{\mathsf{rb}}_N(K) &\simeq \Phi_C(K),\\ \mathrm{E}\gamma^{-1}\mathrm{E}\widetilde{\lambda}^{\mathsf{rb}}_N(K) &\simeq \Psi_C(K),\end{aligned}$$

for $C = \epsilon(\mathbf{k}_B)[1]$, with $B = \{(x,y) \in V \times_N \dot{V}\,;\, x = \lambda y,\ \text{for some } \lambda \geqslant 0\}$ a closed subset of $V \times_N \dot{V}$.

Proof. Since the proofs are similar, let us only discuss the first isomorphism.

Consider the morphisms

$$\begin{array}{ccccccc} \dot{V} & \xleftarrow{q_1} & \dot{V} \times (\mathbb{R}_{\geqslant 0})_\infty & \xrightarrow{\tilde{\gamma}} & V^{\mathsf{rb}}_N & \xrightarrow{p} & V \\ \uparrow{\scriptstyle \tilde{q}_1} & \swarrow{\scriptstyle \tilde{j}} & \uparrow{\scriptstyle i_0} & \square & \uparrow{\scriptstyle i} & & \\ \dot{V} \times \mathbb{R}_\infty & \xleftarrow{\tilde{i}_0} & \dot{V} & \xrightarrow{\gamma} & S_N V, & & \end{array} \qquad (\tilde{p} = p\circ\tilde{\gamma}\colon \dot{V} \times (\mathbb{R}_{\geqslant 0})_\infty \to V)$$

where $i_0(x) = (x,0)$. One has

$$\begin{aligned}\mathrm{E}\gamma^{-1}\mathrm{E}\lambda^{\mathsf{rb}}_N(K) &\simeq \mathrm{E}\gamma^{!}\mathrm{E}i^{!}\mathrm{E}p^{-1}K \simeq \mathrm{E}i_0^{!}\mathrm{E}\tilde{\gamma}^{!}\mathrm{E}p^{-1}K\\ &\underset{(*)}{\simeq} \mathrm{E}i_0^{!}\mathrm{E}\tilde{\gamma}^{-1}\mathrm{E}p^{-1}K[1] \simeq \mathrm{E}i_0^{!}\mathrm{E}\tilde{p}^{-1}K[1]\\ &\simeq \mathrm{E}i_0^{!}\mathrm{E}\tilde{j}^{!}\mathrm{E}\tilde{j}_{!!}\mathrm{E}\tilde{p}^{-1}K[1] \simeq \mathrm{E}\tilde{i}_0^{!}\mathrm{E}\tilde{j}_{!!}\mathrm{E}\tilde{p}^{-1}K[1]\\ &\underset{(**)}{\simeq} \mathrm{E}\tilde{q}_{1!!}\mathrm{E}\tilde{j}_{!!}\mathrm{E}\tilde{p}^{-1}K[1] \simeq \mathrm{E}q_{1!!}\mathrm{E}\tilde{p}^{-1}K[1],\end{aligned}$$

where $(*)$ is due to the fact that $\tilde{\gamma}$ is an $(\mathbb{R}^{\times}_{>0})_\infty$-bundle, and $(**)$ holds because $\mathrm{E}\tilde{j}_{!!}\mathrm{E}\tilde{p}^{-1}K$ is $(\mathbb{R}^{\times}_{>0})_\infty$-conic with respect to the action on the second factor of $\dot{V} \times \mathbb{R}_\infty$.

It follows that $\mathrm{E}\gamma^{-1}\mathrm{E}\lambda^{\mathsf{rb}}_N(K) \simeq \Phi_C(K)$ for $C := \mathrm{R}(\tilde{p},q_1)_!\mathbf{k}_{\dot{V}\times\mathbb{R}_{\geqslant 0}}[1]$. Since $(\tilde{p},q_1)$ decomposes into

$$(\tilde{p},q_1)\colon \dot{V}\times\mathbb{R}_{\geqslant 0} \xrightarrow{\sim} B \hookrightarrow V\times_N \dot{V}\,,$$

we have $\mathrm{R}(\tilde{p},q_1)_!\mathbf{k}_{\dot{V}\times\mathbb{R}_{\geqslant 0}} \simeq \mathbf{k}_B$. □

A.3. **Blow-up and vanishing cycles.** Let X be a smooth complex curve, and $a\in X$. Let z be a coordinate on the complex vector line T_aX, and w the dual coordinate on T^*_aX, so that the pairing $T_aX\times T^*_aX\to\mathbb{C}$ is given by $(z,w)\mapsto zw$. Then, the homeomorphism

$$c\colon \dot{T}_aX\to \dot{T}^*_aX,\quad z\mapsto -z^{-1}$$

does not depend on the choice of the coordinate, and induces a homeomorphism

$$c\colon S_aX \xrightarrow{\sim} S^*_aX.$$

Lemma A.7. *For $K\in\mathrm{E}^{\mathrm{b}}_+(\mathrm{I}\mathbf{k}_{\mathbb{V}})$, there is a natural morphism*

$$\mathrm{E}c^{-1}\mathrm{E}\mu^{\mathsf{sph}}_{\{0\}}(K)[1]\to\mathrm{E}\lambda^{\mathsf{rb}}_{\{0\}}(K).$$

Proof. Since c is an isomorphism and γ^{-1} is fully faithful, it is enough to show that there is a natural morphism

$$\mathrm{E}u^{-1}\mathrm{E}\mu_{\{0\}}(K)[1]\to\mathrm{E}c_*\mathrm{E}\gamma^{-1}\mathrm{E}\lambda^{\mathsf{rb}}_{\{0\}}(K).$$

Write $L=\mathrm{E}\nu_{\{0\}}(K)$. By Lemma A.5, it is equivalent to prove that there is a natural morphism in $\mathrm{E}^{\mathrm{b}}_+(\mathrm{I}\mathbf{k}_{(\dot{T}^*_aX)_\infty})$

$$\mathrm{E}u^{-1}L^\wedge[1]\to\mathrm{E}c_*\mathrm{E}\gamma^{-1}\mathrm{E}\lambda^{\mathsf{rb}}_{\{0\}}(L). \tag{A.4}$$

Consider the subsets of $\mathbb{V}\times\dot{\mathbb{V}}^*$

$$F=\{\mathrm{Re}\,zw\leqslant 0\},\quad G=\{\mathrm{Re}\,zw\leqslant 0,\ \mathrm{Im}\,zw=0\}.$$

The inclusion of closed subsets $G\subset F$ gives a morphism

$$\Phi_{\epsilon(\mathbf{k}_F)}(L)\to\Phi_{\epsilon(\mathbf{k}_G)}(L). \tag{A.5}$$

Then we obtain (A.4) by applying $\mathrm{E}u^{-1}$ to (A.5). In fact, on one hand, recalling the notations on the enhanced Fourier-Sato transforms from §2.3, one has

$$\mathrm{E}u^{-1}L^\wedge[1]\simeq\mathrm{E}u^{-1}\Phi_{\epsilon(\mathbf{k}_F)}(L)[1].$$

On the other hand, one has $G\cap(\mathbb{V}\times\dot{\mathbb{V}}^*)=\{(z,w);\ z=-\lambda w^{-1},\ \exists\lambda\geqslant 0\}$. Hence $\mathrm{E}c_*\mathrm{E}\gamma^{-1}\mathrm{E}\lambda^{\mathsf{rb}}_{\{0\}}(L)\simeq\mathrm{E}u^{-1}\Phi_{\epsilon(\mathbf{k}_G)}(L)[1]$ by Lemma A.6. □

Proposition A.8. *Let $K\in\mathrm{E}\text{-}\mathrm{Perv}(\mathrm{I}\mathbf{k}_X)$. Then there are natural isomorphisms in $\mathrm{E}^{\mathrm{b}}_+(\mathrm{I}\mathbf{k}_{S_aX})$*

$$\mathrm{E}\widetilde{\lambda}^{\mathsf{rb}}_{\{a\}}(K)\xleftarrow{\sim}\mathrm{E}c^{-1}\mathrm{E}\mu^{\mathsf{sph}}_{\{a\}}(K)[1]\xrightarrow{\sim}\mathrm{E}\lambda^{\mathsf{rb}}_{\{a\}}(K).$$

In particular, $\mathrm{E}\lambda^{\mathsf{rb}}_{\{a\}}(K) \simeq \mathrm{E}\widetilde{\lambda}^{\mathsf{rb}}_{\{a\}}(K)$ *is of sheaf type, and its associated sheaf is a local system.*

Proof. (i) Let us show that the first isomorphism follows by duality from the second one.

One has

$$\mathrm{E}\widetilde{\lambda}^{\mathsf{rb}}_{\{a\}}(K) \simeq \mathrm{E}\widetilde{\lambda}^{\mathsf{rb}}_{\{a\}}(\mathrm{D^E D^E}K) \underset{(*)}{\simeq} \mathrm{D^E}\big(\mathrm{E}\lambda^{\mathsf{rb}}_{\{a\}}(\mathrm{D^E}K)[-1]\big),$$
$$\mathrm{E}c^{-1}\mathrm{E}\mu^{\mathsf{sph}}_{\{a\}}(K) \simeq \mathrm{E}c^{-1}\mathrm{E}\mu^{\mathsf{sph}}_{\{a\}}(\mathrm{D^E D^E}K) \underset{(**)}{\simeq} \mathrm{D^E}\big(\mathrm{E}c^{-1}\mathrm{E}\mu^{\mathsf{sph}}_{\{a\}}(\mathrm{D^E}K)[-1]\big).$$

where $(*)$ follows from Lemma A.4, and $(**)$ from [5, Lemma 4.5].

(ii) Let us prove the first isomorphism. Set $\mathbb{V} = T_aX$ and $\mathbb{V}^* = T^*_aX$. By Proposition 3.7, one has $\mathrm{E}\nu_{\{a\}}(K) \simeq e(F)$ for some $F \in \mathrm{Perv}(\mathbf{k}_\mathbb{V}) \cap \mathrm{D}^{\mathrm{b}}_{\mathbb{R}^\times_{>0}}(\mathbf{k}_\mathbb{V})$. By [5, Lemma 4.10] and Lemma A.5, we may take $X = \mathbb{V}$, $a = 0$, and $K = e(F)$. Hence, we are reduced to prove that the morphism

$$\mathrm{E}c^{-1}\mathrm{E}\mu^{\mathsf{sph}}_{\{0\}}(e(F))[1] \to \mathrm{E}\lambda^{\mathsf{rb}}_{\{0\}}(e(F)),$$

from Lemma A.7, is an isomorphism. One has

$$\mathrm{E}c^{-1}\mathrm{E}\mu^{\mathsf{sph}}_{\{0\}}(e(F)) \simeq e\big(c^{-1}\mu^{\mathsf{sph}}_{\{0\}}(F)\big),$$
$$\mathrm{E}\lambda^{\mathsf{rb}}_{\{0\}}(e(F)) \simeq e\big(\lambda^{\mathsf{rb}}_{\{0\}}(F)\big).$$

Since e is fully faithful, it is enough to show that there is an isomorphism

$$c^{-1}\mu^{\mathsf{sph}}_{\{0\}}(F)[1] \xrightarrow{\sim} \lambda^{\mathsf{rb}}_{\{0\}}(F),$$

which can be checked at the level of stalks.

The underlying real vector spaces to $\mathbb{V}$ and $\mathbb{V}^*$ are in duality by the pairing $\langle v, w\rangle = \mathrm{Re}(zw)$. For $\Gamma \subset \mathbb{V}^*$, the set

$$\Gamma^\circ = \{v \in \mathbb{V};\ \langle v, w\rangle \geqslant 0 \text{ for any } w \in \Gamma\}$$

is called the polar cone of Γ.

For $\theta \in S_aX = S_0\mathbb{V}$, by [9, Theorems 4.2.3, 4.3.2] one has

$$\begin{aligned}
\big(\nu^{\mathsf{sph}}_{\{0\}}(F)\big)_\theta &\simeq \varinjlim_{\Lambda, r} \mathrm{RHom}\,(\mathbf{k}_{\Lambda\cap\{|z|<r\}}, F) \\
&\underset{(*)}{\simeq} \varinjlim_{\Lambda} \mathrm{RHom}\,(\mathbf{k}_{\Lambda}, F), \\
\big(\mu^{\mathsf{sph}}_{\{0\}}(F)\big)_{c(\theta)} &\simeq \varinjlim_{\Gamma, r} \mathrm{RHom}\,(\mathbf{k}_{\Gamma^\circ\cap\{|z|<r\}}, F) \\
&\underset{(*)}{\simeq} \varinjlim_{\Gamma} \mathrm{RHom}\,(\mathbf{k}_{\Gamma^\circ}, F),
\end{aligned}$$

where Λ runs over the open convex proper cones in $\mathbb{V}$ containing θ, Γ runs over the open convex proper cones in $\mathbb{V}^*$ containing $c(\theta)$, and $r \to 0+$. Here, the isomorphisms $(*)$ are due to the fact that F is conic.

It then follows from Lemma A.3 (i) that one has

$$\big(\lambda^{\mathsf{rb}}_{\{0\}}(F)\big)_\theta[-1] \simeq \varinjlim_{\Lambda} \mathrm{RHom}\,(\mathbf{k}_{\mathbb{V}\setminus\Lambda}, F).$$

For any Λ as above, taking $\Gamma = c(\Lambda)$, one has $\lambda_2 := \mathbb{V}\setminus\Lambda \supset \Gamma^\circ =: \lambda_1$. Hence, it is enough to prove

$$\mathrm{RHom}\,(\mathbf{k}_{\lambda_2\setminus\lambda_1}, F) \simeq 0.$$

Consider the maps

$$\mathbb{V} \xleftarrow{\ j\ } \dot{\mathbb{V}} \xrightarrow{\ q\ } S_0\mathbb{V} = \dot{\mathbb{V}}/\mathbb{R}^\times_{>0}.$$

Let $\varnothing \subsetneq I_k \subsetneq S_0\mathbb{V}$ be the closed connected subset such that $\dot{\lambda}_k = q^{-1}(I_k)$, for $k = 1, 2$. Let L be a local system on S_0V such that $j^{-1}F \simeq q^{-1}L[1]$. Then, one has

$$\begin{aligned}\mathrm{RHom}\,(\mathbf{k}_{\lambda_2\setminus\lambda_1}, F) &\simeq \mathrm{RHom}\,(\mathrm{R}j_!\,\mathbf{k}_{\dot{\lambda}_2\setminus\dot{\lambda}_1}, F) \simeq \mathrm{RHom}\,(\mathbf{k}_{\dot{\lambda}_2\setminus\dot{\lambda}_1}, q^{-1}L[1])\\ &\simeq \mathrm{RHom}\,(\mathbf{k}_{\dot{\lambda}_2\setminus\dot{\lambda}_1}, q^{!}L) \simeq \mathrm{RHom}\,(\mathrm{R}q_!\,\mathbf{k}_{\dot{\lambda}_2\setminus\dot{\lambda}_1}, L)\\ &\simeq \mathrm{RHom}\,(\mathbf{k}_{I_2\setminus I_1}[-1], L).\end{aligned}$$

The last term vanishes, since

$$V \simeq \mathrm{RHom}\,(\mathbf{k}_{I_1}[-1], L) \xrightarrow{\sim} \mathrm{RHom}\,(\mathbf{k}_{I_2}[-1], L) \simeq V,$$

where V is the stalk of L. □

Remark A.9. Let us use notations as in Lemma A.7 and its proof. Then, for $L = \mathrm{E}\nu_{\{0\}}K$ with $K \in \text{E-Perv}(\mathrm{I}\,\mathbf{k}_{\mathbb{V}})$, the morphism (A.5) is an isomorphism. In fact, on one hand, $\mathrm{E}u^{-1}$(A.5) = (A.4) is an isomorphism by Proposition A.8. On the other hand,

$$\mathrm{E}o^{-1}\Phi_{\epsilon(\mathbf{k}_F)}(L) \simeq \mathrm{E}\tau_* L \simeq \mathrm{E}o^{-1}\Phi_{\epsilon(\mathbf{k}_G)}(L),$$

where $o\colon \{0\} \to \mathbb{V}$ is the linear embedding, and $\tau\colon \mathbb{V} \to \{0\}$ its transpose.

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