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On the cohomology of moduli spaces of trigonal curves

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Abstract

The main object of this thesis is the study of some moduli spaces of curves from the point of view of one of the most important topological invariants: rational cohomology, i.e. singular cohomology with rational coefficients.

The moduli space we want to to study is the moduli space $\mathcal{T}_g$ of trigonal curves of genus g. Its coarse moduli space is a quasi-projective algebraic variety which parametrizes complex trigonal curves of fixed genus g, up to isomorphism. A trigonal curve is defined as a smooth irreducible non-hyperelliptic curve admitting a linear system g^1_3 or, equivalently, a degree 3 map to $\mathbf{P}^1$. Hence, $\mathcal{T}_g$ naturally sits inside the moduli space $\mathcal{M}_g$ of complex smooth irreducible curves of genus g, as a locally closed subvariety. Moreover, given a curve in $\mathcal{T}_g$, its trigonal structure defines a natural embedding in some rational geometrically ruled surface, called *Hirzebruch surface.* The degree of the surface is defined as the *Maroni invariant* of the trigonal curve and it determines the *Maroni stratification* of $\mathcal{T}_g$ into locally closed subvarieties.

In order to study the cohomology of $\mathcal{T}_g$, we will study first that of each stratum. We will see that Maroni strata are quotients of complements of discriminants in a given complex vector space by the action of an algebraic group. To compute the cohomology of complements of discriminants we will use Gorinov-Vassiliev's method, specifically Tommasi's adaptation of the method. From the cohomology of the complement of a discriminant and that of the algebraic group acting on it, one can deduce the cohomology of the corresponding stratum thanks to a theorem of Peters and Steenbrink.

In chapter I we will recall the main properties of trigonal curves, together with the techniques mentioned above. These will be applied first in chapter II in order to obtain a full description of the rational cohomology of $\mathcal{T}_5$. Then, in chapter III, we will use the same techniques to generalize the result obtained in the previous chapter to higher genera. This will give us a description of the cohomology of $\mathcal{T}_g$, with $g \geq 6$, in a certain range. In particular, we will prove that its cohomology ring stabilizes to its tautological ring. Penev and Vakil proved that the tautological ring of $\mathcal{T}_g$ coincides with its Chow ring, which is known from a work by Canning and Larson. Finally, in chapter IV, we will discuss the stabilization of the cohomology ring of moduli spaces of smooth curves of higher gonality, also embedded in a given Hirzebruch surface.

Riassunto

L'argomento principale di questa tesi è lo studio di alcuni spazi di moduli di curve dal punto di vista di uno dei più importanti invarianti topologici: la coomologia razionale, i.e. la coomologia singolare con coefficienti razionali.

Lo spazio di moduli che studieremo è lo spazio di moduli $\mathcal{T}_g$ delle curve trigonali di genere g, una varietà quasi-proiettiva che parametrizza curve trigonali complesse di genere fissato g, a meno di isomorfismo. Una curva trigonale è definita come una curva liscia irriducibile non iperellittica che ammette un sistema lineare g^1_3 o, equivalentemente, una mappa di grado 3 sulla retta proiettiva $\mathbf{P}^1$. Dunque, $\mathcal{T}_g$ è naturalmente contenuto nello spazio di moduli $\mathcal{M}_g$ di curve complesse lisce irriducibili di genere g, come una sottovarietà localmente chiusa. Inoltre, data una curva in $\mathcal{T}_g$, la sua struttura trigonale definisce un'immersione naturale in una superficie razionale geometricamente rigata, cioè una *superficie di Hirzebruch*. Il grado di queste superfici è chiamato *invariante di Maroni* della curva e questo definisce la *stratificazione di Maroni* di $\mathcal{T}_g$ in sottovarietà localmente chiuse.

Al fine di studiare la coomologia di $\mathcal{T}_g$, studieremo prima quella di ciascuno strato. Vedremo che gli strati di Maroni sono quozienti del complementare di un discriminante in un dato spazio vettoriale complesso per l'azione di un gruppo algebrico. Per calcolare la coomologia di complementari di discriminanti useremo il metodo di Gorinov-Vassiliev, precisamente la versione del metodo di Tommasi. A partire dalla coomologia del complementare di un discriminante e quella del gruppo algebrico che agisce su di esso, si deduce la coomologia del corrispondente strato grazie ad un teorema di Peters e Steenbrink.

Nel capitolo I ricorderemo le principali proprietà delle curve trigonali, assieme alle tecniche appena menzionate. Queste tecniche verranno applicate prima nel capitolo II, al fine di ottenere una descrizione completa della coomologia razionale di $\mathcal{T}_5$. Successivamente, nel capitolo III, utilizzeremo le stesse tecniche per generalizzare il risultato ottenuto nel capitolo precedente per generi più alti. Questo ci darà una descrizione della coomologia di $\mathcal{T}_g$, con $g \geq 6$, in un certo intervallo. In particolare, dimostreremo che il suo anello di coomologia si stabilizza al suo anello tautologico. Penev e Vakil hanno dimostrato che l'anello tautologico di $\mathcal{T}_g$ coincide con il suo anello di Chow, il

quale è noto da un lavoro di Canning e Larson. Infine, nel capitolo IV, discuteremo la stabilizzazione dell'anello di coomologia di spazi di moduli di curve lisce di gonalità più alta, anch'esse immerse in una data superficie di Hirzebruch.

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Contents

Introduction

Overview

The moduli space $\mathcal{M}_g$ of complex non-singular curves of genus g is a central object in algebraic geometry. Nonetheless only few general statements about its geometry are known. More precisely, the existence of the coarse moduli space $\mathcal{M}_g$ of complex dimension $3g-3$ that parametrizes isomorphism classes of complex non-singular curves of genus g was first known thanks to Mumford's Geometric Invariant Theory [Mum65]. Later, for $g \geq 2$, Deligne and Mumford [DM69] not only proved that $\mathcal{M}_g$ is irreducible but they also introduced the important notion of a Deligne-Mumford stack. In particular, Knudsen [Knu83] then proved that $\mathcal{M}_g$ is quasi-projective.

These moduli spaces are singular and, for $g \geq 4$, their singular loci correspond to isomorphism classes of curves having non-trivial automorphisms, [ACGH85], [Cor87]. On the other hand, if we use the language of stacks, it turns out that the moduli spaces $\mathcal{M}_g$ are smooth irreducible DM-stacks.

The moduli spaces $\mathcal{M}_g$ have been extensively studied in the last few decades and some results about their geometry are well known. Harer, Arbarello and Cornalba, in [Har83] and [AC87], computed the Picard group of the moduli stack of smooth curves of genus $g \geq 3$ and they proved that it is a free abelian group generated by a single element. Harris, Mumford and Eisenbud proved in [HM82],[Har84],[EH87], that $\mathcal{M}_g$ is of general type for all $g \geq 24$, and it has Kodaira dimension $K_g = 3g-3$ if $g \geq 24$ and $K_{23} \geq 1$. More recently, Farkas improved this result proving that $K_{23} \geq 2$, [Far00], and, together with Jensen and Payne, they proved that $\mathcal{M}_{22}$ is also of general type, [FJP20]. It is also know that $\mathcal{M}_g$ is unirational for $g \leq 14$, [AC81], [Ser81], [CR84], [Ver05].

Another way to understand these spaces is by computing some topological invariants, such as their cohomology groups. In particular we are interested in their cohomology with rational coefficients and since there is an isomorphism between rational cohomology groups of a DM-quotient stack and of its underlying coarse moduli space, [Edi13], we will abuse notation and denote by $\mathcal{M}_g$ both the DM-stack and the coarse moduli space, without distinguishing them. In the last few decades, there has been a considerable progress in the understanding of the rational cohomology ring of $\mathcal{M}_g$. What is known until know, for general values of g, is mostly due to Harer, Mumford, Madsen and Weiss. Harer proved in [Har85] that the cohomology ring $H^i(\mathcal{M}_g; \mathbf{Q})$ is independent of the genus g for $g \geq 3i - 1$. Harer's stability bound was later refined by Ivanov and Boldsen, [Iva89], [Bol12], who proved that the optimal bound is $2g \geq 3i + 2$. This allows us to define the stable cohomology ring $H^\bullet(\mathcal{M}; \mathbf{Q})$ as $H^\bullet(\mathcal{M}_g; \mathbf{Q})$ for a sufficiently large g. Mumford also conjectured in [Mum83] that the stable cohomology ring is generated by tautological classes. This conjecture was later proved by Madsen and Weiss in [MW07] using topological techniques.

The tautological ring of the moduli space of curves

The tautological ring, denoted by $R^\bullet(\mathcal{M}_g)$ is the subring of the Chow ring $A^\bullet(\mathcal{M}_g)$ of the moduli space of smooth curves of genus g generated by *tautological classes* $\kappa_i \in A^i(\mathcal{M}_g)$. These are classes which "naturally come from geometry". Specifically, they are defined as Chern classes of some natural vector bundles on $\mathcal{M}_g$.

Let $\mathcal{C}_g = \mathcal{M}_{g,1}$ be the *universal curve*, or equivalently, the moduli space of 1-pointed smooth curves of genus g. Let us also denote by $\pi : \mathcal{C}_g \to \mathcal{M}_g$ the natural morphism forgetting the marked point, by ω_π its relative dualizing sheaf and $K = c_1(\omega_\pi) \in A^1(\mathcal{C}_g)$. Then

$$\kappa_i := \pi_*(K^{i+1}) \in A^i(\mathcal{M}_g).$$

For their properties and relations in the tautological ring we refer to [Fab99].

These classes were first introduced by Mumford, who proved in [Mum83] that the tautological ring is generated by the kappa classes $\kappa_1, \ldots, \kappa_{g-2}$.

Furthermore, Mumford conjectured that the stable cohomology of $\mathcal{M}_g$ coincides with the image of the tautological ring through the cycle-class map $A^\bullet(\mathcal{M}_g) \to H^{2\bullet}(\mathcal{M}_g; \mathbf{Q})$. Initially this conjecture has been partially proved by Miller [Mil86], then Madsen and Weiss [MW07] provided a complete proof.

Furthermore, for fixed values of g, we have a complete description of the rational cohomology ring of $\mathcal{M}_g$ with $g = 2, 3, 4$, due to the works of Mumford [Mum83], Looijenga [Loo93] and Tommasi [Tom05b], respectively. Unfortunately, for $g \geq 5$ the full rational cohomology ring of $\mathcal{M}_g$ is still unknown.

The moduli space of trigonal curves

One way to approach this problem is to compute first the cohomology of some loci inside $\mathcal{M}_g$. Let us assume $g \geq 3$, the moduli space $\mathcal{M}_g$ has a standard stratification given by gonality. The gonality of a curve C is the smallest positive integer d such that C has a g^1_d and $\mathcal{M}_g$ can be stratified as

$$\mathcal{M}^1_{g,2} \subseteq \mathcal{M}^1_{g,3} \subseteq \cdots \subseteq \mathcal{M}_g,$$

where $\mathcal{M}^1_{g,d} = \{[C] \in \mathcal{M}_g | C \text{ has a } g^1_d\}$ is an irreducible variety of dimension $2d+2g-5$ if $d \leq g/2+1$ and $\mathcal{M}^1_{g,d} = \mathcal{M}_g$ as soon as $d \geq g/2+1$, [ACG11, XXI.11]. In other words, if the Brill-Noether number $2d - g - 2$ is negative then the general curve of genus g has no g^1_d. Thus, curves with negative Brill-Noether number correspond to special points in the moduli space $\mathcal{M}_g$ and these are indeed the curves we will be interested in.

In fact, the aim of this thesis is to study the rational cohomology of the locus $\mathcal{T}_g = \mathcal{M}^1_{g,3} \backslash \mathcal{M}^1_{g,2} \subset \mathcal{M}_g$ of trigonal curves, i.e. smooth non-hyperelliptic curves with a g^1_3, of genus g. The (coarse) moduli space $\mathcal{T}_g$, for $g \geq 4$, is then an irreducible variety of dimension $2g + 1$ with finite quotient singularities, which correspond to isomorphism classes of curves having a non-trivial automorphism group, [ELSV01].

Moreover, Arbarello and Cornalba proved that $\mathcal{T}_g$ is unirational for any g, [AC81], and later Stankova proved in [SF00] that its rational Picard group is freely generated by the tautological class κ_1.

Let us observe that it makes sense to consider tautological classes on $\mathcal{T}_g$ as well. Indeed, when considering a subvariety of $\mathcal{M}_g$, we will say that a class is tautological if it comes from a tautological class of $\mathcal{M}_g$, through the pullback of the restriction map.

For $g \leq 4$, the moduli space $\mathcal{T}_g$ is not really interesting since it is either empty or it coincides with the complement of the hyperelliptic locus $\mathcal{H}_g$, whose rational cohomology is completely known. Thus, $g = 5$ represents the first case in which the cohomology of $\mathcal{T}_g$ cannot be automatically determined from that of $\mathcal{H}_g$ and of $\mathcal{M}_g$. Note also that the moduli space $\mathcal{T}_g$ has the structure of a DM-stack, by being a locally closed substack of a DM-stack, hence, also in this case, we will not distinguish between the DM-stack and its underlying coarse moduli space.

Outline of the results

The results that we will produce in this thesis are based on the canonical embedding of trigonal curves in Hirzebruch surfaces and on the relation between the rational cohomology of their moduli spaces and the cohomology of complements of discriminants.

In fact, the computation of the rational cohomology of $\mathcal{T}_g$ can be first reduced to that of the strata of the so-called Maroni stratification, which is defined by the degree of the Hirzebruch surfaces that naturally contain trigonal curves as divisors.

In the genus 5 case, which will be discussed in chapter II, we will see that this stratification consists of only one stratum, therefore the cohomology of this stratum will automatically give that of the whole moduli space $\mathcal{T}_5$. For higher genera, the stratification consists of more strata, thus we will have to consider all of them.

To be more precise, the cohomology of each stratum is strictly related to that of the complement of the discriminant of some complex vector space. This vector space V is the vector space of global sections of a vector bundle over the Hirzebruch surface $\mathbb{F}$ defining the corresponding stratum. Then, any element $f \in V$ can be associated to a subvariety in $\mathbb{F}$ defined by the vanishing locus of f and the discriminant Σ is the subspace in V of elements defining singular subvarieties, or the whole of $\mathbb{F}$ in the case $f = 0$.

With these definitions we will see that the cohomology of each stratum is precisely the cohomology of the quotient of the complement of the discriminant by the action of a given algebraic group. We will also prove, using a theorem of Peters and Steenbrink [PS03], that the cohomology of this quotient is isomorphic to the tensor product of the cohomology of the total space and that of a subgroup of the group acting on it. Finally, the cohomology of complements of discriminants will be computed using Gorinov-Vassiliev's method [Vas99],[Gor05],[Tom05b], which allows us to compute the Borel-Moore homology of Σ by constructing a simplicial resolution of it, starting from a classification of all singular loci of the elements that it contains. By Alexander's duality this is equivalent to the cohomology of its complement $V \setminus \Sigma$.

We will apply these techniques to study the rational cohomology ring of $\mathcal{T}_g$ and provide a complete description for the case $g = 5$, while, for higher genera, we will prove that $H^i(\mathcal{T}_g;\mathbf{Q})$ is independent of g for $g >> i$. By comparing the description that we obtain with the results of [PV15a], [CL21b] and [CL21a] we deduce that the cohomology ring $H^i(\mathcal{T}_g;\mathbf{Q})$, with i in the stable range, coincides with the image of tautological ring, through the cycle class map. Thus, analogously to moduli spaces of smooth curves, this allows us to define the stable cohomology ring $H^\bullet(\mathcal{T};\mathbf{Q})$ as $H^\bullet(\mathcal{T}_g;\mathbf{Q})$ for g sufficiently large.

Finally, we will also study the stabilization of the cohomology of some other moduli spaces of curves, defined as smooth sections of a vector bundle defined on a given Hirzebruch surface. More precisely, we will prove the stabilization of their cohomology, by extending the result obtained for the moduli space of trigonal curves.

Chapter I

Preliminaries

1 Notation and conventions

Symbol	Description
$\mathbf{C}^n$	n-dimensional complex vector space
$\mathbf{A}^n$	n-dimensional complex affine space
$\mathbf{P}^n$	n-dimensional complex projective space
$\mathbf{P}(w_1, w_2, w_3)$	weighted projective plane of weight $w = (w_1, w_2, w_3)$
$\mathfrak{S}_n$	symmetric group on n elements
$\mathbb{S}_\lambda$	irreducible representation of $\mathfrak{S}_n$ associated with the partition $\lambda \dashv n$
GL_n	general linear group of degree n over $\mathbf{C}$
PGL_n	projective linear group of degree n over $\mathbf{C}$
SL_n	special linear group of degree n over $\mathbf{C}$
$F(Z; k)$	space of ordered configurations of k points on Z
$B(Z; k)$	space of unordered configurations of k points on Z
$H^\bullet(Z; \mathcal{L})$	cohomology of Z with coefficients in the local system $\mathcal{L}$
$\tilde{H}^\bullet(Z; \mathcal{L})$	reduced cohomology of Z with coefficients in the local system $\mathcal{L}$
$\bar{H}_\bullet(Z; \mathcal{L})$	Borel-Moore homology of Z with coefficients in the local system $\mathcal{L}$
$\mathsf{HS}_{\mathbf{Q}}$	category of rational pure Hodge structures
$\mathsf{MHS}_{\mathbf{Q}}$	category of rational mixed Hodge structures
$K_0(\mathsf{C})$	Grothendieck group of the abelian category C
$\mathbf{Q}(k)$	rational Tate Hodge structure of weight $-2k$
$\mathbf{L}$	class of the Tate Hodge structure $\mathbf{Q}(-1)$ in $K_0(\mathsf{HS}_{\mathbf{Q}})$

Throughout this thesis, we will work over the field of complex numbers $\mathbf{C}$. Unless otherwise specified, a smooth curve will denote a non-singular, irreducible, projective variety of dimension 1. Moreover, all cohomology groups will be considered with rational coefficients. By Deligne's Hodge theory, cohomology and Borel-Moore homology of complex quasi-projective varieties carry mixed Hodge structures. In particular, we will work with mixed Hodge structures which are extensions of rational Tate Hodge structures. The rational cohomology of a given space can be described using its Hodge-Grothedieck polynomial, whose i-th coefficient corresponds to the degree i cohomology group of the space.

Definition 1. Let $T_\bullet$ be a graded $\mathbf{Q}$-vector space with mixed Hodge structures. Then we define the Hodge-Grothendieck polynomial of $T_\bullet$ as

$$P(T_\bullet;\mathbf{Q}) := \sum_{i\in\mathbf{Z}} [T_i]\, t^i \in K_0(\mathsf{HS}_{\mathbf{Q}})\,[t]\,. \tag{I.1}$$

Notice that the Grothendieck groups $K_0(\mathsf{HS}_{\mathbf{Q}})$ and $K_0(\mathsf{MHS}_{\mathbf{Q}})$ are the same. Furthermore, we will simply write $P(X;\mathbf{Q})$ when $T_\bullet = H^\bullet(X;\mathbf{Q})$ is the cohomology of X. We will write $\bar{P}(X;\mathbf{Q})$ when $T_\bullet = \bar{H}_\bullet(X;\mathbf{Q})$ is the Borel-Moore homology of X, defined as the homology with locally finite support, [Ful98, Chapter 19].

2 Trigonal curves and the Maroni stratification

In this section we present the moduli space we want to study, namely that of trigonal curves. We will also discuss its stratification by the so-called *Maroni invariant*, which is defined by the natural embedding of trigonal curves in Hirzebruch surfaces.

Definition 2. A *trigonal* curve is a smooth non-hyperelliptic curve admitting a g^1_3.

Let us consider first trigonal curves of low genera. For $g = 2$, it is well known that any curve is hyperelliptic. Denote then by C a smooth non-hyperelliptic curve of genus $g \geq 3$. If $g = 3, 4$ then C is trigonal, [Har77, IV.5.5.2]. Precisely, if $g = 3$ then C canonically embeds in $\mathbf{P}^2$ as a quartic curve: projecting from any point of the curve to $\mathbf{P}^1$ defines a g^1_3, hence C has infinitely many g^1_3. If $g = 4$, the canonical embedding of C in $\mathbf{P}^3$ is the complete intersection of a cubic surface either with a non-singular quadric surface or with a quadric cone. Each of the rulings of the quadric surface on which C lies, singular or not, cuts out a g^1_3.

Thus, if we define the moduli space $\mathcal{T}_g$ as the locus of trigonal curves in the moduli space $\mathcal{M}_g$ of smooth curves of genus g, for $g = 3, 4$, the locus $\mathcal{T}_g$ is open dense in $\mathcal{M}_g$ and it coincides with the complement of the hyperelliptic locus $\mathcal{H}_g$. Recall that the moduli space $\mathcal{H}_g$, with $g \geq 2$, has always the cohomology of a point. This follows from the fact that any hyperelliptic curve has precisely $2g + 2$ distinct branch points, and thus the coarse moduli space of $\mathcal{H}_g$ is isomorphic to the quotient of the moduli space $\mathcal{M}_{0,2g+2}$ of $2g + 2$-pointed genus 0 curves by the action of the symmetric group $\mathfrak{S}_{2g+2}$, which has the cohomology of a point by [KL02, Theorem 2.13].

On the other hand, the cohomology of $\mathcal{T}_3$ and $\mathcal{T}_4$ has been completely determined by Looijenga [Loo93, (4.7)] and Tommasi [Tom05b, Theorems 1.2 and 1.3], respectively. Precisely, we have that

$$H^i(\mathcal{T}_3; \mathbf{Q}) = \begin{cases} \mathbf{Q}, & i = 0; \\ \mathbf{Q}(-6), & i = 6; \\ 0, & \text{otherwise}; \end{cases} \tag{I.2}$$

$$H^i(\mathcal{T}_4; \mathbf{Q}) = \begin{cases} \mathbf{Q}, & i = 0; \\ \mathbf{Q}(-1), & i = 2; \\ \mathbf{Q}(-3), & i = 5; \\ 0, & \text{otherwise}. \end{cases} \tag{I.3}$$

Equivalently, their Hodge-Grothendieck polynomials are

$$P(\mathcal{T}_3; \mathbf{Q}) = \mathbf{L}^6 t^6 + 1,$$

$$P(\mathcal{T}_4; \mathbf{Q}) = \mathbf{L}^3 t^5 + \mathbf{L} t^2 + 1.$$

From now on, we assume $g \geq 5$. In this case a non-hyperelliptic curve is not necessarily trigonal [Har77, IV.5.5.3].

The moduli space $\mathcal{T}_g$ of trigonal curves of genus g is then contained in the moduli space $\mathcal{M}_g$ of smooth curves of genus g as a locally closed subset of dimension $2g + 1$.

In fact, let us consider the open dense subset $\mathcal{T}_g^\circ \subset \mathcal{T}_g$ parametrizing curves whose degree 3 cover of $\mathbf{P}^1$ is simply branched. By Riemann-Hurwitz formula, any of these covers must be ramified at $2g + 4$ points. Up to isomorphism, there are only finitely many triple covers with same branch locus. Thus, we have

$$\dim \mathcal{T}_g = 2g + 4 - \dim PGL_2 = 2g + 1.$$

Remark 1. Let us remark that we can relate the moduli space $\mathcal{T}_g$ of trigonal curves to the Hurwitz scheme $\mathcal{H}_{3,g}$, parametrizing pairs (C, α), consisting of a smooth curve C of genus g and a degree 3 cover $\alpha : C \to \mathbf{P}^1$, up to isomorphism. For the main properties of the Hurwitz scheme, we refer to [HM98, I.G]. The scheme $\mathcal{H}_{3,g}$ naturally maps to the moduli space $\mathcal{M}_g$ via the forgetful map $\pi : (C, \alpha) \mapsto C$. Clearly, its image through π must contain the trigonal locus $\mathcal{T}_g$. Conversely, to get $\mathcal{H}_{3,g}$ from $\mathcal{T}_g$, we need to take into account the maps α, or equivalently, the linear systems g^1_3 on each curve. If the g^1_3 is unique, the map π is a 1-1 correspondence and this is always the case when $g \geq 5$, see [ACGH85, III.B-3.(i)]. Thus, for $g \geq 5$, we have $\mathcal{H}_{3,g} \cong \mathcal{T}_g$.

For $g = 3$, we know from the discussion above that any point of the curve identifies a g^1_3, hence $\mathcal{H}_{3,3}$ is isomorphic to the trigonal locus inside the moduli space $\mathcal{M}_{3,1}$ of genus 3 curves with one marked point.

For $g = 4$, the g^1_3 is uniquely determined by the choice of a ruling on the surface on which the curve lies. Thus, the forgetful map $\pi : \mathcal{H}_{3,4} \to \mathcal{T}_4$ is a double cover ramified over the locus of curves whose canonical model lies on a quadric cone, i.e. curves with a vanishing theta-null.

Thus, the rational cohomology of $\mathcal{H}_{3,3}$ and $\mathcal{H}_{3,4}$ can be easily deduced from the results in [BT07], [Tom05b].

2.1 Canonical embedding and Hirzebruch surfaces

As we have already anticipated, our study of trigonal curves is mainly based on the study of the surfaces on which they lie. Therefore this subsection is devoted to a discussion of these surfaces and their geometry.

It is well known from the classical results of Max Noether, Enriques, Babbage and Petri [ACGH85, III.3] that the canonical model of a curve is contained in the intersection of linearly independent quadrics and, if the curve is trigonal, it lies on a rational normal scroll, i.e the image of a rational ruled surface over $\mathbf{P}^1$ through its embedding in some projective space $\mathbf{P}^N$, [Har77, Cor 2.19].

Here is the idea behind this canonical embedding. Let C be a trigonal curve of genus g and K its canonical divisor. Consider then the canonical embedding

$$\phi_K : C \hookrightarrow \mathbf{P}^{g-1},$$

and let $D = p_1 + p_2 + p_3$ be any divisor of the pencil g^1_3. By geometric Riemann-Roch, the dimension of the linear system containing D equals the number of independent linear relations on the points p_i on the canonical curve, i.e.

$$\dim \overline{\phi_K(D)} = 1.$$

Hence the images of p_i are three collinear points and any quadric containing these three points must also contain the whole line passing through them. This means that the canonical curve lies on a rational ruled surface whose ruling cuts out the g^1_3.

Moreover, it is also well known, see for instance [Har77, V.2.13], that any rational ruled surface over $\mathbf{P}^1$ must be isomorphic to a Hirzebruch surface, first introduced by Hirzebruch in [Hir51].

Definition 3. The surface $\mathbb{F}_n := \mathbf{P}(\mathcal{O}_{\mathbf{P}^1} \oplus \mathcal{O}_{\mathbf{P}^1}(n))$, with $n \in \mathbf{Z}_{\geq 0}$ is called the n-th *Hirzebruch surface.*

For $n = 0$, it is easy to see that $\mathbb{F}_0 \cong \mathbf{P}^1 \times \mathbf{P}^1$. For $n \geq 1$, one can provide a description for $\mathbb{F}_n$ starting from the weighted projective plane $\mathbf{P}(1, 1, n)$, defined as

$$\mathbf{C}^3 \backslash \{0\} / \sim,$$

where $(x, y, z) \sim (\lambda x, \lambda y, \lambda^n z)$ for any $\lambda \in \mathbf{C}^*$. Equivalently, $\mathbf{P}(1, 1, n)$ is the projective variety $\operatorname{Proj} \mathbf{C}[x, y, z]$ where $\mathbf{C}[x, y, z]$ is the graded ring with $\deg x = \deg y = 1$ and $\deg z = n$.

By [Har77, V.2.11.4], the Hirzebruch surface $\mathbb{F}_n$ can also be described as the blow up of a cone $\mathbb{C} \subset \mathbf{P}^{n+1}$ over a rational normal curve of degree n at its vertex. In turn, $\mathbb{C}$ is the image of the weighted projective plane $\mathbf{P}(1, 1, n)$, via the embedding

$$\begin{aligned} \mathbf{P}(1, 1, n) &\to \mathbf{P}^{n+1} \\ [x, y, z] &\mapsto \left[x^n, x^{n-1} y, \dots, y^n, z\right]. \end{aligned}$$

Hence, we will think of $\mathbb{F}_n$ as the blow up of $\mathbf{P}(1,1,n)$ at its singular point $[0,0,1]$. Notice that for $n=1$ this description further simplifies, as indeed, $\mathbf{P}(1,1,1)$ is just the usual projective plane $\mathbf{P}^2$.

The Picard group of a Hirzebruch surface and its intersection form are well known. By abuse of notation we will identify curves with their classes in the Picard group.

Proposition 2.1 ([Har77, V.2]). *Let $n \geq 0$, and consider the surjective morphism $\pi : \mathbb{F}_n \cong \mathbf{P}(\mathcal{O}_{\mathbf{P}^1} \oplus \mathcal{O}_{\mathbf{P}^1}(n)) \to \mathbf{P}^1$ defining the ruling of $\mathbb{F}_n$,*

1. $\mathrm{Pic}(\mathbb{F}_n) \cong \mathbf{Z}E_n \oplus \mathbf{Z}F_n$*, where E_n is the image of the section $(0,1)$ of $\mathcal{O}_{\mathbf{P}^1} \oplus \mathcal{O}_{\mathbf{P}^1}(n)$, which is the unique irreducible curve of negative self-intersection when $n>0$, and F_n is any fiber of the ruling;*

2. *E_n, F_n satisfy*
$$E_n^2 = -n, \qquad F_n^2 = 0, \qquad E_n \cdot F_n = 1;$$

3. *$K_n \sim -2E_n + (-2-n)F_n$, where K_n denotes the canonical divisor on $\mathbb{F}_n$.*

Remark 2. When $n=0$, $\mathbb{F}_0 \cong \mathbf{P}^1 \times \mathbf{P}^1$ and in this case, E_0 F_0 are lines, each of a distinct ruling in $\mathbf{P}^1 \times \mathbf{P}^1$, both with trivial self-intersection.

By the geometry of Hirzebruch surfaces that we have just recalled, the canonical embedding ϕ_K can be actually made more precise. Namely we can relate the degree n of the Hirzebruch surface with the genus g of the trigonal curve lying on it, by explicitly describing C in terms of the generators of the Picard group of $\mathbb{F}_n$.

Proposition 2.2. *Let C be a trigonal curve of genus g, then it can be embedded in $\mathbb{F}_n$ as a divisor of class*
$$C \sim 3E_n + \frac{g+3n+2}{2}F_n, \tag{I.4}$$
with $g \equiv n$ mod 2 and $0 \leq n \leq (g+2)/3$.

Proof. Let C be a trigonal curve of genus g and $\alpha : C \to \mathbf{P}^1$ be the degree 3 cover. We have already seen at the beginning of this section that C canonically embeds in some Hirzebruch surface. However, let us also notice that this also follows from the Casnati-Ekedahl factorization theorem, [CE96, Theorem 1.3], for which the degree 3 cover defines a unique $\mathbf{P}^1$-bundle $\pi : \mathbf{P}\mathcal{E} \to \mathbf{P}^1$ and an embedding
$$i : C \hookrightarrow \mathbf{P}\mathcal{E}, \tag{I.5}$$

such that $\alpha = \pi \circ i$ and $\mathcal{E}$ is the dual of the *Tschirnhausen module* for α, [Mir85], which is defined as follows.

For any open $U \subset \mathbf{P}^1$, the homomorphism $\mathcal{O}_{\mathbf{P}^1}(U) \to (\alpha_* \mathcal{O}_C)(U)$ is just the composition by α^{-1} and thus yields a short exact sequence

$$0 \to \mathcal{O}_{\mathbf{P}^1} \xrightarrow{\alpha^\#} \alpha_* \mathcal{O}_C \to \mathcal{E}^\vee \to 0.$$

This short exact sequence splits, i.e. $\alpha_* \mathcal{O}_C \cong \mathcal{O}_{\mathbf{P}^1} \oplus \mathcal{E}^\vee$ and $\mathcal{E}^\vee := \operatorname{coker} \alpha^\#$ is a locally free $\mathcal{O}_{\mathbf{P}^1}$-module of rank 2 and it decomposes as $\mathcal{E}^\vee \cong \mathcal{O}_{\mathbf{P}^1}(-a) \oplus \mathcal{O}_{\mathbf{P}^1}(-b)$ for some $a, b \in \mathbf{Z}$. Without loss of generality we will assume $a \leq b$.

Twisting $\mathcal{E}$ by $\mathcal{O}_{\mathbf{P}^1}(-a)$ gives us an isomorphism $\mathbf{P}(\mathcal{O}_{\mathbf{P}^1}(a) \oplus \mathcal{O}_{\mathbf{P}^1}(b)) \cong \mathbf{P}(\mathcal{O}_{\mathbf{P}^1} \oplus \mathcal{O}_{\mathbf{P}^1}(n))$, with $n := b - a \in \mathbf{Z}_{\geq 0}$, and hence the embedding in the n-th Hirzebruch surface.

The first condition on n follows by an application of the Grothendieck-Riemann-Roch formula to α :

$$\operatorname{ch}(\alpha_* \mathcal{O}_C) \cdot \operatorname{td}(T_{\mathbf{P}^1}) = \alpha_*(\operatorname{ch}(\mathcal{O}_C) \cdot \operatorname{td}(T_C)),$$

$$\operatorname{ch}(\alpha_* \mathcal{O}_C) \left(1 - \frac{1}{2} K_{\mathbf{P}^1} + \frac{1}{12}(K_{\mathbf{P}^1}^2 + c_2(T_{\mathbf{P}}^1))\right) = \alpha_* \left(1 - \frac{1}{2} K_C + \frac{1}{12}(K_C^2 + c_2(T_C))\right).$$

Here $c_i(\cdot)$, $\operatorname{ch}(\cdot)$ and $\operatorname{td}(\cdot)$ denote the i-th Chern class, the Chern character and the Todd class, respectively, and T_X is the tangent bundle of X.

Comparing classes in $H^2(\mathbf{P}^1; \mathbf{Z})$ gives us the equality

$$c_1(\alpha_* \mathcal{O}_C) = -\frac{1}{2} \left(\alpha_* K_C - 3 K_{\mathbf{P}^1}\right).$$

Since α is a degree 3 cover from C to $\mathbf{P}^1$, $K_C = \alpha^* K_{\mathbf{P}^1} + R$, where R denotes the ramification divisor. By the projection formula and Riemann-Hurwitz formula we have that

$$c_1(\mathcal{E}^\vee) = c_1(\alpha_* \mathcal{O}_C) = -\frac{1}{2} \alpha_* R = -g - 2,$$

which proves the equality

$$a + b = g + 2, \tag{I.6}$$

and thus n must have the same parity of g.

By Prop. 2.1.1 we can write $C \sim m_1 E_n + m_2 F_n$ for some $m_1, m_2 \in \mathbf{Z}$ and from the trigonal structure of the curve we require $C \cdot F_n = 3$, hence $m_1 = 3$. Then, by applying the genus formula we have that

$$g = 1 + \frac{1}{2}(C^2 + C \cdot K_n),$$

where $K_n \sim -2E_n + (-2-n)F_n$ by Prop 2.1.3. This gives us $m_2 = \frac{g+3n+2}{2}$. Finally, since C, E_n are both smooth irreducible curves, their intersection number must be non-negative, hence $C \cdot E_n = -3n + \frac{g+3n+2}{2} \geq 0$ and $n \leq \frac{g+2}{3}$. □

2.2 Stratification by the Maroni invariant

The explicit characterization of a trigonal curve as a divisor in a Hirzebruch surface that we just gave, allows us to stratify the moduli space $\mathcal{T}_g$. This stratification is called the *Maroni stratification* and it depends on the degrees of the Hirzebruch surfaces on which the curves lie.

Let C be a trigonal curve of genus g, then, from the previous results, we know that it canonically embeds in some Hirzebruch surface $\mathbb{F}_n$, for a unique integer n, as in Proposition 2.2.

Definition 4. The integer n is called the *Maroni invariant.*

The *Maroni stratification* can then be written as follows

$$\begin{cases} \mathcal{N}_s \subset \cdots \subset \mathcal{N}_0 = \mathcal{T}_g, & \text{if } g \text{ is even,} \\ \mathcal{N}_s \subset \cdots \subset \mathcal{N}_1 = \mathcal{T}_g, & \text{if } g \text{ is odd,} \end{cases} \tag{I.7}$$

where s is the largest index with the same parity as g satisfying $s \leq \left\lfloor \frac{g+2}{3} \right\rfloor$ and for all $o \leq n \leq s$, $g \equiv n (\mathrm{mod}\, 2)$ we denote by $\mathcal{N}_n$ the closed subscheme

$$\mathcal{N}_n := \{[C] \in \mathcal{T}_g |\, C \text{ has Maroni invariant } \geq n\} \subseteq \mathcal{T}_g.$$

Let us conclude this section by computing the dimension of each of these strata.

As a consequence of [Mir85, Theorem 3.6], a triple cover $\alpha : C \to \mathbf{P}^1$ defines a unique pair $(\mathcal{E}^\vee, \Psi)$, up to isomorphism, where $\mathcal{E}^\vee \cong \mathcal{O}_{\mathbf{P}^1}(-a) \oplus \mathcal{O}_{\mathbf{P}^1}(-b)$ is the Tschirnhausen module for α and Ψ can be considered as a section of $\Lambda^2\mathcal{E}^\vee \otimes \operatorname{Sym}^3 \mathcal{E}$. Thus,

$$\dim \mathcal{N}_n = \dim \Gamma(\Lambda^2\mathcal{E}^\vee \otimes \operatorname{Sym}^3 \mathcal{E}) - \dim_{\mathbf{P}^1} Aut(\mathcal{E}) - \dim_{\mathbf{C}} Aut(\mathbf{P}^1). \tag{I.8}$$

Consider the first summand in the right hand side of (I.8):

$$\begin{aligned}\Lambda^2\mathcal{E}^\vee \otimes \operatorname{Sym}^3\mathcal{E} &= (\mathcal{O}_{\mathbf{P}^1}(-b-a)) \otimes (\mathcal{O}_{\mathbf{P}^1}(3a) \oplus \mathcal{O}_{\mathbf{P}^1}(2a+b) \oplus \mathcal{O}_{\mathbf{P}^1}(a+2b) \oplus \mathcal{O}_{\mathbf{P}^1}(3b)) \\ &= \mathcal{O}_{\mathbf{P}^1}(2a-b) \oplus \mathcal{O}_{\mathbf{P}^1}(a) \oplus \mathcal{O}_{\mathbf{P}^1}(b) \oplus \mathcal{O}_{\mathbf{P}^1}(2b-a).\end{aligned}$$

Let us recall that a, b are such that $a \leq b$, $b - a = n \geq 0$ and $b + a = g + 2 \geq 0$. So, in particular all sheaves above are twisted by non-negative integers. Thus, the dimension of the first summand in (I.8) is

$$\begin{aligned}\dim \Gamma(\Lambda^2\mathcal{E}\vee \otimes \operatorname{Sym}^3\mathcal{E}) &= \binom{2a-b+1}{1} + \binom{a+1}{1} + \binom{b+1}{1} + \binom{2b-a+1}{1} \\ &= 2b + 2a + 4.\end{aligned}$$

As regards the second summand, recall that, by extending operations on vector spaces, there is a bijection between morphisms of vector bundles $\mathcal{E}, \mathcal{F}$ over the same space and sections of the associated Hom-bundle, which in turn is isomorphic to the tensor bundle $\mathcal{E}^\vee \otimes \mathcal{F}$. Then,

$$Aut_{\mathbf{P}^1}\mathcal{E} \cong \Gamma(\mathcal{E}^\vee \otimes \mathcal{E}) = \mathcal{O}_{\mathbf{P}^1} \oplus \mathcal{O}_{\mathbf{P}^1}(a-b) \oplus \mathcal{O}_{\mathbf{P}^1}(b-a) \oplus \mathcal{O}_{\mathbf{P}^1},$$

and all sheaves above but $\mathcal{O}_{\mathbf{P}^1}(a-b)$, with $a \neq b$, have positive dimension for any value of a, b satisfying the usual inequalities. Precisely,

$$\dim Aut_{\mathbf{P}^1}\mathcal{E} = b - a + 3 + \delta_{a,b},$$

where $\delta_{a,b}$ denotes the Kronecker delta.

Thus, together with the fact that $Aut(\mathbf{P}^1) = PGL_2$ has complex dimension 3, yields

$$\dim \mathcal{N}_n = b + 3a - 2 - \delta_{a,b}.$$

By recalling once more that $n = b - a$ and $b + a = g + 2$, one can rewrite the above equation as

$$\dim \mathcal{N}_n = 2g + 2 - n - \delta_{0,n}.$$

Let us observe that the open dense subscheme in $\mathcal{T}_g$ parametrizing curves with minimal Maroni invariant is $\mathcal{N}_1 \backslash \mathcal{N}_3$ if g is odd, or $\mathcal{N}_0 \backslash \mathcal{N}_2$ if g is even. For any $n \geq 0$, $\mathcal{N}_{n+2}$ in

(I.7) is closed of codimension 2 in $\mathcal{N}_n$, with the exception of $\mathcal{N}_2$, which is a divisor in $\mathcal{N}_0$.

3 Gorinov-Vassiliev's method

In this section we present the Gorinov-Vassiliev's method, which will be used in the following chapters in order to compute the cohomology of complements of discriminants. The method was first developed by Vassiliev in [Vas99], then generalized by Gorinov in [Gor05] and finally by Tommasi in [Tom05b]. We will use Tommasi's adaptation of the method, which extends the constructions of Vassiliev and Gorinov to the language of cubical spaces, allowing us to get more information about the mixed Hodge structure on the cohomology of the complement of discriminant we are interested in. The method is already written out in [Tom05b, Section2.1], but we will rewrite it here, not only for the sake of completeness, but also with the aim of fixing the notation that will be maintained in the following chapters.

Let Z be a projective variety and let V be a vector space of sections on a vector bundle over Z. We define the *discriminant* Σ as the closed subset in V of sections which are singular or do not have the expected dimension. Denote by X the complement of Σ in V.

As we have already mentioned in the introduction, computing the cohomology of the complement of a discriminant is equivalent to compute the Borel-Moore homology of the discriminant and this is due to Alexander duality:

$$\tilde{H}^\bullet(V\backslash\Sigma;\mathbf{Q}) \cong H^{\bullet+1}(V, V\backslash\Sigma;\mathbf{Q}) \cong \bar{H}_{2v-1-\bullet}(\Sigma;\mathbf{Q})(-v), \tag{I.9}$$

where $v := \dim_{\mathbf{C}} V$ and $\tilde{H}^\bullet(-;\mathbf{Q})$ denotes the reduced cohomology.

Therefore we will compute the Borel-Moore homology of Σ instead. This can be achieved by constructing a simplicial resolution of Σ, starting from a collection of families $X_1, \ldots, X_N$ of elements in Σ.

Let $K \subset Z$ be a subset. We will say that K is a *configuration* in Z if it is compact and non-empty. For any $f \in V$, let K_f be the set of singular points of the section f on Z. Set $K_0 = Z$ and define the linear space $L(K) := \{f \in V : K_f \neq \emptyset\}$ for any configuration $K \subset Z$. Then we require the families $X_1, \dots, X_N$ to satisfy the following conditions.

LIST I.1

1. For any $f \in \Sigma$, K_f belongs to some X_i.
2. If $K \in X_i$, $L \in X_j$, $K \subsetneq L$, then $i < j$.
3. $X_i \cap X_j = \emptyset$ if $i \neq j$.
4. Any $K \in \overline{X}_i \backslash X_i$ belongs to some X_j with $j < i$.
5. For any $i = 1, \dots, N$, $L(K)$ has the same dimension d_i for all $K \in X_i$.
6. For every i the space $\mathscr{T}_i = \{(z, K) \in Z \times X_i : z \in K\}$ with the evident projection, is the total space of a locally trivial bundle over X_i.
7. Suppose X_i consists of finite configurations. Then for all K, L such that $L \in X_i$ and $K \subsetneq L$, K belongs to some X_j with $j < i$.

From such X_i's, we then define the cubical spaces we will work with.
Let $I \subset \underline{N} = \{1, \dots, N\}$ and consider the simplex $\Delta_\bullet$, where

$$\Delta_I = \{g : I \to [0,1] \,|\, \sum_{i \in I} g(i) = 1\}.$$

For any $I, J \in \underline{N}$ such that $I \subset J$, we will have natural maps $e_{IJ} : \Delta_I \to \Delta_J$ given by extending $g \in \Delta_I$ to 0 on $J \backslash I$.

We define the $\underline{N}$-cubical spaces

$$\Lambda_I := \{K \in \prod_{i \in I} X_i : \text{if } K_i \in X_i, K_j \in X_j, i < j, \text{ then } K_i \subset K_j\};$$

$$\overline{\Lambda}_I := \{K \in \prod_{i \in I} \overline{X}_i : \text{if } K_i \in X_i, K_j \in X_j, i < j, \text{ then } K_i \subset K_j\};$$

$$\mathcal{X}_I := \{(f, K) \in \Sigma \times \Lambda_I : K_f \supset K_{\max I}\}, \quad \text{if } I \neq \emptyset; \qquad \mathcal{X}_\emptyset := \Sigma;$$

$$\overline{\mathcal{X}}_I := \{(f, K) \in \Sigma \times \overline{\Lambda}_I : K_f \supset K_{\max I}\}, \quad \text{if } I \neq \emptyset; \qquad \overline{\mathcal{X}}_\emptyset := \Sigma.$$

The natural forgetful maps $\phi_{IJ} : \Lambda_J \to \Lambda_I$, $\overline{\phi}_{IJ} : \overline{\Lambda}_J \to \overline{\Lambda}_I$, $\varphi_{IJ} : \mathcal{X}_J \to \mathcal{X}_I$, $\overline{\varphi}_{IJ} : \overline{\mathcal{X}}_J \to \overline{\mathcal{X}}_I$ give the spaces above the structure of cubical spaces over the index set $\underline{N}$. Consider then their geometric realization, which is defined for $\Lambda_\bullet$ as the map

$$|\epsilon| : |\Lambda_\bullet| \to \Lambda_\emptyset$$

induced by the natural augmentation on the quotient

$$|\Lambda_\bullet| = \left(\bigsqcup_{I \subset \{1,\ldots,N\}} \Lambda_I \times \Delta_I \right) / \sim,$$

where $(K, g) \sim (K', g')$ if and only if $K' = \phi_{IJ}(K)$ and $g' = e_{IJ}(g)$, and similarly for the other cubical spaces.

We then construct a surjective map

$$\phi : |\overline{\Lambda}_\bullet| \to |\Lambda_\bullet|$$

as follows: let $(K, g) \in \overline{\Lambda}_I \times \Delta_I$ and let $[K, g]$ be its corresponding class in $|\overline{\Lambda}_\bullet|$. Then by conditions 3 and 4, for each K_i, $i \in I$, there is a unique family $X_{k(i)}$ containing K_i. We define $\phi([K, g])$ as the class in $|\Lambda_\bullet|$ of the element $(L, h) \in \Lambda_J \times \Delta_J$ where

$$J := \{k(i) | \, i \in I\},$$

$$L := \prod_{k \in J} L_k; \quad L_k = K_i \text{ for any } i \text{ s.t. } k(i) = k,$$

$$h : J \to [0, 1] \,, \quad h(k) := \sum_{i \in I | k(i) = k} g(i).$$

Similarly, define a surjective map

$$\varphi : |\overline{\mathcal{X}}_\bullet| \to |\mathcal{X}_\bullet|.$$

We consider the spaces $|\overline{\Lambda}_\bullet|, |\overline{\mathcal{X}}_\bullet|$ with the quotient topology under the equivalence relation $\sim$ of the direct topology of the $\overline{\Lambda}_I, \overline{\mathcal{X}}_I$, and on $|\Lambda_\bullet|, |\mathcal{X}_\bullet|$ the topology induced by ϕ, φ, respectively.

Proposition 3.1 ([Gor05]). *The geometric realization $|\mathcal{X}_\bullet| \to \mathcal{X}_\emptyset = \Sigma$ is a homotopy equivalence and induces an isomorphism on the Borel-Moore homology groups.*

From the theory of cubical spaces [PS08, Section 5.3], the Borel-Moore homology of the spaces $\mathcal{X}_I$ has a mixed Hodge structure which will naturally be induced on the Borel-Moore homology of $|\mathcal{X}_\bullet|$, and to that of Σ.

The geometric realizations constructed above admit increasing filtrations. Precisely,

$$\mathrm{Fil}_i|\Lambda_\bullet| := \mathrm{Im}\left(|\Lambda_\bullet|_{\underline{i}}| \hookrightarrow |\Lambda_\bullet|\right),$$

where $|\Lambda_\bullet|_{\underline{i}}|$ is the geometric realization of the cubical spaces restricted to the index set $\underline{i}$, and analogously for $|\mathcal{X}_\bullet|$. Define then locally closed subsets

$$\Phi_i := \mathrm{Fil}_i|\Lambda_\bullet|\backslash \mathrm{Fil}_{i-1}|\Lambda_\bullet|, \qquad F_i := \mathrm{Fil}_i|\mathcal{X}_\bullet|\backslash \mathrm{Fil}_{i-1}|\mathcal{X}_\bullet|.$$

Proposition 3.2 ([Tom05b]). *The filtration $Fil_i|\mathcal{X}_\bullet|$ defines a spectral sequence that converges to the Borel-Moore homology of Σ, whose $E^1_{p,q}$-term is isomorphic to $\bar{H}_{p+q}(F_p; \mathbf{Q})$.*

Finally, as we have already mentioned, the Borel-Moore homology of each stratum F_i can be computed by considering their description as fiber bundles over the family X_i.

Proposition 3.3 (Gor05).

1. *For any $i = 1, \ldots, N$, the stratum F_i is a complex vector bundle of rank d_i over Φ_i, which is a locally trivial fibration over X_i.*
2. *If X_i consists of configurations of m points, the fiber of Φ_i over any $K \in X_i$ is a $(m-1)$-dimensional open simplex, which changes orientation under the homotopy class of a loop in X_i interchanging a pair of points in K.*
3. *If $X_N = \{Z\}$, F_N is the open cone with vertex a point (corresponding to the configuration Z), over $Fil_{N-1}|\Lambda_\bullet|$.*

3.1 A generalized version of Leray-Hirsch theorem

The moduli spaces $\mathcal{N}_n$ we are interested in are actually related to a geometric quotient. Although this will be made more precise later, let us present the theorem of Peters and Steenbrink that we have already mentioned in the introduction. They proved indeed that the Leray-Hirsch theorem can be extended from fiber bundles to geometric quotients by reductive groups.

Let G be an affine reductive algebraic group acting on X, with finite stabilizers. For any $x \in X$, denote by $\rho_x : G \to X$ the orbit map and by $\phi : X \to X/G$ the geometric quotient. The cohomology ring $H^\bullet(G)$ is well known: it is an exterior algebra generated by classes η_i of odd degree $2r_i - 1$ with $i = 1, \ldots, \operatorname{rank} G$.

Theorem 3.4 ([PS03, Theorem 3]). *Suppose that, for all $i = 1, \ldots, \operatorname{rank} G$, there are subschemes $Y_i \subset \Sigma$ of pure codimension r_i in V whose fundamental classes map to a non-zero multiple of η_i under the composition*

$$\bar{H}_{2(v-r_i)}(Y_i) \to \bar{H}_{2(v-r_i)}(\Sigma) \xrightarrow{\sim} H^{2r_i-1}(X) \xrightarrow{\rho^*} H^{2r_i-1}(G).$$

Denote the image of $[Y_i]$ in $H^\bullet(X;\mathbf{Q})$ by y_i, then the map $a \otimes \eta_i \mapsto \phi^* a \smile y_i$, $a \in H^\bullet(X/G;\mathbf{Q})$ extends to an isomorphism of graded $\mathbf{Q}$-vector spaces

$$H^\bullet(X/G;\mathbf{Q}) \otimes H^\bullet(G;\mathbf{Q}) \xrightarrow{\sim} H^\bullet(X;\mathbf{Q}).$$

4 Homological lemmas and configuration spaces

We close this first chapter with some homological lemmas which will be repeatedly used in our following computations, in order to compute the Borel-Moore homology of the families X_i that we have defined in the previous section. The families X_i consist of configuration spaces, so let us first give some definitions and properties for the case of configurations of a finite number of points on a given variety.

Definition 5. Let Z be a projective variety. The space of ordered configurations of k points in Z is defined as

$$F(Z,k) = Z^k \setminus \bigcup_{1 \le i < j \le k} \{(z_1, \ldots, z_k) \in Z^k | z_i = z_j\}.$$

The quotient by the natural action of the symmetric group $\mathfrak{S}_k$ is denoted by $B(Z,k)$ and it is the space of unordered configurations of k points in Z.

In the next chapter, we will consider configurations spaces in $Z = \mathbf{P}^2 \setminus \{P\}$ with P a fixed point in $\mathbf{P}^2$. In this case, we will need to give a further definition.

Definition 6. Fix $P \in \mathbf{P}^2$. A configuration of k points in $\mathbf{P}^2 \setminus \{P\}$ will be called *general* if its points are in general position, i.e. no three points lie on the same line and no two points lie on the same line through P. The open subsets of $F(\mathbf{P}^2 \backslash \{P\}, k)$ and $B(\mathbf{P}^2 \backslash \{P\}, k)$ consisting of general configurations will be denoted by $\tilde{F}(\mathbf{P}^2 \backslash \{P\}, k)$ and $\tilde{B}(\mathbf{P}^2 \backslash \{P\}, k)$, respectively.

Recall also from the previous section, and in particular from Proposition 3.3 that, for families consisting of a finite set of points, the fiber of the bundle $\Phi_i \to X_i$ may be non-orientable. In these cases we will have to consider the Borel-Moore homology in some local system of rank 1. More precisely,

Definition 7. For any subset $Y \subseteq B(Z, k)$, the local system $\pm\mathbf{Q}$ over Y is the one locally isomorphic to $\mathbf{Q}$ that changes its sign under any loop defining an odd permutation in a configuration from Y. We will denote by $\bar{H}_\bullet(Y; \pm\mathbf{Q})$ the *Borel-Moore homology* of Y with twisted coefficients, or the *twisted Borel-Moore homology* of Y, and by $\bar{P}(Y; \pm\mathbf{Q})$ its Hodge-Grothendieck polynomial, defined as in (I.1).

Moreover, the Borel-Moore homology of configuration spaces on a projective variety can be deduced from that of configuration spaces defined over an N-dimensional affine or projective space, which are well known.

Lemma 4.1 ([Vas99, Lemma 2]). *a.* $\bar{H}_\bullet(B(\mathbf{C}^N, k); \pm\mathbf{Q})$ *is trivial for any* $k \geq 2$.

b. $\bar{H}_\bullet(B(\mathbf{P}^N, k); \pm\mathbf{Q}) = H_{\bullet - k(k-1)}(G(k, \mathbf{C}^{N+1}); \mathbf{Q})$, *where* $G(k, \mathbf{C}^{N+1})$ *is the Grassmann manifold of* k*-dimensional subspaces in* $\mathbf{C}^{N+1}$. *In particular* $\bar{H}_\bullet(B(\mathbf{P}^N, k); \pm\mathbf{Q})$ *is trivial if* $k > N + 1$.

From this, it is not difficult to recover the Borel-More homology of other configuration spaces.

Remark 3. In particular, the twisted Borel-Moore homology of a configuration space $B(X, k)$ can immediately be read off from Lemma 4.1 for any space X which admits a stratification whose strata are affine spaces. Such a stratification induces a stratification on $B(X, k)$, whose strata record the number of points in each stratum of X.

Lemma 4.2 ([Tom05a, Lemma 2.14]). *The Hodge-Grothendieck polynomial of* $\bar{H}_\bullet(B(\mathbf{C}^*, k); \pm\mathbf{Q})$ *is* $t^k + \mathbf{L}^{-1} t^{k+1}$ *for any* $k \geq 1$. *If we consider the action of* $\mathfrak{S}_2$ *on* $\mathbf{C}^*$ *induced by* $\tau \mapsto \frac{1}{\tau}$, *we have that the Borel-Moore homology classes of even degree are invariant and those of odd degree are anti-invariant.*

Lemma 4.3. *The Hodge-Grothendieck polynomial of* $\bar{H}_\bullet(B(\mathbf{P}^2 \backslash \{P\}, 2); \pm\mathbf{Q})$ *is* $\mathbf{L}^{-3} t^6$. $\bar{H}_\bullet(B(\mathbf{P}^2 \backslash \{P\}, k); \pm\mathbf{Q})$ *is trivial for* $k \geq 3$.

Proof. $\mathbf{P}^2\backslash\{P\}$ can be decomposed into the disjoint union of spaces S_1, S_2, isomorphic respectively to $\mathbf{C}^2$ and $\mathbf{C}$. Then, to any configuration of points in $B(\mathbf{P}^2\backslash\{P\}, k)$ we can associate an ordered partition (a_1, a_2), where a_i is the number of points contained in S_i. We consider each possible partition of k as defining a stratum in $B(\mathbf{P}^2\backslash\{P\}, k)$, and order each strata by lexicographic order of the index of partition. All strata with any $a_i \geq 2$ have no twisted Borel-Moore homology by Lemma 4.1.a, so the second part of the Lemma is proved. When $k = 2$, the only admissible partition is $(1, 1)$ that is a stratum isomorphic to $\mathbf{C}^3$, hence it has twisted Borel-Moore homology $\mathbf{Q}(3)$ in degree 6 and trivial homology in all other degrees. □

Lemma 4.4. *The Hodge-Grothendieck polynomial of* $\bar{H}_\bullet(\tilde{F}(\mathbf{P}^2\backslash\{P\}, 2); \mathbf{Q})$ *is* $\mathbf{L}^{-4}t^8 + \mathbf{L}^{-3}t^6$.

Proof. By definition, $\tilde{F}(\mathbf{P}^2\backslash\{P\}, 2)$ consists of pairs of points, lying on a line not passing through P. The space of lines not passing through P is $(\mathbf{P}^2\backslash\{P\})^\vee \cong \mathbf{C}^2$. Therefore $\tilde{F}(\mathbf{P}^2\backslash\{P\}, 2)$ is a $\mathbf{C}^2$-bundle over the space of ordered pairs of points on a line, i.e. $F(\mathbf{P}^1, 2)$. The Borel-Moore homology of the base space is $\mathbf{Q}(2)$ in degree 4, $\mathbf{Q}(1)$ in degree 2 and zero in all other degrees. Then, the Borel-Moore homology of $\tilde{F}(\mathbf{P}^2\backslash\{P\}, 2)$ is $\mathbf{Q}(4)$ in degree 8, $\mathbf{Q}(3)$ in degree 6 and zero in all other degrees. □

Lemma 4.5. *The Hodge-Grothendieck polynomial of* $\bar{H}_\bullet(\tilde{F}(\mathbf{P}^2\backslash\{P\}, 3); \mathbf{Q})$ *is* $\mathbf{L}^{-6}t^{12} + \mathbf{L}^{-5}t^{11} + \mathbf{L}^{-4}t^9 + \mathbf{L}^{-3}t^8$.

Proof. The space $\tilde{F}(\mathbf{P}^2\backslash\{P\}, 3)$ consists of triples of points, each contained on a distinct line through P, such that there is no line containing all of them. In other words, $\tilde{F}(\mathbf{P}^2\backslash\{P\}, 3)$ is the complement of the space Y_1 of triples of points all lying on a line not passing through P, in the space Y_2 of triples of points all lying on distinct lines through P. The space Y_1 is a $\mathbf{C}^2$-bundle over $F(\mathbf{P}^1, 3)$ and its Borel-Moore homology is $\mathbf{Q}(5)$ in degree 10, $\mathbf{Q}(3)$ in degree 7 and zero in all other degrees. The space Y_2 is, instead, a $\mathbf{C}^3$-bundle over $F((P)^\vee, 3) \cong F(\mathbf{P}^1, 3)$. Its Borel-Moore homology is $\mathbf{Q}(6)$ in degree 12, $\mathbf{Q}(4)$ in degree 9 and zero in all other degrees. Finally, by considering the Gysin exact sequence induced by the inclusion $\tilde{F}(\mathbf{P}^2\backslash\{P\}, 3) = Y_2\backslash Y_1 \hookrightarrow Y_2$, we have that the Borel-Moore homology of $\tilde{F}(\mathbf{P}^2\backslash\{P\}, 3)$ is $\mathbf{Q}(6)$ in degree 12, $\mathbf{Q}(5)$ in degree 11, $\mathbf{Q}(4)$ in degree 9, $\mathbf{Q}(3)$ in degree 8 and zero in all other degrees. □

Lemma 4.6. *There are isomorphisms*

$$\bar{H}_\bullet(\tilde{B}(\mathbf{P}^2\backslash\{P\}, 2); \pm\mathbf{Q}) \xrightarrow{\sim} \bar{H}_\bullet(B(\mathbf{P}^2\backslash\{P\}, 2); \pm\mathbf{Q})$$

$$\bar{H}_\bullet(\tilde{B}(\mathbf{P}^2\backslash\{P\},3);\pm\mathbf{Q}) \xrightarrow{\sim} \bar{H}_\bullet(B(\mathbf{P}^2\backslash\{P\},3);\pm\mathbf{Q})$$

induced by the natural inclusions.

Proof. Let us consider the inclusions $\tilde{B}(\mathbf{P}^2\backslash\{P\},k) \hookrightarrow B(\mathbf{P}^2\backslash\{P\},k)$, $k = 2,3$. The complement $B(\mathbf{P}^2\backslash\{P\},2)\backslash\tilde{B}(\mathbf{P}^2\backslash\{P\},2)$ is the space of pairs of points lying on the same line through P : it is fibered over $(P)^\vee \cong \mathbf{P}^1$ with fiber isomorphic to $B(\mathbf{C},2)$. On the other hand, $B(\mathbf{P}^2\backslash\{P\},3)\backslash\tilde{B}(\mathbf{P}^2\backslash\{P\},3)$ is the union of 3 locally closed strata: the space of triples lying on the same line not passing through P, the space of triples lying on the same line through P, and the space consisting of triples where exactly 2 points lie on the same line through P. These fiber spaces have fibers $B(\mathbf{P}^1,3)$, $B(\mathbf{C},3)$ and $B(\mathbf{C},2)\times\mathbf{C}^2$, respectively. By Lemma 4.1 all these fibers have trivial twisted Borel-Moore homology. □

Lemma 4.7 ([Gor05, Corollary 3.5])**.** *Let $p : N' \to N$ be a finite sheeted covering of manifolds, and let $\mathcal{L}$ be a local system of coefficients on N'. Then $\bar{H}_\bullet(N',\mathcal{L}) = \bar{H}_\bullet(N,p(\mathcal{L}))$, where $p(\mathcal{L})$ denotes the direct image of the system $\mathcal{L}$.*

Let us, once more, stress the fact that our aim is to compute the Borel-Moore homology of families of singular configurations of elements in our discriminant. Hence we will need to compute the Borel-Moore homology of configurations of k points on a Hirzebruch surface $\mathbb{F}_n$, for any $n \geq 0$, and this can be deduced from the above lemmas by Remark 3.

In fact, $\mathbb{F}_n$ can be stratified into two affine cells, one isomorphic to $\mathbf{P}^2\backslash\{P\} \cong \mathbf{A}^2\cup\mathbf{A}^1$ and the other isomorphic to $\mathbf{P}^1 \cong \mathbf{A}^1\cup\mathbf{A}^0$. Then, as a consequence of Lemma 4.1, we have

Lemma 4.8. *For $n \geq 0$,*

$$\bar{H}_i(B(\mathbb{F}_n,1);\pm\mathbf{Q}) = \begin{cases} \mathbf{Q}, & i=0,\\ 2\mathbf{Q}(1), & i=2,\\ \mathbf{Q}(2), & i=4,\\ 0, & \textit{otherwise}; \end{cases}$$

$$\bar{H}_i(B(\mathbb{F}_n, 2); \pm\mathbf{Q}) = \begin{cases} 2\mathbf{Q}(1), & i = 2, \\ 2\mathbf{Q}(2), & i = 4, \\ 2\mathbf{Q}(3), & i = 6, \\ 0, & otherwise; \end{cases}$$

$$\bar{H}_i(B(\mathbb{F}_n, 3); \pm\mathbf{Q}) = \begin{cases} \mathbf{Q}(2), & i = 4, \\ 2\mathbf{Q}(3), & i = 6, \\ \mathbf{Q}(4), & i = 8, \\ 0, & otherwise; \end{cases}$$

$$\bar{H}_i(B(\mathbb{F}_n, 4); \pm\mathbf{Q}) = \begin{cases} \mathbf{Q}(4), & i = 8, \\ 0, & otherwise; \end{cases}$$

$$\bar{H}_\bullet(B(\mathbb{F}_n, k); \pm\mathbf{Q}) = 0, \qquad \forall k \geq 5.$$

Note also that the twisted Borel-Moore homology groups computed above agree with the ones computed by Tommasi in [Tom05b, Lemma 2.13] for $\mathbb{F}_0 \cong \mathbf{P}^1 \times \mathbf{P}^1$.

Finally let us consider the case of families corresponding to configurations containing curves. These types of families give no contribution to the Borel-Moore homology of the discriminant.

Lemma 4.9 ([Tom05b, Lemma 2.17])**.** *Let Z be a projective variety and suppose we have the following families of configurations in Z :*

$$\begin{aligned} X_1 =& B(Z, 1); \\ X_2 =& \{\{p, q\} \in B(Z, 2) : p, q \in l, l \text{ a line in } Z\}; \\ X_3 =& \{\{p, q, r\} \in B(Z, 3) : p, q, r \in l, l \text{ a line in } Z\}; \\ X_4 =& \{\text{lines in } Z\}. \end{aligned}$$

Construct the cubical space $\Lambda_\bullet$, its geometric realization and the filtration as in section 3. Then the space Φ_4 has trivial Borel-Moore homology.

As it has already been discussed in [Tom05b, Remark 1.18 and Lemma 2.19], a slight modification of the above lemma gives us an analogous result also for families of singular configuration consisting of the union of a rational curve and a fixed finite number of points and of the union of two rational curves, meeting at one point.

Chapter II

Rational cohomology of $\mathcal{T}_5$

This chapter is based on [Zhe21a].

1 Introduction and results

Let us recall once more that for $g = 3, 4$, the rational cohomology of $\mathcal{T}_g$ has already been computed by Looijenga in [Loo93] for $g = 3$, and by Tommasi in [Tom05b] for $g = 4$. In these two cases, $\mathcal{T}_g$ coincides with $\mathcal{M}_g \backslash \mathcal{H}_g$, where $\mathcal{H}_g$ is the moduli space of smooth hyperelliptic curves of genus g. For $g = 3$, any non-hyperelliptic curve admits infinitely many pencils of degree 3, while, for $g = 4$, any non-hyperelliptic curve admits either one or two of them.

On the other hand, when $g \geq 5$ a non-hyperelliptic curve is not necessarily trigonal. In particular, $\mathcal{T}_5$ represents the first case where the cohomology of $\mathcal{T}_g$ cannot be automatically deduced from that of $\mathcal{H}_g$ and $\mathcal{M}_g$. In fact, $\mathcal{M}_5$ can be decomposed into the disjoint union of the moduli spaces of hyperelliptic curves $\mathcal{H}_5$, of trigonal curves $\mathcal{T}_5$, and the one parametrizing curves that are the complete intersection of three linearly independent smooth quadric hypersurfaces in $\mathbf{P}^4$, which will be denoted by $\mathcal{Q}_5$. Therefore, knowing the rational cohomology of $\mathcal{T}_5$ represents an advance not only in the understanding of that of $\mathcal{M}_5$, which is unknown at present, but hopefully also of the cohomology of $\mathcal{T}_g$, for any $g \geq 5$.

What is known about $\mathcal{T}_g \cup \mathcal{H}_g$ until now is mostly due to the works of Stankova, Bolognesi and Vistoli. Stankova computed in [SF00] the rational Picard group of the closure $\overline{\mathcal{T}}_g \subseteq \overline{\mathcal{M}}_g$, while Bolognesi and Vistoli computed in [BV12] the integral Picard group of $\mathcal{T}_g \cup \mathcal{H}_g$. Later, Patel and Vakil established that the rational Chow ring $A^*_{\mathbf{Q}}(\mathcal{T}_g)$ is generated by a single class in codimension 1, [PV15a].

More recently, for $g = 5$, Wennink, in [Wen20], counted the number of points of $\mathcal{T}_5$ over any finite field $\mathbf{F}_q$ with q points:

$$|\mathcal{T}_5(\mathbf{F}_q)| = q^{11} + q^{10} - q^8 + 1.$$

By [vdBE05], [Ber08, Theorem 3.2], this determines the Euler characteristic of $\mathcal{T}_5$ in $K_0(\mathsf{HS}_{\mathbf{Q}})$, the Grothendieck group of rational Hodge structures.

We will refine Wennink's result and compute the rational cohomology of $\mathcal{T}_5$ with its mixed Hodge structure.

Theorem 1.1. *The rational cohomology of $\mathcal{T}_5$ is*

$$H^i(\mathcal{T}_5; \mathbf{Q}) = \begin{cases} \mathbf{Q}, & i = 0; \\ \mathbf{Q}(-1), & i = 2; \\ \mathbf{Q}(-3), & i = 5; \\ \mathbf{Q}(-11), & i = 12; \\ 0, & \textit{otherwise}; \end{cases}$$

where $\mathbf{Q}(-k)$ denotes the Tate Hodge structure of weight $2k$.

The whole rational cohomology of $\mathcal{T}_5$ can also be expressed in terms of its Hodge-Grothendieck polynomial. By Theorem 1.1, then

$$P(\mathcal{T}_5; \mathbf{Q}) = \mathbf{L}^{11}t^{12} + \mathbf{L}^3t^5 + \mathbf{L}t^2 + 1,$$

where $\mathbf{L}$ denotes the class of the Tate Hodge structure $\mathbf{Q}(-1)$.

Moreover, since the moduli space $\mathcal{H}_g$, $g \geq 2$, has always the rational cohomology of a point, we can also prove the following

Corollary 1.2. *The rational cohomology of* $\mathcal{T}_5 \cup \mathcal{H}_5$ *is*

$$H^i(\mathcal{T}_5 \cup \mathcal{H}_5; \mathbf{Q}) = \begin{cases} \mathbf{Q}, & i = 0; \\ \mathbf{Q}(-1), & i = 2; \\ \mathbf{Q}(-2), & i = 4; \\ \mathbf{Q}(-3), & i = 5; \\ \mathbf{Q}(-11), & i = 12; \\ 0, & \textit{otherwise.} \end{cases}$$

Theorem 1.1 and Corollary 1.2 are consistent with the known results on the cohomology of $\mathcal{M}_5$. In particular, the maximal weight class of the cohomology of $\mathcal{T}_5$ can be identified with the top weight cohomology class of $\mathcal{M}_5$, described by Chan, Galatius and Payne in [CGP21] and [CGP20]. They proved indeed that the cohomology $H^{4g-6}(\mathcal{M}_g; \mathbf{Q})$ is non-zero for $g = 5$ and that, by studying the dual complex of the boundary divisor in $\overline{\mathcal{M}}_g$, the top graded piece on the cohomology of $\mathcal{M}_5$ is such that

$$\dim \mathrm{Gr}^W_{6g-6} H^i(\mathcal{M}_5; \mathbf{Q}) = \begin{cases} 1, & i = 14; \\ 0, & i \neq 14. \end{cases}$$

The stratification of $\mathcal{M}_5$

$$\mathcal{T}_5 \cup \mathcal{H}_5 \overset{\text{closed}}{\hookrightarrow} \mathcal{M}_5 \overset{\text{open}}{\hookleftarrow} \mathcal{Q}_5$$

induces a Gysin exact sequence in Borel-Moore homology

$$\cdots \to \bar{H}_{k+1}(\mathcal{Q}_5; \mathbf{Q}) \to \bar{H}_k(\mathcal{T}_5 \cup \mathcal{H}_5; \mathbf{Q}) \to \bar{H}_k(\mathcal{M}_5; \mathbf{Q}) \to \bar{H}_k(\mathcal{Q}_5; \mathbf{Q}) \to \ldots$$

Since $\mathcal{Q}_5$ is affine and $\dim \mathcal{Q}_5 = 12$, [FL08, Theorem 4.1], $\bar{H}_{10}(\mathcal{M}_5; \mathbf{Q}) = \bar{H}_{10}(\mathcal{T}_5; \mathbf{Q})$, and Poincaré duality gives $H^{14}(\mathcal{M}_5; \mathbf{Q}) = H^{12}(\mathcal{T}_5; \mathbf{Q})$.

This also determines part of the cohomology of $\mathcal{M}_5$:

Corollary 1.3. $H^{14}(\mathcal{M}_5; \mathbf{Q}) = \mathbf{Q}(-12)$ *and* $H^i(\mathcal{M}_5; \mathbf{Q}) = 0$ *for any* $i \geq 13$, $i \neq 14$.

The proof of Theorem 1.1 relies on Gorinov-Vassiliev's method, presented in the previous chapter.

2 Moduli space $\mathcal{T}_5$ from a geometric quotient

In this section we define our moduli space $\mathcal{T}_5$ as the quotient of the complement of a discriminant by the action of an algebraic group. We recall from (I.4) that any trigonal curve of genus g may be embedded, via the canonical embedding, in a Hirzebruch surface $\mathbb{F}_n$ as a divisor of class

$$C \sim 3E_n + \frac{g+3n+2}{2} F_n,$$

where E_n is the exceptional divisor and F_n is the class of any fiber of the ruling and the Maroni invariant n has to satisfy $g \equiv n \bmod 2$ and $0 \leq n \leq \frac{g+2}{3}$.

Thus, for $g = 5$, the Maroni stratification consists of only one stratum and any trigonal curve lies on the Hirzebruch surface $\mathbb{F}_1$, as an element of the linear system $|3E_1 + 5F_1|$. We will also drop the subscript and simply write E, F, in the rest of this chapter. The Hirzebruch surface $\mathbb{F}_1$ is the blow up of the projective plane at one point.

There is an additional equivalent description of trigonal curves of genus 5.

Proposition 2.1. *There is a one-to-one correspondence between isomorphism classes of trigonal curves of genus 5 and of projective plane quintics having exactly one ordinary node or cusp.*

Proof. Given a trigonal curve C, and hence a g^1_3, one can show that the linear system $|K-D|$ is a base point free g^2_5, where $D \in g^1_3$ and K is the canonical divisor. This defines a morphism $C \to \mathbf{P}^2$ such that the image has degree 5, with precisely one singularity of delta invariant 1.

Conversely, a plane projective curve of degree 5 with one singularity that is a node or a cusp has arithmetic genus 5, and each line through the singular point meets the curve in three other points, counting multiplicity, defining a g^1_3. $\square$

Let us then consider a projective plane quintics having exactly a node or a cusp and let $P \in \mathbf{P}^2$ denote this singular point. Let $Z = \mathbb{F}_1$ and define V to be the vector space of global sections of $\mathcal{O}_Z(3E + 5F)$. Let X be the open subset of sections defining smooth curves and the discriminant Σ is the complement of X in V.

The vector space V is isomorphic to the vector space of polynomials defining plane curves of degree 5 having at least a singular point at P. Therefore its dimension is 18. Indeed, the dimension of the vector space of polynomials defining plane quintics is $\dim \mathbf{C}[x_0, x_1, x_2]_5 = 21$. A polynomial $f \in \mathbf{C}[x_0, x_1, x_2]_5$ is singular at a point $P \in \mathbf{P}^2$ iff

$\frac{\partial f}{\partial x_i}(P) = 0$, $i = 0, 1, 2$. Hence the dimension of the vector space of polynomials defining plane quintics with a fixed singularity is $21 - 3 = 18$.

This can be also proved as follows. Let us consider a general plane quintic having at least a node or a cusp at the point P that we will blow up. Without loss of generality we may assume $P = [1, 0, 0]$. The plane quintic curve is then defined by a polynomial $f \in \mathbf{C}[x_0, x_1, x_2]$ having degree ≤ 3 with respect to the variable x_0. For any such curve we can consider a projection with center P: fix a line l not passing through P, for instance $l := \{[0, y_1, y_2]\}$, and take the map sending all points of the curve distinct from P to the point of intersection between the line connecting the point to P and l.

The preimage of any point through this map is given by points of the curve on the same line through P, which has parametric equation

$$r : \begin{cases} x_0 = t_0 \\ x_1 = t_1 y_1 \\ x_2 = t_1 y_2 \end{cases} , \qquad [t_0, t_1] \in \mathbf{P}^1.$$

Since any line through P corresponds to a line of the ruling in the blow up, any curve we want to consider can be embedded in the blow up via the mapping

$$[t_0, t_1 y_1, t_1 y_2] \hookrightarrow [t_0, t_1 y_1, t_1 y_2] \times [y_1, y_2]$$

and it has to satisfy $f(t_0, t_1 y_1, t_1 y_2) = 0$. Therefore it has equation

$$t_1^2(t_0^3 g_2(y_1, y_2) + t_0^2 t_1 g_3(y_1, y_2) + t_0 t_1^2 g_4(y_1, y_2) + t_1^3 g_5(y_1, y_2)),$$

where each g_i is a homogeneous polynomial of degree i. Counting the number of parameters will indeed give 18.

The automorphism group of the homogeneous coordinate ring of $Bl_P \mathbf{P}^2$ is the set of automorphisms of the graded ring $\mathbf{C}[x_0, x_1, x_2]$ that fix the point that is blown up, i.e.

$$G = \left\{ \begin{bmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{bmatrix} \in GL_3 \right\} \supset \mathbf{C}^* \times GL_2.$$

Note that ignoring the second and the third entry in the first row of each of the matrices in G means contracting the vector space $\mathbf{C}\,[x_1, x_2]_1 \cong \mathbf{C}^2$ to a point. Therefore G is homotopy equivalent to $\mathbf{C}^* \times GL_2$.

Note also that G contains the normal unipotent subgroup

$$\left\{ \begin{bmatrix} 1 & * & * \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \in GL_3 \right\},$$

hence it is not reductive and we cannot construct our moduli space as a GIT quotient by G. However, we can consider its reductive part $\mathbf{C}^* \times GL_2$, consider the stack quotient $[X/(\mathbf{C}^* \times GL_2)]$ and compute its cohomology, or equivalently the cohomology of its coarse moduli space, the affine quotient variety $X/(\mathbf{C}^* \times GL_2)$. The space $X/(\mathbf{C}^* \times GL_2)$ parametrizes isomorphism classes of pairs (C, L), with C a trigonal curve of genus 5 and L its unique g^1_3, plus a hyperplane section H corresponding to the line l not meeting P that we defined before, when computing the dimension of V. By duality, H is a point in $\mathbf{C}^2$, therefore $[X/(\mathbf{C}^* \times GL_2)]$ is a $\mathbf{C}^2$-bundle over $\mathcal{T}_5$, in the orbifold sense, and they have the same rational cohomology.

We will first consider the reductive subgroup $\{1\} \times GL_2 \subset \mathbf{C}^* \times GL_2$ and the quotient stack $[X/GL_2]$. Then, we will compute its cohomology by using the generalized version of the Leray-Hirsch theorem. Finally, we will consider the orbifold $\mathbf{C}^*$-bundle

$$[X/GL_2] \xrightarrow{\mathbf{C}^*} [X/(\mathbf{C}^* \times GL_2)]\,,$$

and deduce the cohomology of the base space from the Leray spectral sequence associated to this bundle.

Let us stress once more that a quotient stack and its underlying coarse moduli space have the same rational cohomology, therefore, by abuse of notation, we will drop the brackets.

2.1 Generalized Leray-Hirsch theorem

We want to prove that there exists an isomorphism of graded $\mathbf{Q}$-vector spaces with mixed Hodge structures

$$H^\bullet(X/GL_2;\mathbf{Q}) \otimes H^\bullet(GL_2;\mathbf{Q}) \cong H^\bullet(X;\mathbf{Q}).$$

By Theorem I.3.4 it suffices to prove the surjectivity of the orbit map on cohomology

$$\begin{array}{cccc} \rho^* : & H^i(X;\mathbf{Q}) & \longrightarrow & H^i(GL_2;\mathbf{Q}) \\ & \wr\Vert & & \wr\Vert \\ & \bar{H}_{2\dim V-i-1}(\Sigma;\mathbf{Q}) & & \bar{H}_{2\dim M-i-1}(D;\mathbf{Q}), \end{array}$$

where M denotes the space of 2×2 matrices and D the discriminant of GL_2 in M. We know that the cohomology of GL_2 has generators in degrees $i=1,3$, and the generators of $\bar{H}_\bullet(D;\mathbf{Q})$ are $[D]\in \bar{H}_6(D;\mathbf{Q})$ and $[R]\in\bar{H}_4(D;\mathbf{Q})$, where we can assume R to be the subvariety of matrices with only zeros in the first column. Moreover, from the spectral sequence that will be exhibited in Table II.4, $\bar{H}_{34}(\Sigma;\mathbf{Q}) = \langle[\Sigma]\rangle$, and $\bar{H}_{32}(\Sigma;\mathbf{Q}) = \langle[\Sigma_1],[\Sigma_2]\rangle$, where Σ_1 is the subspace in V of polynomials defining curves having a singular point on E, and Σ_2 is the subspace in V of polynomials defining a singularity on a fixed line of the ruling F.

Let us consider the extension $\tilde{\rho}: M\to V$ of the orbit map $\rho: GL_2\to X$. Fix $f\in X$ and consider the image of an element in R :

$$A=\begin{pmatrix} 0 & b \\ 0 & d\end{pmatrix} \mapsto A\cdot f(x_0,x_1,x_2) = f(x_0,bx_2,dx_2) = \alpha(b,d)x_2^2 h_3(x_0,x_2),$$

where $\alpha(b,d)$ is some constant and h_3 is the product of 3 lines through the point $[0,1,0]$.

So, elements of R are mapped to polynomials whose zero loci are the union of a double line of the ruling of fixed equation $y_2=0$ and three lines through a point of the ruling $([t_0,t_1,0],[1,0])$. Similarly, elements in D are mapped to curves which are the union of any double line of the ruling and three lines through a point of that ruling.

Hence we can deduce that $\rho^*([\Sigma])$ is a non-zero multiple of $[D]$, while the preimage of $[R]$ through ρ^* must be a non-trivial linear combination of $[\Sigma_1],[\Sigma_2]$, proving the surjectivity of the map in cohomology.

3 Application of Gorinov-Vassiliev's method

In this section we apply Gorinov-Vassiliev's method, introduced in section I.3, proving thus Theorem 1.1. First of all, we produce a list of all the possible configurations of singularities of genus 5 curves in $Bl_P\mathbf{P}^2$, meeting the exceptional divisor E at least twice. To do so, we recall that we are considering curves in $\mathbb{F}_1$ which are elements of the linear system $|3E + 5F|$.

Since all singularities are obtained as degenerations of nodes, we will first consider only such singularities. Assume that the curve is irreducible. By computing the arithmetic genus we get an upper bound for the number of singularities. For instance, by the *genus formula*, we have that

$$g(3E + 5F) = 1 + \frac{1}{2}((3E + 5F)^2 + (3E + 5F) \cdot K) = 5,$$

where K is the canonical divisor on $\mathbb{F}_1$. So we can have at most 5 ordinary double points.

Then, we will consider all the possible ways in which the curve can be reducible. Here we will have to take into account not only the singularities of each irreducible component, but also all the intersections between them.

Finally, we will consider all the possible degenerations of the singularities obtained in this way (points can be on the exceptional divisor or points in general position can become collinear, etc...) and all the subsets of finite configurations.

For any configuration of singularities, the elements in V which are singular at least at that configuration form a vector space and we can compute its codimension. By ordering all the configurations obtained by increasing codimension, and then by increasing number of points, we will get a list of configurations indexed by (j). By defining X_j as the space of configurations of type (j) we will get a sequence of families of configurations that will satisfy conditions 1-7 in List I.1. We denote by c_j the codimension in V of the vector space of elements which are singular at least at the configuration of type (j).

For the complete list see List A.1. Here we report instead only a shorter version of this list. We can omit, for instance, all configurations containing rational curves. These, in fact, give no contribution to the Borel-Moore homology of the discriminant by Lemma I.4.9. We also combine similar configurations that give no contribution.

In the following, we call a configuration of points *general* if it is a configuration of points in general position, where no point is contained in E and no two points lie in the same line of the ruling. When we consider configurations containing both 'general' and 'not general' points, we also require the general points to be in general position with respect to the others. However, we allow the general points to belong to the same lines of the ruling defined by points that are on E, in the same configuration space. Indeed, in this case, the configuration space in which k general points are defined is $B(\mathbf{P}^2 \setminus \{P\}, k)$, as in the case without points on E.

For each item below, we write in square brackets the codimension c_j of the vector space of elements in V, which are singular at least at the corresponding configuration.

We will also use the following notation:

line of the ruling	it is an element in $\lvert F\rvert$, i.e. the strict transform of a line in $\mathbf{P}^2$ passing through P;
line	it is an element in $\lvert E+F\rvert$, i.e. the strict transform of a line in $\mathbf{P}^2$ not passing through P;
conic $\mathcal{C}_P$	it is an element in $\lvert E+2F\rvert$, i.e. the strict transform of a conic in $\mathbf{P}^2$ passing through P;
conic $\mathcal{C}$	it is an element in $\lvert 2E+2F\rvert$, i.e. the strict transform of a conic in $\mathbf{P}^2$ not passing through P.

1. A point on the exceptional divisor E; [3]
2. A general point; [3]
3. Two points on E; [5]
4. Two (or three) points on a line of the ruling; [6 (7)]
5. A point on E + a general point; [6]
6. Two general points; [6]
7. Three points or more on E; [6]
8. Two points on E + a general point; [8]
9. A point *(that can be either on E, or general)* + two (or three) points on a line of the ruling; [9 (10)]
10. A point on E + two general points;[9]
11. Three (or four) points on a line L; [9 (10)]
12. Three (or resp. four, five) general points; [9 $(12, 15)$]
13. A point on E + two (or three) points on a line F of the ruling + the point of intersection between F and E; [9 (10)]
14. Three points on E + a general point; [9]
15. Two points on E + two (or three) points on a line of the ruling; [10 (11)]

16. Two points on E + two (or three) points on a line F of the ruling + the point of intersection between F and E; [10 (11)]
17. Two points on E + two general points; [11]
18. Three points on E + two (or three) points on a line of the ruling; [11 (12)]
19. Three points on E + two (or three) points on a line F of the ruling + the point of intersection between F and E; [11 (12)]
20. Two points on each of two lines of the ruling (or resp. two points on a ruling and three points on the other one, or three points on each of two rulings); [12 $(13, 14)$]
21. Two general points + two (or three) points on a line of the ruling; [12 (13)]
22. A point *(that can be either on E, or general)* + three (or four) points on a line L; [12 (13)]
23. A point on E + three (or four) general points; [12 (15)]
24. A point on E + two (or three) points on a line F of the ruling + a general point; [12 (13)]
25. A point on E + two (or three) points on a line F of the ruling + a general point + the point of intersection between F and E; [12 (13)]
26. Three points on E + two general points; [12]
27. Two points on E + three (or four) points on a line L; [14 (15)]
28. Two points on E + three general points; [14]
29. Two points on a line F of the ruling + three points on a line L; [14]
30. Two points on a line F of the ruling + three points on a line L + the point of intersection between F and L; [14]
31. Five (or six) points on a conic $\mathcal{C}_P$ $(\mathcal{C})$; [14 (17)]
32. Two points on each of two lines F_1, F_2 of the ruling + the intersection points with E + a point on E; [14]
33. Two points on each of two lines + the point of intersection; [15]
34. Two points on each of two lines of the ruling (or two points on a ruling and three on the other one) + a general point; [15 (16)]
35. A point on E + two general points + two or more points on a line of the ruling; [15]
36. Three general points + two (or three) points on a line of the ruling; [15 (16)]
37. A point on E + a general point + three (or four) points on a line; [15 (16)]
38. Two general points + three (or four) points on a line; [15 (16)]
39. Three points on E + three (or four) points on a line; [15 (16)]

40. Three points on E + three general points; [15]
41. Two points on E + two general points + two points on a line F of the ruling + the point of intersection between E and F; [16]
42. Five points on a conic $\mathcal{C}_P$ + a general point; [17]
43. Three points on E + four points on a conic $\mathcal{C}_P$; [17]
44. Two points on a ruling F + three points on a line L + the intersection point between F and L + a general point; [17]
45. Three points on each of two rulings + a general point; [17]
46. Three points on each of two lines + the point of intersection; [17]
47. 7 points: three points of intersection between two conics $\mathcal{C}_P$ and $\mathcal{C}'_P$, one of which is on E + four points of intersection with a line; [17]
48. 7 points: three points of intersection between two conics $\mathcal{C}_P$ and $\mathcal{C}'_P$, none of which are on E + four points of intersection with a line; [17]
49. 7 points: four points of intersection between two conics $\mathcal{C}$, $\mathcal{C}_P$ + three points of intersection with a line of the ruling; [17]
50. 8 points: a point on E + two points on each of two rulings F_1, F_2 + the points of intersection between F_1 and E, and F_2 and E + a general point; [17]
51. 8 points: two points on E + three points on a line L + the intersection points of a line F of the ruling with E and L + another point on F; [17]
52. 8 points: three points of intersection of two conics $\mathcal{C}_P$ and $\mathcal{C}'_P$, each meeting a line F of the ruling and E at one point + the point of intersection between F and E; [17]
53. 8 points: two points of intersection between two lines F_1, F_2 of the ruling and a line L + 6 points of intersection with a conic $\mathcal{C}$ meeting each line at two distinct points; [17]
54. 8 points: three points of intersection between a line F of the ruling and two lines L_1, L_2 + five points of intersection with a conic $\mathcal{C}_P$ meeting each line twice and F only once, outside E; [17]
55. 8 points: three points of intersection between a line F of the ruling and two lines L_1, L_2 + five points of intersection with a conic $\mathcal{C}_P$ meeting each line twice and F at the intersection point with E; [17]
56. 9 points: four points of intersection between E, two lines F_1, F_2 of the ruling and a line L + five points of intersection with a conic $\mathcal{C}_P$ meeting L twice and E, F_1, F_2 once; [17]

57. 9 points: three points of intersection between E and three lines F_1, F_2, F_3 of the ruling + 6 points of intersection with a conic $\mathcal{C}$ meeting each F_i at two distinct points; [17]
58. 9 points: the points of intersection between two lines of the ruling and three general lines; [17]
59. 10 points: the points of intersection between E, three lines of the ruling and two lines; [17]
60. The whole $Bl_P\mathbf{P}^2$. [18]

3.1 Non-trivial configurations

Since simplicial bundles are non-orientable, we will consider the Borel-Moore homology with coefficients in the local system $\pm\mathbf{Q}$.

We also recall from Lemmas I.4.1 and I.4.3 that configurations with at least three points on a rational curve, configurations with at least two points on a rational curve minus a point, and configurations with at least three general points give no contribution. Thus, among the first 41 configurations, only the following have non-trivial Borel-Moore homology:

(A) A point on E. [3]
(B) A general point. [3]
(C) Two points on E. [5]
(D) A point on E + a general point. [6]
(E) Two general points. [6]
(F) One general point + two points on E. [8]
(G) Two general points + one point on E. [9]
(H) Two general points + two points on E. [11]

We will also prove in Subsection II.3.3 that there are only four other configurations having non-trivial Borel-Moore homology:

(I) 7 points: configuration 47. [17]
(J) 7 points: configuration 48. [17]
(K) 8 points: configuration 55. [17]
(L) Whole $Bl_P\mathbf{P}^2$. [18]

Recall that we are studying singular configurations of curves that are equivalent to plane projective quintics having at least one singularity. So, we can deduce their Borel-Moore homology by considering their equivalent description in the projective plane. This equivalent description is obtained by fixing a point P that is the one that, when blown-up, will give us the corresponding curve in $\mathbb{F}_1$.

Note that the configuration spaces that we will consider in the following are empty unless they are defined as the singular locus of the plane quintics that will be described.

Columns (A)-(H)

From Proposition I.3.3, if X_j consists of configurations of m points, then the stratum F_j is a $\mathbf{C}^{18-c_j} \times \mathring{\Delta}_{m-1}$-bundle over X_j, where $\mathring{\Delta}_{m-1}$ is an $(m-1)$-dimensional open simplex. Therefore we get the following results.
The space F_A is a $\mathbf{C}^{15}$-bundle over $X_A \cong \mathbf{P}^1$.
The space F_B is a $\mathbf{C}^{15}$-bundle over $X_B \cong \mathbf{P}^2 \backslash \{P\}$.
The space F_C is a $\mathbf{C}^{13} \times \mathring{\Delta}_1$-bundle over $X_C \cong B(\mathbf{P}^1, 2)$.
The space F_D is a $\mathbf{C}^{12} \times \mathring{\Delta}_1$-bundle over $X_D \cong \mathbf{P}^1 \times \mathbf{P}^2 \backslash \{P\}$.
The space F_E is a $\mathbf{C}^{12} \times \mathring{\Delta}_1$-bundle over $X_E \cong B(\mathbf{P}^2 \backslash \{P\}, 2)$.
The space F_F is a $\mathbf{C}^{10} \times \mathring{\Delta}_2$-bundle over $X_F \cong \mathbf{P}^2 \backslash \{P\} \times B(\mathbf{P}^1, 2)$.
The space F_G is a $\mathbf{C}^{9} \times \mathring{\Delta}_2$-bundle over $X_G \cong B(\mathbf{P}^2 \backslash \{P\}, 2) \times \mathbf{P}^1$.
The space F_H is a $\mathbf{C}^{7} \times \mathring{\Delta}_3$-bundle over $X_H \cong B(\mathbf{P}^2 \backslash \{P\}, 2) \times B(\mathbf{P}^1, 2)$.

Column (I)+(J)

Each configuration in X_I is defined as the singular locus of the blow up at P of a plane quintic defined by two irreducible and reduced conics tangent at P and a line meeting the conics at four distinct points, as in Figure II.1.

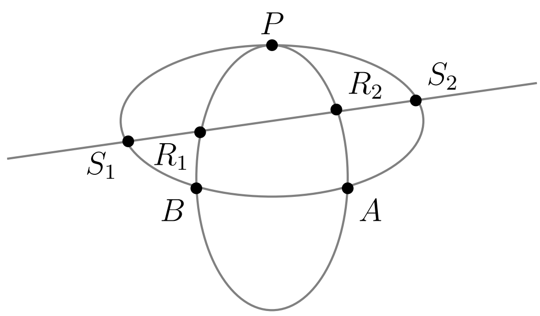

FIGURE II.1: Configuration of type (I).

On the other hand, configurations of type (J) arise from blowing up P, where P is now one of the four points of intersection between two irreducible and reduced conics that, together with a line not meeting the conics at any of the points of intersection, define the plane projective quintic curve in Figure II.2.

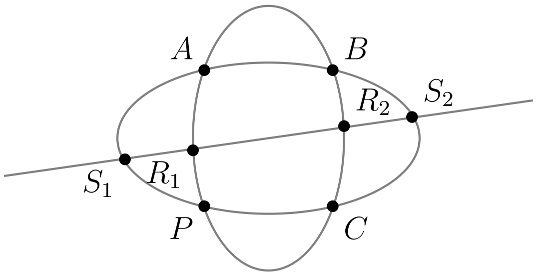

FIGURE II.2: Configuration of type (J).

By noticing that the configuration space X_I is contained in the closure of X_J (by allowing one of the points A, B, C to lie on the exceptional divisor E of $\mathbb{F}_1$) we can consider a bigger configuration space containing both of them, which we will denote by X_{I+J}.

The space X_{I+J} consists of 7 singular points $A, B, C, R_1, R_2, S_1, S_2$. The points A, B, C are in general position, also with respect to P, and only one of them is allowed to lie on E, i.e to coincide with P. The points R_1, R_2, S_1, S_2 are four distinct points of intersection between two distinct conics passing through A, B, C, P and a line l not passing through any of these points.

We can fiber X_{I+J} over the space parametrizing the points A, B, C and the choice of the line:

$$X_{I+J} \to \mathcal{B} := \{(\{A, B, C\}, l) : A, B, C, l \text{ as in the description above}\}.$$

The fiber of this map, which we denote by $\mathcal{Z}$, will then be the space of pairs of conics passing through the four points and not tangent to the line. Note that $\mathcal{Z}$ is exactly the same fiber space considered in [Gor05, Section 4.2] in Column 38.

As both conics have to satisfy 4 linear independent conditions (that consist either in the passage through 4 distinct points or 3 points plus the tangency condition), each of them is uniquely determined by a point on the line l. We denote these points by S_1, R_1, as in the figures. Recall that there are exactly two conics in the pencil with base locus $A+B+C+P$ that are tangent to l. Let T_1, T_2 be the points of intersection between l and the two tangent conics. Any other conic will meet l at two distinct points. Exchanging these two intersection points defines an involution on $l \cong \mathbf{P}^1$ that fixes T_1, T_2, and by choosing an appropriate coordinate system we can assume that $T_1 = [1, 0]$, $T_2 = [0, 1]$ and that the involution will be $[1, t] \mapsto [1, -t]$.

Therefore we can set $S_1 = [1, t]$, $S_2 = [1, -t]$, $R_1 = [1, s]$, $R_2 = [1, -s]$ and the space $\mathcal{Z}$ parametrizing the two conics of the configuration will be a quotient of

$$(t, s) \in \tilde{\mathcal{Z}} := \mathbf{C}^2 \backslash (\{t = 0\} \cup \{s = 0\} \cup \{s = t\} \cup \{s = -t\}).$$

We note that $\{t = 0\} \cup \{s = 0\} \cup \{s = t\} \cup \{s = -t\}$ is the disjoint union of four copies of $\mathbf{C}^*$ and one point, so we have that the Borel-Moore homology of $\tilde{\mathcal{Z}}$ is $\bar{H}_4(\tilde{\mathcal{Z}}) = \mathbf{Q}(2)$, $\bar{H}_3(\tilde{\mathcal{Z}}) = 4\mathbf{Q}(1)$, $\bar{H}_2(\tilde{\mathcal{Z}}) = 3\mathbf{Q}$ and $\bar{H}_q(\tilde{\mathcal{Z}}) = 0$ for any $q \leq 1$ or $q \geq 5$.

To get the Borel-Moore homology of $\mathcal{Z}$, we need first to consider the following involutions of $\tilde{\mathcal{Z}}$:

- $i :$ $(t, s) \mapsto (s, t)$ exchanges the two points S_1 and R_1, hence the two conics;
- $j :$ $(t, s) \mapsto (\frac{1}{t}, \frac{1}{s})$ exchanges 0 and ∞. Therefore it acts as the involution on l that exchanges the two tangency points;
- $k :$ $(t, s) \mapsto (t, -s)$ exchanges R_1 and R_2. (Note that k has the same action on homology as $k' : (t, s) \mapsto (-t, s)$ so we can consider only one of them).

By studying the action of i, j, k on the stratification of $\mathbf{C}^2 \backslash \tilde{\mathcal{Z}}$ into four copies of $\mathbf{C}^*$ and a point, we obtain:

Lemma 3.1. *The action of i, j, k on the Borel-Moore homology classes of $\tilde{\mathcal{Z}}$ is as given in Table II.1.*

Table II.1

	i	j	k
degree 4	+	+	+
degree 3	+	+	+
	+	+	−
	+	−	+
	−	−	+
degree 2	+	−	+
	+	−	−
	−	+	+

Proof. We consider the following classes of degree 2 in $(\{t=0\}\cup\{s=0\}\cup\{s=t\}\cup\{s=-t\})$:
$[t=0]+[s=0]$,
$[t=0]-[s=0]$,
$[s=t]+[s=-t]$,
$[s=t]-[s=-t]$.
By computing the actions of i,j,k we get the signs written in row '*degree 3*'. To get the signs for the row '*degree 2*' we recall that the Borel-Moore homology of each copy of $\mathbb{C}^*$, with the local system induced by the involution that exchanges $0 \leftrightarrow \infty$, is $\mathbb{S}_2$ in degree 2 and $\mathbb{S}_{1^2}$ in degree 1, so, the signs for the involutions i,k will be the same, while we need to invert the sign for the involution j. Note that there should be a class that is invariant for all i,j,k, but this is the class of the point defining $(\{t=0\}\cup\{s=0\}\cup\{s=t\}\cup\{s=-t\})$ together with four copies of $\mathbb{C}^*$. □

Recall that by exchanging the two conics we are actually exchanging the points R_1 with S_1, and R_2 with S_2. So, in order to get the Borel-Moore homology of $\mathcal{Z}$ from that of $\tilde{\mathcal{Z}}$, we have to consider the invariant classes with respect to the involution i. On the other hand, if we exchange R_1 with R_2, the two conics are not necessarily swapped. So, we also require the classes to be anti-invariant with respect to the action of k.

By Table II.1, $\bar{H}_q(\mathcal{Z})$ is $\mathbf{Q}$ in degree 2, $\mathbf{Q}(1)$ in degree 3, and zero in all other degrees. Therefore, the spectral sequence of the bundle $X_{I+J} \to \mathcal{B}$ will have two rows, determined by the homology classes of the base space:

in degree 3: defined by the Borel-Moore homology of $\mathcal{B}$ with constant coefficients;

in degree 2: defined by the Borel-Moore homology of $\mathcal{B}$ with non-constant coefficients $\mathcal{J}$.

Here, the system of coefficients is determined by the action of the involution j on the corresponding class in $\bar{H}_\bullet(\mathcal{Z})$. In fact, by moving the line l around one of the three points A,B,C, the points T_1,T_2 are exchanged.

To compute the Borel-Moore homology of the base space $\mathcal{B}$, we consider a covering $\tilde{\mathcal{B}}$, where the points A,B,C are ordered. Thus, there is a natural action of the symmetric group $\mathfrak{S}_3$ on $\tilde{\mathcal{B}}$ and we can recover the Borel-Moore homology of $\mathcal{B}$ by taking the $\mathfrak{S}_3$-anti-invariant classes of the Borel-Moore homology of $\tilde{\mathcal{B}}$.

$\bar{H}_\bullet(\tilde{\mathcal{B}};\mathbf{Q})$: Note that $\tilde{\mathcal{B}}$ can be thought of as a fiber space over the space parametrizing three lines through P, and the line l, not passing through P, which is $F(\mathbf{P}^1,3)\times\mathbf{C}^2$. Denote by r_A, r_B, r_C the three lines containing the points A, B, C, respectively. After an appropriate change of coordinates, we may assume

$$r_A : x_2 = 0, \qquad r_B : x_1 = 0, \qquad r_C : x_1 - x_2 = 0, \qquad l : x_0 = 0.$$

Then, the fiber of $\tilde{\mathcal{B}}$ over (r_A, r_B, r_C, l) is the space parametrizing the points A, B, C and can be identified with a subset in $\mathbf{C}^3$: the point $(u, v, w)\in\mathbf{C}^3$ corresponds to the choices

$$A = [1, u, 0]\,, \qquad B = [1, 0, v]\,, \qquad C = [1, w, w]\,.$$

We need then to remove the locus where the three points are collinear, which is a quadric cone of equation $uw + vw - uv = 0$.

Thus, $\bar{H}_\bullet(\tilde{\mathcal{B}};\mathbf{Q})$ is invariant with respect to the involution $u \leftrightarrow v$, and by noticing that this involution corresponds to the exchange of a couple of points among A, B, C, it is also invariant with respect to the $\mathfrak{S}_3$-action.

$\bar{H}_\bullet(\tilde{\mathcal{B}};\mathcal{J})$: In order to compute the Borel-Moore homology of $\tilde{\mathcal{B}}$ with non-constant coefficients, we will consider the subsets

$$\tilde{\mathcal{B}}_J \overset{\text{open}}{\subseteq} \tilde{\mathcal{B}} \qquad \text{and} \qquad \tilde{\mathcal{B}}_I \overset{\text{closed}}{\subseteq} \tilde{\mathcal{B}},$$

where

$$\tilde{\mathcal{B}}_J = \{((A, B, C), l)\in\tilde{\mathcal{B}} | A, B, C \neq P\}$$

defines configurations of type (J), and

$$\tilde{\mathcal{B}}_I = \{((A, B, C), l)\in\tilde{\mathcal{B}} |\text{ one of } A, B, C \text{ is equal to } P\}$$

defines those of type (I).

Consider the projection onto the triples (A, B, C),

$$\tilde{\mathcal{B}}_J \xrightarrow{Y_J} \{(A, B, C) | P \notin \overline{AB}, \overline{BC}, \overline{AC}; A, B, C \text{ not collinear}\},$$

where $Y_J \cong \mathbf{P}^2 \setminus \{\text{four lines in general position}\}$. By studying the preimage of Y_J in the double cover of $\mathbf{P}^2$ ramified along four lines we notice that there is one only class in its Borel-Moore homology with non-constant coefficients, that is $\mathbf{Q}(1)$ in degree 2, and it is $\mathfrak{S}_3$-invariant.

Similarly,

$$\tilde{\mathcal{B}}_I \xrightarrow{Y_I} \{(A, B, C)| \text{ one of } A, B, C \text{ belongs to } E\},$$

where $Y_I \cong \mathbf{P}^2 \setminus \{\text{three lines in general position, one with mult. 2}\}$. Here, because of the double component, the covering is not normal, and its normalization is the double cover of $\mathbf{P}^2$, ramified over the two simple lines. The fiber Y_I and its double cover have the same homology with rational coefficients, thus the Borel-Moore homology with non-constant coefficients is trivial.

Finally, by considering the Gysin exact sequence associated to inclusions

$$\tilde{\mathcal{B}}_I \overset{\text{closed}}{\hookrightarrow} \tilde{\mathcal{B}} \overset{\text{open}}{\hookleftarrow} \tilde{\mathcal{B}}_J$$

we get that $\bar{H}_\bullet(\tilde{\mathcal{B}}; \mathcal{J})$ also has no $\mathfrak{S}_3$-anti-invariant classes.

Since both homologies have only $\mathfrak{S}_3$-invariant classes, the Borel-Moore homology of $\mathcal{B}$, both with constant and non-constant coefficients, will be trivial. Therefore we can conclude that the whole configuration space, and consequently F_{I+J}, has trivial Borel-Moore homology.

Column (K)

Configurations of type (K) are the singular locus of the blow up at P of a conic tangent to a line at P and two other lines in the projective plane. Note that we can assume the conic to be irreducible, since we have already considered the reducible case that is configuration 59. Let us define the space of configurations of the same type with the only exception that we let P free in $\mathbf{P}^2$: $\mathcal{K} := \{(P, f) \in \mathbf{P}^2 \times \mathbf{P}(\Sigma) | f \text{ has a node in } P$ and its singular points define a configuration of type (K)$\}$. Then we can consider the space X_K as the fiber of the bundle

$$\begin{aligned} \mathcal{K} &\to \mathbf{P}^2 \\ (P, f) &\mapsto P. \end{aligned}$$

Let us consider such a configuration. In the projective plane, this is defined by the point P, the intersection point of a conic $\mathcal{C}$ through P, its tangent at P, and 2 general lines r, s, not meeting at P or any other point of the conic. We denote by $E_i, i = 1, \ldots, 4$ the four points of intersection of the two lines and the conic, and by A, B the intersection points with the tangent line to the conic. We label the points as in the following figure.

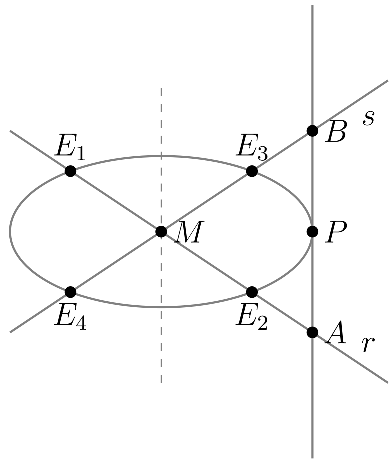

FIGURE II.3: Configuration of type (K).

Up to projective transformations, we may assume the E_i to be the projective frame of $\mathbf{P}^2$: $E_1 = [1, 0, 0]$, $E_2 = [0, 1, 0]$, $E_3 = [0, 0, 1]$, and $E_4 = [1, 1, 1]$. Then we can consider another fiber bundle

$$\mathcal{K} \xrightarrow{PGL_3} Y,$$

where $Y := \{(P, A, B) | P \in \mathbf{P}^2 \setminus \bigcup \overline{E_i E_j}; \{A, B\} = \mathcal{T}_P\mathcal{C} \cap (r \cup s)\}$, and $\mathcal{T}_P\mathcal{C}$ is the tangent line of C at the point P. Note that, once we have fixed the points P, E_i, $i = 1, \ldots, 4$, and hence the lines r, s and the conic, the points A, B are uniquely determined.

Thus, Y is isomorphic to the space $\mathbf{P}^2 \setminus \bigcup \overline{E_i E_j}$, that is isomorphic to the moduli space $\mathcal{M}_{0,5}$ of genus 0 curves with 5 marked points. In fact, for $n \geq 3$,

$$\mathcal{M}_{0,n} = \{(t_0, \ldots, t_{n-3}) \in (\mathbf{P}^1)^{n-3} | t_i \neq 0, 1, \infty, \text{ and } t_i \neq t_j\}.$$

By the equivariant Hodge-Euler characteristic of $\mathcal{M}_{0,5}$, which is computed in [Get95], we have that the Borel-Moore homology of $\mathcal{M}_{0,5}$ is generated by the following classes:

$\mathbf{Q}(2) \otimes \mathbb{S}_5$ in degree 4;

$\mathbf{Q}(1) \otimes \mathbb{S}_{3,2}$ in degree 3;

$\mathbf{Q} \otimes \mathbb{S}_{3,1^2}$ in degree 2;

where by $- \otimes \mathbb{S}_\lambda$ we mean that we are considering the local system of coefficients corresponding to the irreducible representation of $\mathfrak{S}_5$, associated to the partition λ of 5. On Y there is a natural action of the dihedral group D_4, that is the group of symmetries of a square, defined by the points E_i. So, when computing its Borel-Moore homology, we need to consider local systems of coefficients defined by the action of D_4. The action of D_4 can be embedded in the symmetric group $\mathfrak{S}_4$ by sending each symmetry to the corresponding permutation of vertices. Restricting to $\mathfrak{S}_4$, we get the following representations:

$$\begin{aligned} &\mathbb{S}_5 \to \mathbb{S}_4 \\ &\mathbb{S}_{3,2} \to \mathbb{S}_{3,1} \oplus \mathbb{S}_{2,2} \\ &\mathbb{S}_{3,1^2} \to \mathbb{S}_{3,1} \oplus \mathbb{S}_{2,1^2}. \end{aligned} \tag{II.1}$$

We then consider the character table of D_4, plus the lines of the character table of $\mathfrak{S}_4$ corresponding to the irreducible representations in (II.1), that can be found in [Ser77], displayed in Table II.2.

TABLE II.2: Character table of D_4 and some irreducible characters of $\mathfrak{S}_4$.

	e	(12)(34)	(1324)	(12)	(13)(24)	
ψ_1	1	1	1	1	1	
ψ_2	1	1	1	-1	-1	
ψ_3	1	1	-1	1	-1	
ψ_4	1	1	-1	-1	1	
χ	2	-2	0	0	0	
$\mathbb{S}_4$	1	1	1	1	1	$=\psi_1$
$\mathbb{S}_{3,1}$	3	-1	-1	1	-1	$=\chi + \psi_3$
$\mathbb{S}_{2,2}$	2	2	0	0	2	$=\psi_1 + \psi_4$
$\mathbb{S}_{2,1^2}$	3	-1	1	-1	-1	$=\chi + \psi_2$

Hence we can write the Borel-Moore homology groups of $\mathcal{M}_{0,5}$ as D_4- representations:

$\mathbf{Q}(2) \otimes \psi_1$ in degree 4;

$\mathbf{Q}(1) \otimes (\psi_1 + \psi_3 + \psi_4 + \chi)$ in degree 3;

$\mathbf{Q} \otimes (\psi_2 + \psi_3 + \chi^{\oplus 2})$ in degree 2.

Now, we only need to consider the term involving the representation that corresponds to a local system of coefficients obtained by the restriction of $\pm\mathbf{Q}$ on $\pi_1(B(\mathbf{P}^2, 8)) = \mathfrak{S}_8$ to the fundamental group of our configuration space, represented in Figure II.3. As we noticed before, the latter group is $D_4 \subset \mathfrak{S}_8$. In fact, it has to fix the points P, M and the points A, B are uniquely determined by the choice of the points E_i, whose permutations define the symmetric group $\mathfrak{S}_4 \subset \mathfrak{S}_8$. Since we also require $E_1, E_2 \in r$ and $E_3, E_4 \in s$ we get indeed D_4. So the local system we are looking for is the restriction of the sign representation of $\mathfrak{S}_8$ to D_4 whose trace can be computed as follows.

- e: Clearly the identity will be mapped to $+1$;
- $(12)(34)$: the element $(12)(34)$ acts by exchanging the two points on each of the two lines: $E_1 \leftrightarrow E_2$, $E_3 \leftrightarrow E_4$ and thus will give a $+1$;
- (1324) : the element (1324) corresponds to a rotation by $\pi/2$ of the E_i that is an odd permutation of the E_i. It also interchanges the two lines and hence the points A, B, giving a $+1$;
- (12) : the element (12) is the transposition of two points on the same line, moving no other point, so it will be mapped to -1;
- $(13)(24)$: finally, the element $(13)(24)$ corresponds to the symmetry with respect to the dashed line, that is an even permutation. This interchanges again the two lines, and hence A, B, giving -1.

By comparing this to the character table of D_4, we get that the local system we want to consider is the one defined by the representation ψ_2, that we will denote by W. Hence the Borel-Moore homology of Y with coefficients in W is $\mathbf{Q}$ in degree 2 and zero in all other degrees. We can compute the Borel-Moore homology of $\mathcal{K}$ just by tensoring that of Y with the one of PGL_3. The latter can be computed by duality from its cohomology: $\bar{H}_{16}(PGL_3; \mathbf{Q}) = \mathbf{Q}(8)$, $\bar{H}_{13}(PGL_3; \mathbf{Q}) = \mathbf{Q}(6)$, $\bar{H}_{11}(PGL_3; \mathbf{Q}) = \mathbf{Q}(5)$, $\bar{H}_8(PGL_3; \mathbf{Q}) = \mathbf{Q}(3)$, and it is zero in all other degrees. Therefore, $\bar{H}_{18}(\mathcal{K}; \mathbf{Q}) = \mathbf{Q}(8)$, $\bar{H}_{15}(\mathcal{K}; \mathbf{Q}) = \mathbf{Q}(6)$, $\bar{H}_{13}(\mathcal{K}; \mathbf{Q}) = \mathbf{Q}(5)$, $\bar{H}_{10}(\mathcal{K}; \mathbf{Q}) = \mathbf{Q}(3)$, and it is zero in all the other degrees.

Finally we compute the Borel-Moore homology of X_K from the fibration

$$\mathcal{K} \to \mathbf{P}^2.$$

Since $\mathbf{P}^2$ is simply connected, there is a first quadrant spectral sequence

$$E^2_{p,q} = \bar{H}^p(\mathbf{P}^2) \otimes \bar{H}^q(X_K) \Rightarrow \bar{H}^{p+q}(\mathcal{K};\mathbf{Q}),$$

	0	1	2	3	4
14	$\mathbf{Q}(6)$		$\mathbf{Q}(7)$		$\mathbf{Q}(8)$
13	$\mathbf{Q}(5)$	$\nwarrow$	$\mathbf{Q}(6)$	$\nwarrow$	$\mathbf{Q}(7)$
12					
11	$\mathbf{Q}(4)$		$\mathbf{Q}(5)$		$\mathbf{Q}(6)$
10	$\mathbf{Q}(3)$	$\nwarrow$	$\mathbf{Q}(4)$	$\nwarrow$	$\mathbf{Q}(5)$
9					

where the differentials $d^2_{2,10}$, $d^2_{4,10}$, $d^2_{2,13}$, $d^2_{4,13}$ must all be isomorphisms in order to obtain the Borel-Moore homology of the total space that we computed above. Therefore, since the space F_K is a $\mathbf{C} \times \mathring{\Delta}_7$-bundle over X_K, the Hodge-Grothendieck polynomial of $\bar{H}_\bullet(F_K;\mathbf{Q})$ must be $\mathbf{L}^{-7}t^{23} + \mathbf{L}^{-6}t^{22} + \mathbf{L}^{-5}t^{20} + \mathbf{L}^{-4}t^{19}$.

Column (L)

As a consequence of Proposition I.3.3, F_L is an open cone and its Borel-Moore homology can be obtained from the spectral sequence in Table II.3.

Table II.3

	A	B	C	D	E	F	G	H	K
12									$\mathbf{Q}(6)$
11									$\mathbf{Q}(5)$
10									
9									$\mathbf{Q}(4)$
8									$\mathbf{Q}(3)$
7									
6									
5									
4									
3				$\mathbf{Q}(3)$			$\mathbf{Q}(4) \leftarrow$	$\mathbf{Q}(4)$	
2		$\mathbf{Q}(2)$			$\mathbf{Q}(3)$	$\mathbf{Q}(3)$			
1	$\mathbf{Q}(1)$			$\mathbf{Q}(2)^2$			$\mathbf{Q}(3)$		
0		$\mathbf{Q}(1)$	$\mathbf{Q}(1)$			$\mathbf{Q}(2)$			
-1	$\mathbf{Q}$			$\mathbf{Q}(1)$					

Its columns coincide with those of the main spectral sequence, which is the one converging to $\bar{H}_\bullet(\Sigma;\mathbf{Q})$, shifted by twice the dimension of the complex vector bundle that defines each column. The space $V\setminus\Sigma$ is affine of dimension 18, hence $H^i(V\setminus\Sigma) \cong \bar{H}_{35-i}(\Sigma)$ must be trivial for any $i > 18$. For this reason, the differential $d^1_{H,3} : E^1_{H,3} \to E^1_{G,3}$ is non-trivial, and in the second page of the spectral sequence all differentials in the columns $(A) - (G)$ are also non-trivial.

We can then conclude that the Hodge-Grothendieck polynomial of $\bar{H}_\bullet(F_L; \mathbf{Q})$ is $\mathbf{L}^{-6}t^{22} + \mathbf{L}^{-5}t^{21} + \mathbf{L}^{-4}t^{19} + \mathbf{L}^{-3}t^{18}$.

3.2 Main spectral sequence

The first page of the spectral sequence converging to $\bar{H}_\bullet(\Sigma, \mathbf{Q})$ is given in Table II.4.

Table II.4

	A	B	C	D	E	F	G	H	K	L
32		$\mathbf{Q}(17)$								
31	$\mathbf{Q}(16)$									
30		$\mathbf{Q}(16)$								
29	$\mathbf{Q}(15)$									
28										
27				$\mathbf{Q}(15)$						
26			$\mathbf{Q}(14)$		$\mathbf{Q}(15)$					
25				$\mathbf{Q}(14)^2$						
24										
23				$\mathbf{Q}(13)$						
22						$\mathbf{Q}(13)$				
21							$\mathbf{Q}(13)$			
20						$\mathbf{Q}(12)$				
19							$\mathbf{Q}(12)$			
18										
17								$\mathbf{Q}(11)$		
16										
15										
14									$\mathbf{Q}(7)$	
13									$\mathbf{Q}(6)$	
12										$\mathbf{Q}(6)$
11									$\mathbf{Q}(5)$	$\mathbf{Q}(5)$
10									$\mathbf{Q}(4)$	
9										$\mathbf{Q}(4)$
8										$\mathbf{Q}(3)$

Following Section II.2.1, the cohomology of X must contain a copy of the cohomology of GL_2. So, all differentials in Table II.4 are trivial. Applying then the isomorphism induced by the cap product with the fundamental class of the discriminant

$$\tilde{H}^\bullet(X;\mathbf{Q}) \cong \bar{H}_{35-\bullet}(\Sigma;\mathbf{Q})(-18)$$

we compute the whole cohomology of X and that of X/GL_2. The Hodge-Grothendieck polynomial of the cohomology of X/GL_2 is $\mathbf{L}^{12}t^{13} + \mathbf{L}^{11}t^{12} + \mathbf{L}^4t^6 + \mathbf{L}^3t^5 + \mathbf{L}^2t^3 + 1$.

We finally consider the fibration

$$X/GL_2 \to X/(\mathbf{C}^* \times GL_2).$$

There is a first quadrant cohomology spectral sequence starting with E_2 and converging to $H^\bullet(X/GL_2;\mathbf{Q})$:

$$E_2^{p,q} = H^p(X/(\mathbf{C}^* \times GL_2); H^q(\mathbf{C}^*;\mathbf{Q})) \Rightarrow H^{p+q}(X/GL_2;\mathbf{Q}).$$

Since we know the cohomology of the total space and of the fibre, we can compute the cohomology of the base space from the second page of the spectral sequence represented in Table II.5, with non-trivial differential $d_2^{0,1} : E_2^{0,1} \to E_2^{2,0}$ because the term $\mathbf{Q}(-1)$ is not appearing in the cohomology of X/GL_2.

TABLE II.5

1	$\mathbf{Q}(-1)$		$\mathbf{Q}(-2)$			$\mathbf{Q}(-4)$							$\mathbf{Q}(-12)$
0	$\mathbf{Q}$		$\mathbf{Q}(-1)$			$\mathbf{Q}(-3)$							$\mathbf{Q}(-11)$
	0	1	2	3	4	5	6	7	8	9	10	11	12

The choice of this spectral sequence is due to [CL21a, Theorem 1.1]. An alternative spectral sequence would give two additional classes $\mathbf{Q}(-2) \in H^3(\mathcal{T}_5;\mathbf{Q})$, $\mathbf{Q}(-2) \in H^4(\mathcal{T}_5;\mathbf{Q})$, but this is impossible since the rational Chow ring of $\mathcal{T}_5$ is trivial in degree bigger or equal than 2.

Therefore, the Hodge-Grothendieck polynomial of the cohomology of the base space, and hence of the moduli space of trigonal curve of genus 5, is $\mathbf{L}^{11}t^{12} + \mathbf{L}^3t^5 + \mathbf{L}t^2 + 1$.

3.3 Trivial configurations

As we promised in the computation of the spectral sequences in Tables II.3 and II.4, we now consider the remaining configurations and prove that they have trivial twisted

Borel-Moore homology.

Configurations (42)-(43)

Both these configurations are equivalent to the configurations of singularities of a plane quintic that is the union of a conic and a singular cubic. To be more precise, in the first type, the two curves meet each other at 6 distinct points and P is any of the points of intersection, while in the second one they intersect at the singular point of the cubic, that is P.

FIGURE II.4: Configurations of type (42)-(43).

Both configuration spaces can be fibered over the space of conics through P. If we denote the conic by $\mathcal{C}$, the fibers will be respectively isomorphic to $B(\mathcal{C}\backslash\{P\},5)$ and $B(\mathcal{C}\backslash\{P\},4)$. They both have trivial twisted Borel-More homology by Lemma I.4.1.

Configurations (44)-(45)-(46)

These configurations are all obtained by blowing up a singular point in the configuration of type 37 in [Gor05], defined by the intersection points of two lines and a cubic curve in $\mathbf{P}^2$ having one singular point. To be more precise, configurations of these types correspond to the blow up at P, where P has to be an ordinary double point.

In the first type, P is defined as the point of intersection between a line and the cubic, in the second one, it is the point of intersection between the two lines, and finally it is the singular point of the cubic. Note that, in the first two configuration spaces, the cubic need not to be irreducible: it can decompose into three concurrent lines or into the union of a conic and a line tangent to it. However, this cannot happen for configuration (46), otherwise P would not be an ordinary double point. The two reducible cases, with P not an ordinary double point, define configurations (59) and (55), respectively. Configuration (55) was already considered as configuration (K), while configuration (59) will be considered later.

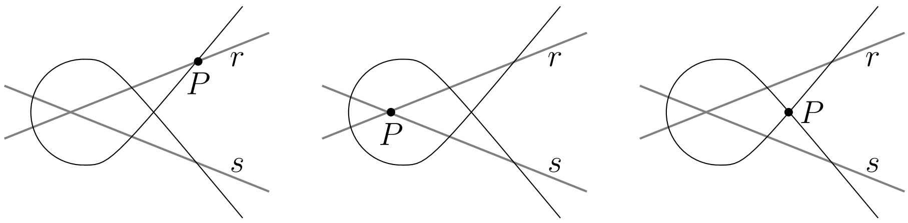

FIGURE II.5: Configurations of type (44)-(45)-(46).

Configuration spaces of type (44), (45), (46) can then all be fibered over the space parametrizing the two lines r, s. The fibers are isomorphic to the quotient of $B(\mathbf{C}^*, 2) \times B(\mathbf{C}, 3)$, $B(\mathbf{C}, 3) \times B(\mathbf{C}, 3)$ and $B(\mathbf{C}, 3) \times B(\mathbf{C}, 3)$, respectively, by the involution given by exchanging the two lines. Since they have all trivial twisted Borel-Moore homology, the homology of the configuration spaces will be trivial as well.

Configuration (49)

Configurations of type (49) are obtained by the same plane curve considered for type (J), where P is defined as the point of intersection of a conic and the line.

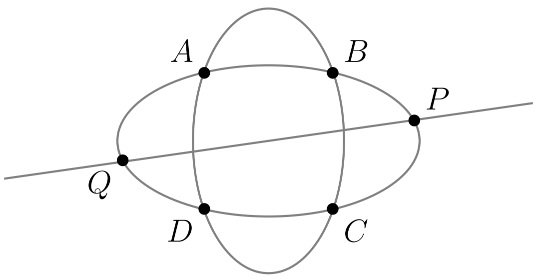

FIGURE II.6: Configuration of type (49).

Then $X_{(49)}$ can be fibered over $\tilde{B}(\mathbf{P}^2\backslash\{P\}, 4) \ni \{A, B, C, D\}$. Once these points are fixed, we notice that the conic $\mathcal{C}$ passing also through P is uniquely determined. Therefore the fiber Y is itself a fiber bundle over the space $L \cong \mathbf{P}^1\backslash\{5 \text{ points}\}$ of lines not passing through any of the points A, B, C, D and not tangent to $\mathcal{C}$. The fiber $\mathcal{Z}$ is defined as the space of conics, passing through $\{A, B, C, D\}$, not tangent to $l \in L$ and different from $\mathcal{C}$.

The space $\mathcal{Z}$ is isomorphic to $\mathbf{P}^1\backslash\{0, 1, \infty\} \cong \mathbf{C}\backslash\{0, 1\}$. Moreover, determining a conic in $\mathcal{Z}$ is equivalent to choosing a point in l that is different to P, Q and the 2 points of tangency T_1, T_2 in l. Thus, $\bar{H}_\bullet(\mathcal{Z}, \pm\mathbf{Q})$ is $\mathbf{Q}$ in degree 1 and 0 in all other degrees. Note also that, when moving l around A, for instance, the points of tangency in l are swapped. Therefore $\pi_1(L)$ acts on $\bar{H}_1(\mathcal{Z}, \pm\mathbf{Q})$ anti-invariantly and the Borel-Moore homology of the fiber Y is defined by that of L with non-trivial coefficient system:

$$\bar{H}_\bullet(L; \bar{H}_1(\mathcal{Z})) = \mathbf{Q}, \qquad \text{in degree } 0.$$

Finally, notice that we are considering a local system on L that changes its sign under the action of any loop in $\mathbf{P}^1$ around any point removed. Therefore any $\gamma \in \tilde{B}(\mathbf{P}^2 \backslash \{P\}, 4)$ transposing a pair of points must act on the fiber as the multiplication by -1. This means that the local system induced by the fiber on $\tilde{B}(\mathbf{P}^2 \backslash \{P\}, 4)$ is $\pm\mathbf{Q}$ and by Lemmas I.4.3 and I.4.6 the twisted Borel-Moore homology of $X_{(49)}$ will be trivial.

Configurations (50)-(51)

In all the following configuration spaces P has to be a triple point. More precisely, they are defined by blowing up the following curves at P.

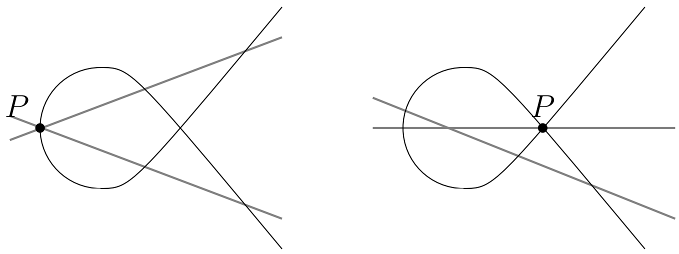

FIGURE II.7: Configurations of type (50)-(51).

We can fiber the configuration spaces over the space parametrizing the pairs of lines. The fiber spaces will be then isomorphic to a quotient of $B(\mathbf{C}, 2) \times B(\mathbf{C}, 2)$ and $\mathbf{C}^* \times B(\mathbf{C}, 3)$, respectively, and they both have trivial twisted Borel-Moore homology.

Configuration (52)

As above, P must be again a triple point. In particular, configurations of type (52) are defined by two distinct conics meeting at P and three additional points A, B, C, and a line l through P, not meeting any of A, B, C.

Then, the configuration space can be fibered over the space $\tilde{B}(\mathbf{P}^2 \backslash \{P\}, 3) \times (\mathbf{P}^1 \backslash \{3 \text{ points}\}) \ni \{(\{A, B, C\}, l)\}$, parametrizing the intersection points between the two conics and the choices for the line l. Once we have fixed l, two points on it will

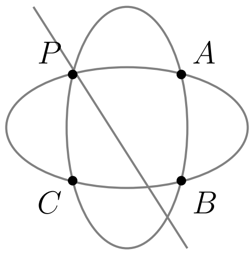

FIGURE II.8: Configuration of type (52).

uniquely determine the two conics. Hence, the fiber space will be $B(\mathbf{C}, 2)$ whose Borel-Moore homology will be considered with constant coefficients. In fact, when we exchange the two conics we are actually exchanging 2 couples of points in the configuration space: the two points lying on the line, and the two points of intersection between the exceptional divisor and the strict transforms of the two conics. On the other hand, there is a natural action of $\mathfrak{S}_3$ on the base space. By noticing that both factors have no $\mathfrak{S}_3$-anti-invariant classes in their homologies[1], the total space will have trivial twisted Borel-Moore homology.

Configurations (53)-(54)

These configurations are obtained by blowing up a point of intersection between two lines and a point of intersection between a line and a conic, respectively, in the set of 9 distinct points in $\mathbf{P}^2$ defined as follows. Three points A, B, C are in general position, defined as the intersection points of three distinct lines. The other six points are defined as the intersection points between the three lines $\overline{AB}, \overline{BC}, \overline{AC}$ and a conic not tangential to the lines. This is configuration 39 in [Gor05].

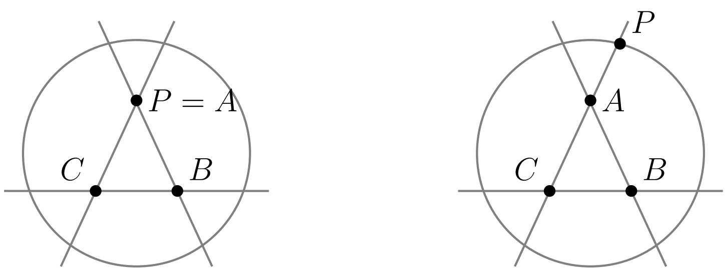

FIGURE II.9: Configurations of type (53)-(54).

[1]For the factor $\tilde{B}(\mathbf{P}^2 \setminus \{P\}, 3)$ this follows by Lemmas I.4.3 and I.4.6. While for the second factor, this can be deduced by computing the Borel-Moore homology of $\mathbf{P}^1 \setminus \{3 \text{ points}\}$ in terms of $\mathfrak{S}_3$-representations, that is $\mathbb{S}_3(1)$ in degree 2 and $\mathbb{S}_{2,1}$ in degree 1.

As in [Gor05], we want to fiber both configuration spaces over the spaces parametrizing the points of intersection between the three lines. When we choose P as one of these points, e.g. A, the total space will be fibered over $B(\mathbf{P}^2\backslash\{P\},2)$ instead of $B(\mathbf{P}^2,3)$. On the other hand, when P is the intersection point between the conic and a line, the configuration space is fibered over a quotient of $B(\mathbf{P}^2\backslash\{P\},2)\times\mathbf{C}^* \ni (\{B,C\},A)$. The fiber space, denoted by Y in [Gor05], will be in both cases the same, i.e. a fiber bundle over $B(\mathbf{C}^*,2)\times B(\mathbf{C}^*,2)$, the configuration space of 2 points on each of $\overline{AB},\overline{BC}$, excluding A,B,C, with fiber isomorphic to $\mathbf{C}^*$ minus a point. Therefore it will have the same Borel-Moore homology, which is $\mathbf{Q}$ in degree 5, $\mathbf{Q}(1)^2$ in degree 6, $\mathbf{Q}(2)$ in degree 7 and zero in all other degrees.

However we will have to consider the action of the fundamental group of the new base space that is either $B(\mathbf{P}^2\backslash\{P\},2)$, or it contains it as a factor of a product. The fundamental group will then be the restriction of the symmetric group $\mathfrak{S}_3$ to $\{B,C\}$: $\mathfrak{S}_2$. Thus, we only need to consider local systems of coefficients corresponding to the restrictions of the representations of $\mathfrak{S}_3$: trivial and sign representation will restrict respectively to trivial and sign representation on $\mathfrak{S}_2$, while the 2-dimensional irreducible representation restricts to the direct sum of the trivial and sign representation.

We have that $\bar{P}(B(\mathbf{P}^2\backslash\{P\},2),\mathbf{Q})=\mathbf{L}^{-4}t^8$ and $\bar{P}(B(\mathbf{P}^2\backslash\{P\},2),\pm\mathbf{Q})=\mathbf{L}^{-3}t^6$, by Lemmas I.4.3, I.4.4 and I.4.7. So we will get a similar E^2-term of the spectral sequence, with the only difference that the action of $\mathfrak{S}_2$ on $\bar{H}_6(Y;\pm\mathbf{Q})=\mathbf{Q}(1)^2$ now must be reducible.

$$\begin{tikzcd}
7 & \mathbf{Q}(5) & & & & \\
6 & \mathbf{Q}(4) & & \mathbf{Q}(5) \arrow[ul] & & \\
5 & & & \mathbf{Q}(4) \arrow[ul] & & \\
 & 6 & 7 & 8 & 9 & 10
\end{tikzcd}$$

The differentials $d^2_{8,5}:E^2_{8,5}\to E^2_{6,6}$, $d^2_{8,6}:E^2_{8,6}\to E^2_{6,7}$ must be non-trivial because otherwise we would get non-trivial classes in the main spectral sequence whose corresponding cohomology is not divisible by that of GL_2, contradicting section II.2.1. Therefore, also these configuration spaces have trivial twisted Borel-Moore homology.

Configurations (56)-(57)

These configuration spaces are the configurations of singularities obtained by blowing up the following curves at P.

FIGURE II.10: Configurations of type (56)-(57).

Consider first configurations of type (56). Similarly to configurations (53), (54), we fiber the space over the points of intersection between the lines, that is $\tilde{B}(\mathbf{P}^2\backslash\{P\},2)$. These two points, together with P, uniquely determine the three lines. The other 4 points left, together with P, will uniquely determine the conic. The fiber space is then isomorphic to the quotient of $\mathbf{C}^* \times \mathbf{C}^* \times B(\mathbf{C}^*, 2)$ by the involution exchanging the first two factors. Thus it has twisted Borel-Moore homology equal to $\mathbf{Q}$ in degree 4 and $\mathbf{Q}(1)$ in degree 5. The fundamental group of the base space acts by exchanging the two points, and thus induces an $\mathfrak{S}_2$-action on the line not passing through P that is the one described in Lemma I.4.2. Therefore, the Borel-Moore homology class in degree 5 must be anti-invariant under such an action, while the class in degree 4 will be invariant. By applying Lemmas I.4.3, I.4.4 and I.4.6 we have that the second page of the spectral sequence must have the following form:

$$\begin{array}{c|ccc}
5 & \mathbf{Q}(4) & \longleftarrow & \\
4 & & & \mathbf{Q}(4) \\
\hline
 & 6 & 7 & 8
\end{array}$$

where the differential will be an isomorphism.

The second configuration space can be fibered over the space parametrizing the three lines through P. Since it suffices to fix 5 points on those lines to determine the conic, the fiber space will be a quotient of $B(\mathbf{C}, 2) \times B(\mathbf{C}, 2) \times \mathbf{C}$, which has trivial twisted Borel-Moore homology.

Configurations (58)-(59)

These configuration spaces are both defined by 5 lines in the projective plane. The first is obtained by blowing up any of the singular points of 5 lines in general position, while the second one is obtained by blowing up the point of intersection of three concurrent lines, in a plane quintic defined by such three lines and two additional lines meeting at a point outside the other lines.

FIGURE II.11: Configurations of type (58)-(59).

We can fiber both configuration spaces over the set of lines meeting at P, thus:

$$X_{(58)} \to B(\mathbf{P}^1, 2) \qquad \text{and} \qquad X_{(59)} \to B(\mathbf{P}^1, 3).$$

The fiber spaces will then be the spaces of the remaining lines defining the configuration that are, respectively, $B(\mathbf{C}^2, 3)$ and $B(\mathbf{C}^2, 2)$, by duality. Both have trivial twisted Borel-Moore homology by Lemma I.4.1.

Chapter III

Stable cohomology of $\mathcal{T}_g$

This chapter is based on [Zhe21b].

1 Introduction and results

In this chapter, we give a description of the stable rational cohomology of $\mathcal{T}_g$ by studying the loci of trigonal curves lying on each Hirzebruch surface, hence each stratum in the Maroni stratification. We will use again Gorinov-Vassiliev's method [Vas99], [Gor05], [Tom05b], which reduces the computation of the cohomology of complements of discriminants to the study of a filtration on a geometric realization of the discriminant, based on a classification of its singular loci. In particular we won't consider the whole classification, but only the families of singular configurations having low codimension in the vector space in which the discriminant is defined.

Our starting point will be the result in [Zhe21a], where we computed the rational cohomology of the moduli space of trigonal curves of genus 5. For $g = 5$, in fact, all trigonal curves lie on the first Hirzebruch surface $\mathbb{F}_1$ as smooth divisors. However, $\mathbb{F}_1$ and the other $\mathbb{F}_n$'s contain other trigonal curves of higher genera. For higher values of g, the classification of the singular loci of such curves is more complicated, but we will see that the families of singular configurations we will consider have a description which is analogous to the one we had for $g = 5$. This will allow us to compute the cohomology of trigonal curves lying on $\mathbb{F}_n$ in a certain range.

We will exhibit the procedure for any Hirzebruch surface of degree $n \geq 0$ in order to compute the stable cohomology of the locus of trigonal curves lying on it, defined as $N_n := \{[C] \in \mathcal{T}_g | \, C$ has Maroni invariant $n\}$.

We prove first that the cohomology of each N_n stabilizes and compute it in the stable range.

By considering then the spectral sequence associated to the Maroni stratification of $\mathcal{T}_g$, we show that almost all the cohomology classes of all strata N_n are canceled by non-trivial differentials in the E_1 and E_2 pages.

We finally obtain a description of the stable cohomology of $\mathcal{T}_g$, for g sufficiently large. Precisely,

Theorem 1.1. *The rational cohomology of $\mathcal{T}_g$, in degree $i < \left\lfloor \frac{g}{4} \right\rfloor$, is*

$$H^i(\mathcal{T}_g; \mathbf{Q}) = \begin{cases} \mathbf{Q}, & i = 0; \\ \mathbf{Q}(-1), & i = 2; \\ \mathbf{Q}(-2), & i = 4; \\ 0, & \textit{otherwise.} \end{cases}$$

Remark 4. Note that, if $g \equiv 2 \bmod 4$, the above description of the rational cohomology of $\mathcal{T}_g$ holds for $i \leq \left\lfloor \frac{g}{4} \right\rfloor$.

In [PV15a] Patel and Vakil proved that the rational Chow ring $A^*_{\mathbf{Q}}(\mathcal{T}_g)$ coincides with its tautological ring, denoted $R^*_{\mathbf{Q}}(\mathcal{T}_g)$, which is defined as the subring generated by the pullback of the tautological classes in $A^*_{\mathbf{Q}}(\mathcal{M}_g)$. In particular, they proved that it is generated by a single class in codimension 1, the kappa class κ_1. Then, our main result also implies that

Corollary 1.2. *For g, i as above,*

$$H^i(\mathcal{T}_g; \mathbf{Q}) = \begin{cases} R^{i/2}_{\mathbf{Q}}(\mathcal{T}_g), & i \textit{ even;} \\ 0, & i \textit{ odd.} \end{cases}$$

Remark 5. For $g = 3, 4, 5$, $H^\bullet(\mathcal{T}_g; \mathbb{Q})$ is completely known from [Loo93], [Tom05b], [Zhe21a], respectively. However, in none of these cases the cohomology ring is tautological.

From the proof of our main result, we can also deduce the stable cohomology of the moduli space $\mathcal{T}_g^\dagger$ of framed triple covers, i.e. the moduli space parametrizing pairs (C, α) with C a smooth curve of genus g and α a degree 3 map from C to a fixed $\mathbf{P}^1$. Notice that $\mathcal{T}_g^\dagger$ is the underlying moduli space of the stack $\mathcal{H}_{3,g}^\dagger$, defined in [PV15a].

Corollary 1.3. *Let $g \geq 6$, the rational cohomology of $\mathcal{T}_g^\dagger$, in degree $i < \lfloor \frac{g}{4} \rfloor$, is*

$$H^i(\mathcal{T}_g^\dagger; \mathbf{Q}) = \begin{cases} \mathbf{Q}, & i = 0; \\ \mathbf{Q}(-1), & i = 2; \\ \mathbf{Q}(-3), & i = 5; \\ \mathbf{Q}(-4), & i = 7; \\ 0 & \textit{otherwise.} \end{cases}$$

Remark 6. Let us remark that our results are in contradiction with [PV15a, Theorems A and B]. For a sufficiently large g, the rational Chow ring of $\mathcal{T}_g^\dagger$ is strictly smaller than that of $\mathcal{T}_g$, which has a different description from the one given by Patel and Vakil. In fact, there was an error in the last section of their preprint concerning relations between kappa classes, which has been corrected by Canning and Larson in [CL21a].

The chapter is organized as follows. In section 2 we introduce the main ingredients that we will need in order to prove the results that we have just stated. Then, in section 3 we will apply Gorinov-Vassiliev's method to our setting and we prove in section 4 that the generalized Leray-Hirsch theorem can be applied. Finally, we give a proof of Theorem 1.1 and of Corollary 1.3 in section 5.

2 Trigonal curves as divisors in $\mathbb{F}_n$ and codimensions of spaces of sections

In this section we introduce the notation and some results that we will use throughout this chapter. Recall once more from Prop I.2.2 that any trigonal curve of genus g can be embedded in a Hirzebruch surface $\mathbb{F}_n$ as a divisor of class

$$C \sim 3E_n + \frac{g + 3n + 2}{2} F_n,$$

with Maroni invariant n, such that $g \equiv n \bmod 2$ and $0 \leq n \leq (g + 2)/3$.

Definition 8. Define $V_{d,n}$ to be the vector space of global sections of $\mathcal{O}_{\mathbb{F}_n}(3E_n + dF_n)$. The discriminant locus $\Sigma_{d,n} \subset V_{d,n}$ is the closed subset of sections defining singular curves and $X_{d,n}$ is its complement in $V_{d,n}$.

There is an explicit way to compute the dimension of $V_{d,n}$. Recall that a further description of a Hirzebruch surface $\mathbb{F}_n$, with $n \geq 1$, is given by blowing up the weighted projective space $\mathbf{P}(1,1,n)$ at its singular point $[0,0,1]$:

$$\mathbb{F}_n = Bl_{[0,0,1]}\mathbf{P}(1,1,n),$$

where $\mathbf{P}(1,1,n) = \operatorname{Proj} \mathbf{C}[x,y,z]$ such that $\deg x = \deg y = 1$ and $\deg z = n$.
Then, a polynomial f defining a trigonal curve of degree d in $\mathbf{P}(1,1,n)$ is of the form:

$$f(x,y,z) = \alpha_{d-3n}(x,y)z^3 + \beta_{d-2n}(x,y)z^2 + \gamma_{d-n}(x,y)z + \delta_d(x,y) = 0 \tag{III.1}$$

where $\alpha_{d-3n}(x,y), \beta_{d-2n}(x,y), \gamma_{d-n}(x,y), \delta_d(x,y)$ are homogeneous polynomials in the coordinates x,y, of degrees $d-3n$, $d-2n$, $d-n$, d respectively, with $d \geq 3n$.

We can visualize the coefficients in the following figure:

$$\begin{array}{lccccccccccccccc}
d-3n+1 & & & & \bullet & & \dots & & \dots & & \bullet & & & \\
d-2n+1 & & & \bullet & & \bullet & & \dots & & \bullet & & \bullet & & \\
d-n+1 & & \bullet & & \bullet & & \dots & & \dots & & \bullet & & \bullet & \\
d+1 & \bullet & & \bullet & & \dots & & \dots & & \dots & & \bullet & & \bullet
\end{array} \tag{III.2}$$

where the j-th row, $j = 1, \dots, 4$, corresponds to the coefficients of monomials $x^a y^b z^{3-j+1}$ with $a + b = d - n(3 - j + 1)$. Counting the number of parameters we get that $v_{d,n} := \dim_{\mathbf{C}} V_{d,n} = 4d + 4 - 6n$, when $n \geq 1$. Note that this also agrees when $n = 0$: $V_{d,0}$ is isomorphic to the vector space of polynomials of bidegree $(3,d)$, with $d = \frac{g+2}{2}$, on $\mathbf{P}^1 \times \mathbf{P}^1$, i.e.

$$V_{d,0} \cong \mathbf{C}[x_0, x_1, y_0, y_1]_{3,d} \cong \mathbf{C}^{4(d+1)}.$$

We now want to consider elements in $V_{n,d}$ which are singular at configuration spaces $B(\mathbb{F}_n, N)$, discussed in section I.4. Since we are dealing with projective surfaces, requiring any polynomial $f \in V_{n,d}$ to be singular at any fixed point in $\mathbb{F}_n$ will impose 3 linearly independent conditions. If we require f to be singular at a configuration of N

points we expect the number of imposed conditions to be $3N$ and we will prove that this is indeed what happens when d is sufficiently large with respect to N.

Lemma 2.1. *Fix $N \geq 1$. For any $n \geq 0$, the restriction of*

$$\{(f, p_1, \dots, p_N) \in V_{d,n} \times B(\mathbb{F}_n, N) | p_1, \dots, p_N \in \mathrm{Sing}(f)\} \xrightarrow{\pi} B(\mathbb{F}_n, N)$$

to the locus where at most two points p_i lie on the same line of the ruling is a vector bundle of rank $v_{d,n} - 3N$ provided $d \geq 2N + 3n - 1$ holds.

Remark 7. Before proving the lemma, note that we can further simplify the assumption that no more than two points lie on the same line of the ruling, by considering only the case where they all belong to distinct lines of the ruling. Clearly, curves which are singular at pairs of points in the same line of the ruling are easier to treat and they can be reduced to curves of smaller degree which are singular at points lying on distinct lines of the ruling. The reason behind this is that, in both cases, the vector subspaces in $V_{d,n}$ of curves which are singular at these points have the expected codimension.

In fact, let us consider first a set of $N = 2k$ points consisting of k couples of points on k distinct lines of the ruling. It is easy to show that the vector subspace of curves which are singular at these $2k$ points is non-empty for $d \geq \frac{3}{2}(k+n) - 1$, which is always satisfied when $d \geq 2N + 3n - 1$, and it is given exactly by all polynomials of the form

$$\ell_1 \cdots \ell_k g, \tag{III.3}$$

where $\ell_1, \dots, \ell_k$ are the equations of the k lines of the ruling containing the $2k$ points and g is a section of $\mathcal{O}_{\mathbb{F}_n}(3E_n + (d-k)F_n)$, vanishing at the $2k$ prescribed points. Then, counting parameters as we have done in (III.2), the vector subspace generated by these polynomials has dimension $4(d-k) + 4 - 6n - 2k$, which is non-negative by the assumption $d \geq \frac{3}{2}(k+n) - 1$, hence it has codimension $6k = 3N$ in $V_{d,n}$.

This also holds if we consider a set of N points consisting of $2k$ points, defined as above, together with h points, each lying on a distinct line of the ruling, all different from the k lines of the ruling containing the $2k$ points. In this case the vector subspace of curves which are singular at these N points is given exactly by polynomials of the form (III.3), where we further require g to be singular at the h points. As we will prove below, this last assumption will increase the codimension by $3h$ and thus the codimension in $V_{d,n}$ of the vector subspace generated by these polynomials will be $6k + 3h = 3N$.

Proof. By the above remark, we will assume that all p_i's lie on distinct lines of the ruling. Following the proof of [Tom20, Lemma 4], let us fix a set of N distinct points $\{p_1, \ldots, p_N\} \subset \mathbb{F}_n$. Assume first that $n \geq 1$, and consider the evaluation map

$$\mathbf{C}[x, y, z]_d \xrightarrow{ev} M_{3,N}(\mathbf{C})$$

which assigns to each $f(x, y, z) = \alpha(x, y)z^3 + \beta(x, y)z^2 + \gamma(x, y)z + \delta(x, y)$ in the weighted polynomial ring $\mathbf{C}[x, y, z]_d$, with $\deg x = \deg y = 1$ and $\deg z = n$, a $3 \times N$ matrix whose i-th column is defined by

$$\begin{cases} \begin{pmatrix} \partial f/\partial x(p_i) \\ \partial f/\partial y(p_i) \\ \partial f/\partial z(p_i) \end{pmatrix} & \text{if } p_i \in \mathbb{F}_n \backslash E_n; \\ \begin{pmatrix} \partial \alpha/\partial x_0(p_i) \\ \partial \alpha/\partial y_0(p_i) \\ \beta(p_i) \end{pmatrix} & \text{if } p_i \in E_n; \end{cases}$$

where x_0, y_0 denote the coordinates in $E_n \cong \{([0,0,1], [x_0, y_0]) \in \{[0,0,1]\} \times \mathbf{P}^1\} \subset Bl_{[0,0,1]}\mathbf{P}(1,1,n) \cong \mathbb{F}_n$. The evaluation map is a linear map and its kernel is exactly the fiber of π over $\{p_1, \ldots, p_N\}$. Therefore, in order to prove the lemma, it is sufficient to show that ev is surjective for $d \geq 2N + 3n - 1$.

Consider first the case where $p_1 \in \mathbb{F}_n \backslash E_n$. After an appropriate change of coordinates we may assume that $p_1 = [1, 0, 0]$. Fix a degree $r \geq 3n+1$ and consider the polynomials

$$\varphi_0 = x^r \ell_2^2 \cdots \ell_N^2,$$

$$\varphi_1 = x^{r-1} y \ell_2^2 \cdots \ell_N^2,$$

$$\varphi_2 = x^{r-n} z \ell_2^2 \cdots \ell_N^2.$$

where $\ell_2, \ldots, \ell_N$ are the equations of the lines of the ruling containing $p_2, \ldots, p_N$. All φ_i vanish with multiplicity 2 at $p_2, \ldots, p_N$ and hence they are all sent to matrices with trivial entries outside the first column. Moreover, since $p_1 \notin \ell_i$, for $i \geq 2$, then

$$\ell_2 \cdots \ell_N(p_1) = a_0 \neq 0; \qquad \frac{\partial \ell_2 \cdots \ell_N}{\partial x}(p_1) = N a_0; \qquad \frac{\partial \ell_2 \cdots \ell_N}{\partial y}(p_1) = a_1 \neq 0.$$

and the evaluations on such polynomials are

$$ev(\varphi_0) = \begin{pmatrix} ra_0^2 + 2Na_0^2 & 0 & \dots \\ 2a_1 & 0 & \dots \\ 0 & 0 & \dots \end{pmatrix},$$

$$ev(\varphi_1) = \begin{pmatrix} 0 & 0 & \dots \\ a_0^2 & 0 & \dots \\ 0 & 0 & \dots \end{pmatrix},$$

$$ev(\varphi_2) = \begin{pmatrix} 0 & 0 & \dots \\ 0 & 0 & \dots \\ a_0^2 & 0 & \dots \end{pmatrix}.$$

Hence $ev(\varphi_0), ev(\varphi_1), ev(\varphi_2)$ are linearly independent generators for the subspace in $M_{3,N}(\mathbf{C})$ of matrices with zeros outside the first column.

Next, consider the case where $p_1 \in E_n$: it is of the form $([0,0,1], [x_0, y_0]) \in Bl_{[0,0,1]}\mathbf{P}(1,1,n)$, and we may assume $p_1 = ([0,0,1], [1,0])$, after an appropriate change of coordinates. We now define

$$\varphi_0 = x^{r-3n} z^3 \ell_2^2 \cdots \ell_N^2,$$

$$\varphi_1 = x^{r-3n-1} y z^3 \ell_2^2 \cdots \ell_N^2,$$

$$\varphi_2 = x^{r-2n} z^2 \ell_2^2 \cdots \ell_N^2,$$

where $\ell_2, \dots, \ell_N$ are defined as in the previous case. The evaluations on these polynomials now are

$$ev(\varphi_0) = \begin{pmatrix} (r + 2N - 3n)a_0^2 & 0 & \dots \\ 2a_0 a_1 & 0 & \dots \\ 0 & 0 & \dots \end{pmatrix},$$

$$ev(\varphi_1) = \begin{pmatrix} 0 & 0 & \dots \\ a_0^2 & 0 & \dots \\ 0 & 0 & \dots \end{pmatrix},$$

$$ev(\varphi_2) = \begin{pmatrix} 0 & 0 & \dots \\ 0 & 0 & \dots \\ a_0^2 & 0 & \dots \end{pmatrix},$$

which are again linearly independent generators for the subspace in $M_{3,N}(\mathbf{C})$ of matrices with zeros outside the first column.

Hence, we have proved that all matrices in $M_{3,N}(\mathbf{C})$ with trivial entries outside the first column belong to the image of ev and by symmetry this can be generalized to all the other columns, proving the surjectivity of ev. This has been proved using polynomials φ_i which have degree $d \geq r + 2(N-1) \geq 2N + 3n - 1$. This bound is actually sharp: if E_n is a component of the curve, then we can refer to (III.2): by counting the number of parameters of a polynomial f defining a curve having E_n as a component and N distinct singular points on E_n, we get that $d \geq 2N+3n-1$, as desired.

Finally, when $n = 0$, recall that $V_{d,0} \cong \mathbf{C}\left[x_0, x_1, y_0, y_1\right]_{3,d}$, hence any polynomial $f \in \mathbf{C}\left[x_0, x_1, y_0, y_1\right]_{3,d}$ is homogeneous of degree 3 in the variables x_0, x_1 and homogeneous of degree d in the variables y_0, y_1, at the same time. By definition, f is singular at a point p if and only if $\frac{\partial f}{\partial x_0}(p) = \frac{\partial f}{\partial x_1}(p) = \frac{\partial f}{\partial y_0}(p) = \frac{\partial f}{\partial y_1}(p) = 0$, and these are actually three independent conditions. In fact, if we denote $p = \left(\left[X_0, X_1\right], \left[Y_0, Y_1\right]\right)$ the coordinates of $p \in \mathbf{P}^1 \times \mathbf{P}^1$, then $f(x_0, x_1, Y_0, Y_1) = g(x_0, x_1) \in \mathbf{C}\left[x_0, x_1\right]_3$ and $f(X_0, X_1, y_0, y_1) = h(y_0, y_1) \in \mathbf{C}\left[y_0, y_1\right]_d$. By Euler's formula both on g and h, the vanishing of any three partial derivatives forces also the fourth one to be zero.

So we can define the evaluation map as

$$\begin{aligned} \mathbf{C}\left[x_0, x_1, y_0, y_1\right]_{3,d} &\xrightarrow{ev} M_{3,N}(\mathbf{C}) \\ f &\mapsto \begin{pmatrix} \partial f/\partial x_0(p_1) & \dots & \partial f/\partial x_0(p_N) \\ \partial f/\partial x_1(p_1) & \dots & \partial f/\partial x_1(p_N) \\ \partial f/\partial y_0(p_1) & \dots & \partial f/\partial y_0(p_N) \end{pmatrix}. \end{aligned}$$

Notice that, since E_0 is now a line of the ruling distinct from the one containing F_0, there is no need to discuss if $p_1 \in E_0$ or not. Choose coordinates such that $p_1 = \left(\left[1, 0\right], \left[1, 0\right]\right) \in \mathbf{P}^1 \times \mathbf{P}^1$ and define

$$\varphi_0 = x_0^r \ell_2^2 \cdots \ell_N^2, \qquad \varphi_1 = x_0^{r-1} x_1 \ell_2^2 \cdots \ell_N^2, \qquad \varphi_2 = x_0^{r-1} y_0 \ell_2^2 \cdots \ell_N^2,$$

where $r \geq 1$ and $\ell_2, \dots, \ell_N$ are again the equations of the lines of a ruling containing $p_2, \dots, p_N$. One can check that the evaluations on these polynomials are again linearly independent generators for the subspace in $M_{3,N}(\mathbf{C})$ of matrices with zeros outside the first column, hence ev is again surjective and the polynomials φ_i now have degree $d \geq r + 2(N-1) \geq 2N-1$, which agrees with what we have proved for $n \geq 1$. $\square$

3 Vassiliev's spectral sequence

In this section, we compute the first columns of the Vassiliev's spectral sequence, hence the stable part of the cohomology of $X_{d,n}$.

First of all, recall that the rational cohomology of $X_{d,n}$ is equivalent to the Borel-Moore homology of $\Sigma_{d,n}$ by Alexander duality, (I.9):

$$\tilde{H}^{\bullet}(V_{d,n} \backslash \Sigma_{d,n}; \mathbf{Q}) \cong \bar{H}_{2v_{d,n}-1-\bullet}(\Sigma_{d,n}; \mathbf{Q})(-v_{d,n}).$$

Then, in order to compute the Borel-Moore homology of $\Sigma_{d,n}$ we apply Gorinov-Vassiliev's method, which consists in constructing a simplicial resolution of $\Sigma_{d,n}$, admitting a filtration such that the Borel-Moore homology of the strata define a spectral sequence converging to that of $\Sigma_{d,n}$.

Precisely, assume that $d \geq 2N + 3n - 1$, let $I \subset \underline{N} = \{1, \dots, N\}$ and define the $\underline{N}$-cubical spaces of section I.3 as $\{\chi_I\}_{I \subset \underline{N}}, \{\overline{\chi}_I\}_{I \subset \underline{N}}$, where, if $I = \{i_1, \dots, i_r\}$ such that $i_j \neq N$ for any $j = 1, \dots, r$,

$$\chi_I := \{(f, y_1, \dots, y_r) \in V_{d,n} \times \prod_{j=1}^{r} B(\mathbb{F}_n, i_j) | y_1 \subset \dots \subset y_r \subset \mathrm{Sing}(f)\},$$

$$\chi_\emptyset := \Sigma_{d,n}, \qquad \text{and} \qquad \chi_{I \cup \{N\}} := \{(f, y_1, \dots, y_r) \in \chi_I | f \in \overline{\Sigma}_N\};$$

$$\overline{\chi}_I := \{(f, y_1, \dots, y_r) \in V_{d,n} \times \prod_{j=1}^{r} \overline{B(\mathbb{F}_n, i_j)} | y_1 \subset \dots \subset y_r \subset \mathrm{Sing}(f)\},$$

$$\overline{\chi}_\emptyset := \Sigma_{d,n}, \qquad \text{and} \qquad \overline{\chi}_{I \cup \{N\}} := \{(f, y_1, \dots, y_r) \in \overline{\chi}_I | f \in \overline{\Sigma}_N\};$$

where $\overline{\Sigma}_N$ denotes the Zariski closure of the locus in $\Sigma_{d,n}$ of polynomials defining curves with at least N distinct singular points. We then construct its geometric realization

$$|\chi_\bullet| = \left(\bigsqcup_{I \subset \{1,\dots,N\}} \chi_I \times \Delta_I \right) / \sim,$$

as in section I.3, and define the increasing filtration

$$\mathrm{Fil}_i|\chi_\bullet| := \mathrm{Im}\left(|\chi_\bullet|_{\underline{i}}| \hookrightarrow |\chi_\bullet| \right),$$

with locally closed subsets

$$F_i := \mathrm{Fil}_i \backslash \mathrm{Fil}_{i-1}.$$

By I.3.2, the filtration Fil_i defines a spectral sequence, called the Vassiliev's spectral sequence. This spectral sequence converges to the Borel-Moore homology of $\Sigma_{d,n}$, whose $E^1_{p,q}$-term is isomorphic to $\bar{H}_{p+q}(F_p; \mathbf{Q})$.

Then, in order to compute the Borel-Moore homology of the discriminant $\Sigma_{d,n}$ we need to consider first the Borel-Moore homology of each F_i. By construction, we have that

$$F_i = \left(\bigsqcup_{I \subset \{1,\dots,N\};\, \max I = i} \chi_I \times \Delta_I \right) / \sim .$$

When $i < N$, by Proposition I.3.3, F_i is a non-orientable simplicial bundle over $\chi_{\{i\}}$ with fiber isomorphic to the interior of a $(i-1)$-dimensional simplex.
Moreover, by Lemma 2.1, since we are working under the assumption that $d \geq 2N + 3n - 1$ holds, $\chi_{\{i\}}$ is a complex vector bundle over $B(\mathbb{F}_n, i)$ of rank $v_{d,n} - 3i$. Putting all together we obtain an explicit formula for the Borel-Moore homology of F_i for $i < N$, namely

$$\bar{H}_\bullet(F_i; \mathbf{Q}) = \bar{H}_{\bullet - 2v_{d,n} + 6i - i + 1}(B(\mathbb{F}_n, i); \pm\mathbf{Q}) \otimes \mathbf{Q}(v_{d,n} - 3i), \tag{III.4}$$

which can be computed by Lemma I.4.8. In particular, for any $n \geq 0$, the configuration spaces $B(\mathbb{F}_n, k)$ have all the same twisted Borel-Moore homology, which is trivial for $k > 4$. Thus, among the first $N - 1$ strata, only $F_1, \dots, F_4$ contribute non-trivially to the Borel-Moore homology of $\Sigma_{d,n}$. They correspond to the following classification of singular configurations:

(1) One point, [3] ;

(2) Two points, [6];

(3) Three points, [9];

(4) Four points, [12] .

By the formula (III.4) and by Lemma I.4.8, we compute the Borel-Moore homology of the associated strata and we get that the first columns of the E^1-page of the spectral sequence will look as the ones represented in Table III.1.

Table III.1

	1	2	3	4	...
$2v_{d,n}-3$	$\mathbf{Q}(v_{d,n}-1)$				
$2v_{d,n}-4$					
$2v_{d,n}-5$	$\mathbf{Q}(v_{d,n}-2)^2$				
$2v_{d,n}-6$					
$2v_{d,n}-7$	$\mathbf{Q}(v_{d,n}-3)$	$\mathbf{Q}(v_{d,n}-3)^2$			
$2v_{d,n}-8$					
$2v_{d,n}-9$		$\mathbf{Q}(v_{d,n}-4)^2$			
$2v_{d,n}-10$					
$2v_{d,n}-11$		$\mathbf{Q}(v_{d,n}-5)^2$	$\mathbf{Q}(v_{d,n}-5)$		
$2v_{d,n}-12$					
$2v_{d,n}-13$			$\mathbf{Q}(v_{d,n}-6)^2$		
$2v_{d,n}-14$					
$2v_{d,n}-15$			$\mathbf{Q}(v_{d,n}-7)$		
$2v_{d,n}-16$					
$2v_{d,n}-17$				$\mathbf{Q}(v_{d,n}-8)$	
$2v_{d,n}-18$					

Remark 8. Notice that this does not agree with the first 5 columns of the spectral sequence obtained in [Zhe21a, Table 3]. Indeed, for $g=5$, we have that the inequality $d \geq 2N+2$ is not satisfied when $N \geq 2$, hence the corresponding configurations do not have the expected codimension.

Remark 9. For $n=0$, the spectral sequence agrees with the first 4 columns of the spectral sequence in [Tom05b, Table 3], twisted by $\mathbf{Q}(v_{d,0}-16)$ in degree $2(v_{d,0}-16)$.

When $i=N$, recall that

$$F_N = \left(\bigsqcup_{I \subset \{1,\dots,N\};\, \max I = N} \chi_I \times \Delta_I \right) / \sim .$$

Following [Tom14, Lemma 18] we can further stratify the stratum F_N as the union of locally closed substrata

$$\phi_0 = \left(\chi_{\{N\}} \times \Delta_{\{N\}}\right) / \sim, \qquad \phi_l = \left(\chi_{I\cup\{N\}} \times \Delta_{I\cup\{N\}}\right) / \sim; \qquad 1 \le l \le N-1.$$

Then, for any of these substratum we have natural maps

$$\phi_0 \to \chi_{\{N\}}, \qquad \phi_l \to \chi_{\{l,N\}},$$

where $\phi_0 \cong \chi_{\{N\}}$ by definition of $\sim$, and the fiber of $\phi_l \to \chi_{\{l,N\}}$ is the interior of a l-dimensional simplex: it is a cone over the fiber of $F_l \to \chi_{\{l\}}$, which by [Gor05, Theorem 3 and Lemma 1] is again the interior of a $(l-1)$-dimensional simplex.
Moreover, for any $f \in \overline{\Sigma}_N$, the projections $(f, p_1, \ldots, p_N) \to f$, $(f, p_1, \ldots, p_N) \mapsto (f, p_1, \ldots, p_l)$ define surjections

$$\{(f, p_1, \ldots, p_N) \in V_{d,n} \times B(\mathbb{F}_n, N) | p_1, \ldots, p_N \in \mathit{Sing}(f)\} \to \chi_{\{N\}},$$

$$\{(f, p_1, \ldots, p_N) \in V_{d,n} \times B(\mathbb{F}_n, N) | p_1, \ldots, p_N \in \mathit{Sing}(f)\} \to \chi_{\{l,N\}},$$

where the domain, by the assumption that $d \ge 2N + 3n - 1$ and by Lemma 2.1, is a vector bundle of rank $v_{d,n} - 3N$ over $B(\mathbb{F}_n, N)$, which has dimension $2N$. Therefore, we have that

$$\dim_{\mathbf{R}} \phi_0 \le 2v_{d,n} - 2N \qquad \text{and} \qquad \dim_{\mathbf{R}} \phi_l \le 2v_{d,n} - 2N + l; \qquad 1 \le l \le N-1.$$

Then, since the largest dimensional stratum is ϕ_{N-1}, we have $\dim_{\mathbf{R}} F_N = \dim_{\mathbf{R}} \phi_{N-1} \le 2v_{d,n} - N - 1$. So the Borel-Moore homology of F_N must vanish in degree $k \ge 2v_{d,n} - N$. As a consequence, when considering the whole spectral sequence, we have that, for $k \ge 2v_{d,n} - N$, the Borel-Moore homology of $\Sigma_{d,n}$ is defined only by the strata $F_1, \ldots F_4$. By Alexander duality, this means that the cohomology of $X_{d,n}$ is given by that of the strata $F_1, \ldots F_4$, for $k < N \le \frac{d-3n+1}{2}$.

4 Group action on Hirzebruch surfaces

In this section we compute the rational cohomology of each stratum of the Maroni stratification, by considering the action of the automorphism group of $\mathbb{F}_n$ for any $n \ge 0$.

Let G_n denote the automorphism group of $\mathbb{F}_n$. When $n \geq 1$, G_n is isomorphic to $Aut(\mathbf{P}(1,1,n))$, which is the group of automorphisms of the weighted graded ring $\mathbf{C}[x,y,z]$ with $\deg x = \deg y = 1$ and $\deg z = n$, fixing the singular point $[0,0,1]$. Such automorphisms are of the form

$$\begin{cases} x \mapsto a_1 x + a_2 y \\ y \mapsto b_1 x + b_2 y \\ z \mapsto cz + q(x,y), \end{cases}$$

where $a_1, a_2, b_1, b_2, c \in \mathbf{C}$ are such that $c(a_1 b_2 - a_1 b_2) \neq 0$ and $q \in \mathbf{C}[x,y]_n$. Note that $\mathbf{C}[x,y]_n \cong \mathbf{C}^{n+1}$ is contractible, therefore G_n is homotopy equivalent to the reductive group $\mathbf{C}^* \times GL_2$.

The coarse moduli space $X_{d,n}/(\mathbf{C}^* \times GL_2)$ parametrizes isomorphism classes of triples (C, L, H), where C is a trigonal curve of genus $g = 2d - 3n - 2$, L is the linear system defining its trigonal structure and H a hyperplane section of $\mathbb{F}_n$. Thus $X_{d,n}/(\mathbf{C}^* \times GL_2)$ is an orbifold $\mathbf{C}^{n+1}$-bundle over $N_n = \{[C] \in \mathcal{T}_g | \, C \text{ has Maroni invariant } n\}$, and we can deduce the stable rational cohomology of the latter from that of $X_{d,n}/(\mathbf{C}^* \times GL_2)$.

Let us observe that, when $n \geq 1$, a generalized version of the Leray-Hirsch theorem can be applied to $H^\bullet(X_{d,n}; \mathbf{Q})$ in order to recover the rational cohomology of $X_{d,n}/GL_2$ from that of $X_{d,n}$ and of GL_2.

Proposition 4.1. *For $n \geq 1$, there is an isomorphism of graded $\mathbf{Q}$-vector spaces with mixed Hodge structures*

$$H^\bullet(X_{d,n}/GL_2; \mathbf{Q}) \otimes H^\bullet(GL_2; \mathbf{Q}) \cong H^\bullet(X_{d,n}; \mathbf{Q}).$$

Proof. By Theorem I.3.4 it is sufficient to prove the surjectivity of the map on cohomology

$$\rho^* : \bar{H}_{2v_{d,n}-i-1}(\Sigma_{d,n}; \mathbf{Q}) \cong H^i(X_{d,n}; \mathbf{Q}) \to H^i(GL_2; \mathbf{Q}) \cong \bar{H}_{2\dim M - i - 1}(D; \mathbf{Q}),$$

induced by the orbit map $\rho : GL_2 \to X_{d,n}$, where M is the space of 2×2 matrices and D is the discriminant in M.

The generators of $\bar{H}_\bullet(D;\mathbf{Q})$ are $[D]$ in degree 6 and $[R]$ in degree 4, where $R \subset D$ is the subvariety of matrices with zeros in the first column. From [Zhe21a, Section 3.1] we already know that, for $n = 1$, ρ^* is surjective: the preimages of the generators $[D]$ and $[R]$ are a non-zero multiple of the class $[\Sigma_{d,n}] \in \bar{H}_{2v_{d,1}-2}(\Sigma_{d,1};\mathbf{Q})$ and a non-trivial linear combination of the classes $\left[\Sigma^{(1)}_{d,n}\right], \left[\Sigma^{(2)}_{d,n}\right] \in \bar{H}_{2v_{d,1}-4}(\Sigma_{d,1};\mathbf{Q})$, respectively, where $\Sigma^{(1)}_{d,n}$ is the subspace of polynomials in $V_{d,n}$ which are singular at a point on E_n and $\Sigma^{(2)}_{d,n}$ is the subspace of polynomials in $V_{d,n}$ which are singular at a point on a line of a ruling. From Table III.1, we observe that the class in degree $2v_{d,n} - 2$ and the two classes in degree $2v_{d,n} - 4$ appear in each Vassiliev's spectral sequence, with $n \geq 1$, therefore ρ^* must be surjective for any $n \geq 1$.

In fact, by recalling that elements of $V_{d,n}$ are polynomials of the form (III.1), if we consider the extension of the orbit map $D \to \Sigma_{d,n}$, and the subvariety R, the latter acts on $V_{d,n}$ by

$$\begin{pmatrix} 0 & c_1 \\ 0 & c_2 \end{pmatrix} \cdot f(x, y, z) = C(c_1, c_2)y^{d-3n} g(y, z),$$

for a fixed $f \in X_{d,n}$, where $C(c_1, c_2) \in \mathbf{C}$ and g is a weighted polynomial in $\mathbf{C}[y, z]_{3n}$, with $\deg y = 1$, $\deg z = n$.

Thus, elements in R are sent to polynomials defining curves which are union of the line of the ruling of equation $y = 0$, with multiplicity $d - 3n$, and some other curve of lower degree passing through a point of this line of the ruling. Similarly elements in D will be sent to polynomials defining curves that are union of some line of the ruling, with multiplicity $d-3n$, and another curve meeting this line of the ruling at some point. Therefore the preimages of $[D]$ and $[R]$ through ρ^* must be a non-zero multiple of $[\Sigma_{d,n}]$ and a non-trivial linear combination of $\left[\Sigma^{(1)}_{d,n}\right], \left[\Sigma^{(2)}_{d,n}\right]$, respectively, as predicted. $\square$

With the above result, we are now able to give a description of the rational cohomology of N_n, in degree $i \leq \left\lfloor \frac{d-3n}{2} \right\rfloor$.

Proposition 4.2. *The rational cohomology of N_n, for $n \geq 1$ and in degree $i \leq \left\lfloor \frac{d-3n}{2} \right\rfloor$, is either*

$$H^i(N_n; \mathbf{Q}) = \begin{cases} \mathbf{Q}, & i = 0, \\ \mathbf{Q}(-1), & i = 2, \\ \mathbf{Q}(-3), & i = 5, \\ \mathbf{Q}(-4), & i = 7, \\ 0, & otherwise; \end{cases} \tag{III.5}$$

or

$$H^i(N_n; \mathbf{Q}) = \begin{cases} \mathbf{Q}, & i = 0, \\ \mathbf{Q}(-1), & i = 2, \\ \mathbf{Q}(-2), & i = 3, \\ \mathbf{Q}(-2), & i = 4, \\ \mathbf{Q}(-3), & i = 5, \\ \mathbf{Q}(-4), & i = 7, \\ 0, & otherwise. \end{cases} \tag{III.6}$$

The rational cohomology of N_0, in degree $i \leq \left\lfloor \frac{g+2}{4} \right\rfloor$, is

$$H^i(N_0; \mathbf{Q}) = \begin{cases} \mathbf{Q}, & i = 0, \\ \mathbf{Q}(-3), & i = 5, \\ 0, & otherwise. \end{cases} \tag{III.7}$$

Proof. Consider first the case $n \geq 1$. By applying Proposition 4.1, the rational cohomology of the quotient $X_{d,n}/GL_2$, in degree $i \leq \left\lfloor \frac{d-3n}{2} \right\rfloor$, will be

$$H^i(X_{d,n}/GL_2; \mathbf{Q}) = \begin{cases} \mathbf{Q}, & i = 0; \\ \mathbf{Q}(-2), & i = 3; \\ \mathbf{Q}(-3), & i = 5; \\ \mathbf{Q}(-5), & i = 8; \\ 0, & \text{otherwise.} \end{cases} \tag{III.8}$$

Consider then the spectral sequence associated to the bundle

$$X_{d,n}/GL_2 \xrightarrow{\mathbf{C}^*} X_{d,n}/(\mathbf{C}^* \times GL_2),$$

which will look either as

Table III.2

$$\begin{tikzcd}[row sep=small, column sep=small]
1 & \mathbf{Q}(-1) \arrow[drr] & & \mathbf{Q}(-2) & & & \mathbf{Q}(-4) \arrow[drr] & & \mathbf{Q}(-5) & & \\
0 & \mathbf{Q} & & \mathbf{Q}(-1) & & & \mathbf{Q}(-3) & & \mathbf{Q}(-4) & & \\
 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \dots
\end{tikzcd}$$

or as

Table III.3

$$\begin{tikzcd}[row sep=small, column sep=small]
1 & \mathbf{Q}(-1) \arrow[drr] & & \mathbf{Q}(-2) \arrow[drr] & \mathbf{Q}(-3) \arrow[drr] & \mathbf{Q}(-3) & \mathbf{Q}(-4) \arrow[drr] & & \mathbf{Q}(-5) & & \\
0 & \mathbf{Q} & & \mathbf{Q}(-1) & \mathbf{Q}(-2) & \mathbf{Q}(-2) & \mathbf{Q}(-3) & & \mathbf{Q}(-4) & & \\
 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \dots
\end{tikzcd}$$

where in both spectral sequences the differentials must be non-trivial because of (III.8).

Both spectral sequences also imply that $H^2(N_n; \mathbf{Q})$ is generated by the Euler class ξ of the $\mathbf{C}^*$-bundle, which is a non-zero multiple of κ_1, by [PV15a]. Moreover, the spectral sequence represented in Table III.3 would imply that $H^4(N_n; \mathbf{Q})$ is generated by $\kappa_1 \cdot \xi = \alpha\kappa_1^2 \neq 0$.

Hence, the choice of the spectral sequence corresponds to verify if $\kappa_1^2 = 0$ holds in $H^4(N_n; \mathbf{Q})$ and we cannot determine this a priori. Thus, from Tables III.2 and III.3 we obtain two possible description of the cohomology of N_n, which are (III.5) and (III.6), respectively.

Here, notice that the stable range $i \leq \left\lfloor \frac{d-3n}{2} \right\rfloor$ is the same obtained for $H^i(X_{d,n}/GL_2; \mathbf{Q})$. Indeed, if we had a non-trivial class in $E_2^{\left\lfloor \frac{d-3n}{2} \right\rfloor, 0}$ in Tables III.2 and III.3, then additional non-trivial classes would also appear in $H^i(X_{d,n}/GL_2; \mathbf{Q})$, for $i \leq \left\lfloor \frac{d-3n}{2} \right\rfloor$.

Finally, when $n = 0$, the group G_0 acting on $\mathbb{F}_0$ is different from those we have considered when $n \geq 1$. However, also in this case a generalized version of Leray-Hirsch theorem can be applied. Indeed, G_0 is exactly the group considered in [Tom05b], Section 3.1]. Precisely G_0 is a reductive group which is isogenous to $\mathbf{C}^* \times SL_2 \times SL_2$, whose cohomology is known. The isogeny $\iota : \mathbf{C}^* \times SL_2 \times SL_2 \to G_0$ is an isogeny between connected algebraic groups and therefore ι induces an isomorphism on rational cohomology.

Thus, the rational cohomology of $X_{d,0}/G_0$ has already been computed in [Tom05b, Section 3.7] and by applying the generalized version of Leray-Hirsch theorem we get (III.7). □

Remark 10. From the rational cohomology of $\mathcal{T}_5$, computed in chapter II, we can establish that, when g is odd, the cohomology of the stratum N_1 is described by (III.5). This is due to the fact that N_1 is open in $\mathcal{T}_g$ and the fundamental class of its complement is a non-zero multiple of κ_1^2. Therefore, from the description of the rational Chow ring of $\mathcal{T}_5$ in [CL21a, Theorem 1.1] we can conclude that $\kappa_1^2 = 0$ in $H^4(N_1; \mathbf{Q})$.

5 Maroni stratification

Recall that $\mathcal{T}_g$ has a natural stratification by the Maroni invariant, (I.7):

$$\mathcal{N}_{\lfloor \frac{g+2}{3} \rfloor} \subset \cdots \subset \mathcal{N}_{g \bmod 2} = \mathcal{T}_g,$$

where $\mathcal{N}_n = \{[C] \in \mathcal{T}_g | C \text{ has Maroni invariant } \geq n\}$ with $0 \leq n \leq \lfloor \frac{g+2}{3} \rfloor$ and $g \equiv n \bmod 2$. Notice that $N_n = \mathcal{N}_n \backslash \mathcal{N}_{n+2}$, so we have indeed computed the cohomology of the strata in the Maroni stratification of $\mathcal{T}_g$, within a certain range.

In order to deduce the cohomology of $\mathcal{T}_g$ from that of the strata, we consider the spectral sequence associated with this stratification. Recall that

$$\dim \mathcal{N}_n = 2g + 2 - n - \delta_{0,n},$$

so that each $\mathcal{N}_{n+2}$ has codimension 2 in $\mathcal{N}_n$ with the sole exception of $\mathcal{N}_2 \subset \mathcal{T}_g$, which is a divisor for g even. Moreover, observe from Proposition 4.2 that N_0 is the only stratum having different cohomology from the other strata. Consequently, we will need to distinguish the cases for g even and odd.

5.1 Case g even

Suppose first that g is even. We can recover the rational cohomology of $\mathcal{T}_g$, in a certain range, from the Gysin spectral sequence in Borel-Moore homology induced by the Maroni stratification (I.7).

Precisely, the E^1- page of the spectral sequence is obtained by considering in each column the Borel-Moore homology of each stratum N_n. We will twist the whole spectral sequence by $\mathbf{Q}(-\dim \mathcal{T}_g)$ in order to get the fundamental class of $\mathcal{T}_g$ in degree 0.

Recall also from Proposition 4.2 that, for $n \geq 2$ we have two possible descriptions for the cohomology of each stratum.

Assume first that the cohomology of each stratum, except N_0, in the Maroni stratification is given by (III.5). The corresponding spectral sequence is represented in Table III.4.

TABLE III.4

		N_6	N_4	N_2	N_0	
...	-4	-3	-2	-1	0	
					$\mathbf{Q}$	0
				$\mathbf{Q}(-1)$		-1
						-2
				$\mathbf{Q}(-2)$		-3
			$\mathbf{Q}(-3)$			-4
					$\mathbf{Q}(-3)$	-5
			$\mathbf{Q}(-4) \leftarrow$	$\mathbf{Q}(-4)$		-6
		$\mathbf{Q}(-5)$				-7
				$\mathbf{Q}(-5)$		-8
		$\mathbf{Q}(-6) \leftarrow$	$\mathbf{Q}(-6)$			-9
	$\mathbf{Q}(-7)$					-10
			$\mathbf{Q}(-7)$			-11
	...	$\mathbf{Q}(-8)$				-12
	...					-13
	...	$\mathbf{Q}(-9)$				-14

Let us consider the differentials highlighted in Table III.4. The targets of these differentials are the 0-th or 2-nd cohomology group of a stratum, which are 1-dimensional by Proposition 4.2. Hence the differentials in the stable range may only have rank 0 or 1.

To check whether they have rank 1 or not, it suffices to study both the fundamental class $[\mathcal{N}_n]$ and the generator of $H^2(\mathcal{N}_n; \mathbf{Q})$, if $n \geq 1$, for each stratum $\mathcal{N}_n$, in $\mathcal{T}_g$. This has been already done in [PV15b] and [PV15a].

Penev and Vakil proved in [PV15b, Theorem 3.3] that any Chow class in $\mathcal{N}_n$ is the restriction of a tautological class on $\mathcal{M}_g$. By abuse of notation we will denote both the tautological class in $R^\bullet(\mathcal{M}_g)$ and its pullback through the restriction map in $R^\bullet(\mathcal{T}_g)$ in the same way.

Patel and Vakil showed indeed that the rational Chow ring of $\mathcal{T}_g$ is generated by the kappa class κ_1 and that the fundamental class $[\mathcal{N}_n]$ is a multiple of the $(n-1)$-th power of κ_1, in [PV15a, Proposition 6.2].

Moreover, for $n \geq 1$, the second cohomology group $H^2(\mathcal{N}_n; \mathbf{Q})$ is generated by the fundamental class of the locus of curves tangent to E_n. Therefore the generator of $H^2(\mathcal{N}_n; \mathbf{Q})$ must also be a multiple of a power of κ_1, precisely of a n-th power.

Then, both the fundamental classes $[\mathcal{N}_n]$ and the class generating $H^2(\mathcal{N}_n; \mathbf{Q})$ must vanish for $n \geq 4$ by [CL21a, Theorem 1.1].

This means that the differentials $d^1_{p,q} : E^1_{p,q} \to E^1_{p-1,q}$ and $d^2_{p,q} : E^2_{p,q} \to E^2_{p-2,q+1}$ must all be of rank 1.

Now, assume instead that there exist at least one stratum $N_{\bar{n}}$, whose cohomology is described by (III.6) and denote by $\bar{n}$ the minimum integer for which this happens.

If $\bar{n} = 2$, then the corresponding spectral sequence will look like the one in Table III.5.

TABLE III.5

	N_4	N_2	N_0	
...	-2	-1	0	
			$\mathbf{Q}$	0
		$\mathbf{Q}(-1)$		-1
				-2
		$\mathbf{Q}(-2)$		-3
	$\mathbf{Q}(-3) \leftarrow$	$\mathbf{Q}(-3)$		-4
		$\mathbf{Q}(-3) \leftarrow$	$\mathbf{Q}(-3)$	-5
	$\mathbf{Q}(-4)$	$\mathbf{Q}(-4)$		-6
$\mathbf{Q}(-5)$	...			-7
	...	$\mathbf{Q}(-5)$		-8
$\mathbf{Q}(-6)$	$\mathbf{Q}(-6)$			-9

Thus we get two extra classes $\mathbf{Q}(-3)$ in degrees 5 and 6, and the latter must be algebraic from the discussion on Table III.3. Therefore it must vanish also from by [CL21a, Theorem 1.1] and means that the differentials highlighted in Table III.5 will be of rank 1, while the other differentials will behave exactly as in the previous case.

Similarly this happens for all the next strata whose cohomology is described by (III.6).

Assume then that $\bar{n} > 2$. In this case, the corresponding spectral sequence will look like the one represented in Table (III.6).

TABLE III.6

$\dots$	$N_{\bar{n}+2}$ $-\frac{\bar{n}}{2}-1$	$N_{\bar{n}}$ $-\frac{\bar{n}}{2}$	$N_{\bar{n}-2}$ $-\frac{\bar{n}}{2}+1$	$\dots$	
					$\dots$
			$\mathbf{Q}(-\bar{n}+3)$		$-2(\bar{n}-1)+\frac{\bar{n}}{2}+3$
					$-2(\bar{n}-1)+\frac{\bar{n}}{2}+2$
			$\mathbf{Q}(-\bar{n}+2)$	$\mathbf{Q}(-\bar{n}+2)$	$-2(\bar{n}-1)+\frac{\bar{n}}{2}+1$
		$\mathbf{Q}(-\bar{n}+1)$			$-2(\bar{n}-1)+\frac{\bar{n}}{2}$
				$\mathbf{Q}(-\bar{n}+1)$	$-2(\bar{n}-1)+\frac{\bar{n}}{2}-1$
		$\mathbf{Q}(-\bar{n})$	$\mathbf{Q}(-\bar{n})$		$-2(\bar{n}-1)+\frac{\bar{n}}{2}-2$
	$\mathbf{Q}(-\bar{n}-1)$	$\leftarrow \mathbf{Q}(-\bar{n}-1)$			$-2(\bar{n}-1)+\frac{\bar{n}}{2}-3$
		$\mathbf{Q}(-\bar{n}-1)$	$\leftarrow \mathbf{Q}(-\bar{n}-1)$		$-2(\bar{n}-1)+\frac{\bar{n}}{2}-4$
	$\mathbf{Q}(-\bar{n}-2)$	$\mathbf{Q}(-\bar{n}-2)$			$-2(\bar{n}-1)+\frac{\bar{n}}{2}-5$
$\mathbf{Q}(-\bar{n}-3)$					$-2(\bar{n}-1)+\frac{\bar{n}}{2}-6$
		$\mathbf{Q}(-\bar{n}-3)$			$-2(\bar{n}-1)+\frac{\bar{n}}{2}-7$

The two additional classes that we might have here are $\mathbf{Q}(-\bar{n}-1)$ in degrees $2\bar{n}+1$ and $2\bar{n}+2$. The second class is again algebraic, hence it must vanish also from by [CL21a, Theorem 1.1]. The differentials highlighted in Table III.6 will be of rank 1, while the other differentials will behave exactly as in the previous case. This can be repeated to all strata N_n, with $n > \bar{n}$ when their cohomology is given by (III.6).

Therefore we may conclude that, in degree $i < \frac{g}{4}$,

$$H^i(\mathcal{T}_g; \mathbf{Q}) = \begin{cases} \mathbf{Q}, & i = 0; \\ \mathbf{Q}(-1), & i = 2; \\ \mathbf{Q}(-2), & i = 4; \\ 0, & \text{otherwise}; \end{cases}$$

where the bound $i < \frac{g}{4}$ is obtained by recalling from the previous section that the cohomology of each strata N_n is known in degree lower than $\frac{d-3n+1}{2}$, where $d = \frac{g+3n+2}{2}$.

For any $0 \leq n \leq \left\lfloor \frac{g+2}{3} \right\rfloor$, we require

$$\begin{aligned} i &< \min\left\{ 2\operatorname{codim}_{\mathcal{T}_g} N_n + \frac{d-3n+1}{2};\ 0 \leq n \leq \left\lfloor \frac{g+2}{3} \right\rfloor \right\} - 1 \\ &= \min\left\{ 2(n+1) + \frac{g+3n+2}{4} + \frac{-3n+1}{2};\ 0 \leq n \leq \left\lfloor \frac{g+2}{3} \right\rfloor \right\} - 1 \\ &= \frac{g}{4}. \end{aligned}$$

5.2 Case g odd

Consider now the odd genus case. The E^1- page of the Gysin spectral sequence in Borel-Moore homology induced by the Maroni stratification (twisted again by $\mathbf{Q}(-\dim \mathcal{T}_g)$), with all strata having cohomology as in (III.5), is represented in Table III.7.

TABLE III.7

		N_7	N_5	N_3	N_1	
...	-4	-3	-2	-1	0	
					$\mathbf{Q}$	0
						-1
					$\mathbf{Q}(-1)$	-2
				$\mathbf{Q}(-2)$		-3
						-4
				$\mathbf{Q}(-3) \leftarrow$	$\mathbf{Q}(-3)$	-5
			$\mathbf{Q}(-4) \leftarrow$			-6
					$\mathbf{Q}(-4)$	-7
			$\mathbf{Q}(-5) \leftarrow$	$\mathbf{Q}(-5)$		-8
		$\mathbf{Q}(-6) \leftarrow$				-9
				$\mathbf{Q}(-6)$		-10
		$\mathbf{Q}(-7) \leftarrow$	$\mathbf{Q}(-7)$			-11
	$\mathbf{Q}(-8) \leftarrow$					-12
	...		$\mathbf{Q}(-8)$			-13
	...	...				-14

For the same reasons discussed in the even genus case, the differentials highlighted in Table III.7 are all of rank 1.

If we also consider the case where there exist at least one stratum whose cohomology is as in (III.6), we also have an analogue of the argument discussed in the even genus case.

Thus, the stable rational cohomology of $\mathcal{T}_g$, with g odd, coincides with the one obtained in the even genus case and precisely, in degree $i < \frac{g-3}{4}$,

$$H^i(\mathcal{T}_g; \mathbf{Q}) = \begin{cases} \mathbf{Q}, & i = 0; \\ \mathbf{Q}(-1), & i = 2; \\ \mathbf{Q}(-2), & i = 4; \\ 0, & \text{otherwise}; \end{cases}$$

where $\frac{g-3}{4} = \min\left\{2(n-1) + \frac{g+3n+2}{4} + \frac{-3n+1}{2};\ 1 \leq n \leq \left\lfloor \frac{g+2}{3} \right\rfloor\right\} - 1$.

Comparing both results, obtained for g even and odd, we get that the rational cohomology $H^\bullet(\mathcal{T}_g; \mathbf{Q})$ stabilizes to its rational Chow ring, and equivalently to its tautological ring, for g sufficiently large.

5.3 Stable cohomology of the moduli space of framed triple covers

Finally, let us conclude by giving the proof of Corollary 1.3.

Proof of Corollary 1.3. Let us revisit the computation of the cohomology of the strata N_n, which were obtained by considering the quotient spaces $X_{d,n}/H_n$, with $H_n = \mathbf{C}^* \times GL_2$ for any $n \geq 1$ and $H_0 = \mathbf{C}^* \times SL_2 \times SL_2$.

Taking first the projectivization $\mathbf{P}X_{d,n}$, and considering then the quotients $\mathbf{P}X_{d,n}/\mathbf{C}^*$, for $n \geq 1$, and $\mathbf{P}X_{d,0}/SL_2$ would give us the rational cohomology, until a certain degree, of a SL_2-cover of N_n that we will denote by $N_n^\dagger$, for any $n \geq 0$.

Consider first $n = 0$. As we have already noticed in section 4, in this case the generalized version of Leray-Hirsch theorem can be applied to the whole $H_0 = \mathbf{C}^* \times SL_2 \times SL_2$, meaning that the cohomology of $X_{d,0}$ is completely divisible by that of $\mathbf{C}^* \times SL_2 \times SL_2$. Therefore the rational cohomology of $N_0^\dagger$ is simply the cohomology of

$X_{d,0}$ divided by that of $\mathbf{C}^* \times SL_2$:

$$H^i(N_0^\dagger;\mathbf{Q}) = \begin{cases} \mathbf{Q}, & i=0; \\ \mathbf{Q}(-2), & i=3; \\ \mathbf{Q}(-3), & i=5; \\ \mathbf{Q}(-5), & i=8; \\ 0, & \text{otherwise}; \end{cases}$$

in degree $i \leq \left\lfloor \frac{d}{2} \right\rfloor$.

For $n \geq 1$, the Leray-Hirsch theorem does not apply to the action of the whole group H_n, but only to the action of GL_2 on $X_{d,n}$. However, let us recall that $N_n^\dagger$ has been defined as an SL_2-cover of N_n. Hence, the cohomology of $N_n^\dagger$ is obtained by tensoring the cohomology of N_n with that of $GL_2 = SL_2 \rtimes \mathbf{C}^*$, and then dividing by cohomology of $\mathbf{C}^*$. Since we have two descriptions for the cohomology of N_n by Proposition 4.2, we also get two for $N_n^\dagger$.

$$H^i(N_n^\dagger;\mathbf{Q}) = \begin{cases} \mathbf{Q}, & i=0; \\ \mathbf{Q}(-1), & i=2; \\ \mathbf{Q}(-2), & i=3; \\ 2\mathbf{Q}(-3), & i=5; \\ \mathbf{Q}(-4), & i=7; \\ \mathbf{Q}(-5), & i=8; \\ \mathbf{Q}(-6), & i=10; \\ 0, & \text{otherwise}; \end{cases} \tag{III.9}$$

or

$$H^i(N_n^\dagger;\mathbf{Q}) = \begin{cases} \mathbf{Q}, & i=0; \\ \mathbf{Q}(-1), & i=2; \\ 2\mathbf{Q}(-2), & i=3; \\ \mathbf{Q}(-2), & i=4; \\ 2\mathbf{Q}(-3), & i=5; \\ \mathbf{Q}(-4), & i=6; \\ \mathbf{Q}(-4), & i=7; \\ \mathbf{Q}(-5), & i=8; \\ \mathbf{Q}(-6), & i=10; \\ 0, & \text{otherwise}; \end{cases} \tag{III.10}$$

in degree $i \leq \left\lfloor \frac{d-3n}{2} \right\rfloor$.

Now, by looking at the Maroni stratification, all these $N_n^\dagger$ can be interpreted as locally closed strata of a moduli space denoted by $\mathcal{T}_g^\dagger$, which is a SL_2-cover of $\mathcal{T}_g$. The cohomology of $\mathcal{T}_g^\dagger$ can be deduced by writing the analogues of Tables III.4 and III.7.

Similarly to what we have done for $\mathcal{T}_g$, let us assume first that the rational cohomology of all strata are given by (III.9). We will later prove that the same result holds even if some (or all) strata are described instead by (III.10) .

The spectral sequences associated with the Maroni stratification $\{N_n^\dagger\}$ of $\mathcal{T}_g^\dagger$, for g even and odd, are represented in Tables III.8 and III.9, respectively.

Table III.8: Spectral sequence converging to $\bar{H}_\bullet(\mathcal{T}_g^\dagger;\mathbf{Q})$ for g even.

		$N_6^\dagger$	$N_4^\dagger$	$N_2^\dagger$	$N_0^\dagger$	
...	-4	-3	-2	-1	0	
					$\mathbf{Q}$	0
				$\mathbf{Q}(-1)$		-1
						-2
				$\mathbf{Q}(-2)$	$\mathbf{Q}(-2)$	-3
			$\mathbf{Q}(-3)$	$\mathbf{Q}(-3)$		-4
					$\mathbf{Q}(-3)$	-5
			$\mathbf{Q}(-4)$	$2\mathbf{Q}(-4)$		-6
		$\mathbf{Q}(-5)$	$\mathbf{Q}(-5)$			-7
				$\mathbf{Q}(-5)$	$\mathbf{Q}(-5)$	-8
		$\mathbf{Q}(-6)$	$2\mathbf{Q}(-6)$	$\mathbf{Q}(-6)$		-9
	$\mathbf{Q}(-7)$	$\mathbf{Q}(-7)$				-10
			$\mathbf{Q}(-7)$	$\mathbf{Q}(-7)$		-11
	$\mathbf{Q}(-8)$	$2\mathbf{Q}(-8)$	$\mathbf{Q}(-8)$			-12
	...	...	...	...		-13

Table III.9: Spectral sequence converging to $\bar{H}_\bullet(\mathcal{T}_g^\dagger;\mathbf{Q})$ for g odd.

		$N_7^\dagger$	$N_5^\dagger$	$N_3^\dagger$	$N_1^\dagger$	
...	-4	-3	-2	-1	0	
					$\mathbf{Q}$	0
						-1
					$\mathbf{Q}(-1)$	-2
				$\mathbf{Q}(-2)$	$\mathbf{Q}(-2)$	-3
						-4
				$\mathbf{Q}(-3)$	$2\mathbf{Q}(-3)$	-5
			$\mathbf{Q}(-4)$	$\mathbf{Q}(-4)$		-6
					$\mathbf{Q}(-4)$	-7
			$\mathbf{Q}(-5)$	$2\mathbf{Q}(-5)$	$\mathbf{Q}(-5)$	-8
		$\mathbf{Q}(-6)$	$\mathbf{Q}(-6)$			-9
				$\mathbf{Q}(-6)$	$\mathbf{Q}(-6)$	-10
		$\mathbf{Q}(-7)$	$2\mathbf{Q}(-7)$	$\mathbf{Q}(-7)$		-11
	$\mathbf{Q}(-8)$	$\mathbf{Q}(-8)$				-12
	...	...	...	...		-13

Let us consider the even genus case. The odd genus case will be analogous.

We would like to study the differentials in Table III.8 and, in order to do so, let us consider the Chow ring of $\mathcal{T}_g^\dagger$.

Patel and Vakil proved in [PV15a, Prop. 6.1] and [PV15a, Vistoli's Theorem] that the Chow ring of $\mathcal{T}_g^\dagger$ is also generated by the tautological class κ_1 and it is related to that of $\mathcal{T}_g$ by

$$A^\bullet(\mathcal{T}_g^\dagger) = A^\bullet(\mathcal{T}_g)/(\mu), \tag{III.11}$$

where μ is a multiple of κ_1^2.

Moreover, by the previous discussion on the cohomology of each stratum $N_n^\dagger$, for $n \geq 1$, the cohomology of $N_n^\dagger$ is isomorphic to the tensor product of the cohomology of N_n and that of SL_2.

The differential $d^1 : E^1_{0,-3} \to E^1_{-1,-3}$ must then be of rank 1. Its target class is in fact the generator of $H^2(N_2^\dagger; \mathbf{Q}) \cong H^2(N_2; \mathbf{Q})$, hence a multiple of the 2-nd power of κ_1 from [PV15a, Prop. 6.1] and the previous discussion on the rank of the differentials in Table III.4. Thus, by (III.11), this class must vanish.

Similarly, the differentials in the E^1-page, such as $d^1_{-1,-6}$, $d^1_{-2,-9}$, $d^1_{-3,-12}, \dots$, etc. must also be of rank 1. The target classes are indeed defined by generators of $H^2(N_n^\dagger; \mathbf{Q}) \cong H^2(N_n; \mathbf{Q})$ which are multiples of $(n-1)$-powers of κ_1 with $n \geq 4$ by [PV15a, Proposition 6.2], and they must vanish by (III.11) and [CL21a, Theorem 1.1].

Consider then the differentials in the E^1-page, having two-dimensional spaces as targets. Precisely, these classes are tensor products of the generator $H^2(N_n; \mathbf{Q})$ by that of $H^3(SL_2; \mathbf{Q})$. So, these differentials are also non-trivial by the argument above.

Finally, the differentials in the E^1-page, such as $d^1_{-1,-4}$, $d^1_{-2,-7}$, $d^1_{-3,-10}, \dots$, etc. might not be of rank 1. In this case, the differentials in the E^2-page, having as targets both the source and the target classes of a trivial differential d^1, must be of rank 1. In fact, these classes are either the generator of the fundamental class of N_n, or its tensor product by the generator $H^3(SL_2; \mathbf{Q})$. Therefore they must also vanish by [PV15a, Proposition 6.2].

In conclusion, for both g even and odd, we have, in degree $i < \left\lfloor \frac{g}{4} \right\rfloor$,

$$H^i(\mathcal{T}_g^\dagger;\mathbf{Q}) = \begin{cases} \mathbf{Q}, & i=0; \\ \mathbf{Q}(-1), & i=2; \\ \mathbf{Q}(-3), & i=5; \\ \mathbf{Q}(-4), & i=7; \\ 0, & \text{otherwise.} \end{cases}$$

$\square$

What is left to do is to show that the same result also holds if any of the strata $N_n^\dagger$ has rational cohomology as in (III.10). We consider again the even genus case and assume that $N_2^\dagger$ is given by (III.10). The associated spectral sequence is represented in Table III.10.

TABLE III.10

		$N_6^\dagger$	$N_4^\dagger$	$N_2^\dagger$	$N_0^\dagger$	
...	-4	-3	-2	-1	0	
					$\mathbf{Q}$	0
				$\mathbf{Q}(-1)$		-1
						-2
				$\mathbf{Q}(-2)$	$\mathbf{Q}(-2)$	-3
			$\mathbf{Q}(-3) \leftarrow$	$2\mathbf{Q}(-3)$		-4
				$\mathbf{Q}(-3) \leftarrow$	$\mathbf{Q}(-3)$	-5
			$\mathbf{Q}(-4)$	$2\mathbf{Q}(-4)$		-6
		$\mathbf{Q}(-5)$	$\mathbf{Q}(-5) \leftarrow$	$\mathbf{Q}(-5)$		-7
				$2\mathbf{Q}(-5) \leftarrow$	$\mathbf{Q}(-5)$	-8
		$\mathbf{Q}(-6)$	$2\mathbf{Q}(-6)$	$\mathbf{Q}(-6)$		-9
	$\mathbf{Q}(-7)$	$\mathbf{Q}(-7)$				-10
			$\mathbf{Q}(-7)$	$\mathbf{Q}(-7)$		-11
	$\mathbf{Q}(-8)$	$2\mathbf{Q}(-8)$	$\mathbf{Q}(-8)$			-12
	...	...	...	...		-13

Notice that, with respect to Table (III.8), we get four additional classes. Two of which are a non-zero multiple κ_1^3 and its tensor product by $H^3(SL_2;\mathbf{Q})$. Thus they must vanish by (III.11) and [CL21a, Theorem 1.1], meaning that the differentials $d^1_{0,-4}$ and $d^1_{0,-8}$ have maximal rank. Consider then the targets of the other differential highlighted in Table

III.10. From the discussion done for Table (III.8), these must also have maximal rank. Summarizing, the differentials are such that the additional classes must also vanish.

These argument can be repeated for all the other strata having cohomology described by (III.10), and generalized to $N_n^\dagger$ with $n > 2$. This yields that the stable cohomology of $\mathcal{T}_g^\dagger$ is indeed the one in Corollary 1.3.

Chapter IV

Further stabilization results for moduli spaces of smooth curves on a fixed Hirzebruch surface

In this final chapter we want to discuss the stabilization of the cohomology of some other moduli spaces of curves. More specifically, we generalize the stabilization proved for the cohomology of $\mathcal{T}_g$ to the cohomology of moduli spaces of smooth curves of higher gonality, embedded in a fixed Hirzebruch surface.

1 Introduction and results

In the previous chapters we have studied the rational cohomology of the moduli space of smooth curves defining an open subset in the vector space of global sections $H^0(\mathbb{F}_n; \mathcal{O}_{\mathbb{F}_n}(3E_n + dF_n))$. In particular we proved in chapter III that the cohomology of this moduli space stabilizes for $d \to \infty$.

In this chapter we generalize this stabilization result to other moduli spaces of curves, defined in a similar way.

In fact, we will still consider curves embedded in a fixed Hirzebruch surface $\mathbb{F}_n$ for some $n \geq 0$.

Definition 9. Let $V^k_{d,n} := H^0(\mathbb{F}_n; \mathcal{O}_{\mathbb{F}_n}(kE_n + dF_n))$. Define then $\Sigma^k_{d,n} \subset V^k_{d,n}$ to be the locus of sections defining singular curves on $\mathbb{F}_n$, and $X^k_{d,n}$ its complement.

Let us also recall that every connected linear algebraic group is isogenous to a reductive group and, in particular, the automorphism group G_n of a Hirzebruch surface is isogenous to the reductive group H_n for any $n \geq 0$, where $H_n = \mathbf{C}^* \times GL_2$ if $n \geq 1$ and $H_0 = \mathbf{C}^* \times SL_2 \times SL_2$.

We can then define the (coarse) moduli space of smooth curves in the linear system $|kE_n + dF_n|$ on a fixed Hirzebruch surface $\mathbb{F}_n$ as underlying space of the quotient $X^k_{d,n}/H_n$, for any $n \geq 0$.

Remark 11. Notice that if $k = 3$, then $V^3_{d,n}$ is exactly the vector space denoted $V_{d,n}$ in chapter III. Therefore, the stabilization of $H^\bullet(X^3_{d,n}/H_n; \mathbf{Q})$ has already been proved. In this case we saw that $X^3_{d,n}/H_n$ is an orbifold $\mathbf{C}^{n+1}$-bundle over the Maroni stratum N_n. Thus $X^3_{d,n}/H_n$ and N_n have the same cohomology and the stable cohomology of $X^3_{d,n}/H_n$ is described in Proposition III.4.2.

The main result of this chapter is the following

Theorem 1.1. *Fix $k > 3$, then $H^i(X^k_{d,n}/H_n; \mathbf{Q}) \cong H^i(X^3_{d,n}/H_n; \mathbf{Q})$ for $i < \frac{d-kn+1}{2}$.*

Remark 12. Notice that the above statement agrees with [EW15, Theorem 9.5], in which Erman and Wood proved that the probability of a curve of bidegree (k, d) on a fixed Hirzebruch surface to be smooth is independent of k for $k \geq 3$ and $d \to \infty$.

The stabilization of the cohomology might also hold for $k = 2$. However this case is radically different from the other ones. It will be sketchily treated in the last section and in particular we will prove that, even if the cohomology might stabilize, the stable ring would be different from that obtained for $H^\bullet(X^3_{d,n}/H_n; \mathbf{Q})$.

2 Cohomology of complements of resultants

Before giving a proof of Theorem 1.1, let us consider the case $k = 1$.

We will prove that, for $k = 1$, the space $X^1_{d,n}/H_n$ has the rational cohomology of a point for any $n \geq 0$.

Assume first $n \geq 1$. Any $f \in V^1_{d,n}$ is of the form $f(x, y, z) = \alpha(x, y)z + \beta(x, y)$ with α, β homogeneous polynomials in $\mathbf{C}[x, y]$ of degrees $d - n$ and d, respectively.

The polynomial f is singular if and only if α and β are not relatively prime, i.e. if they share a common factor or equivalently if their *resultant* is zero.

In fact, in this case, the discriminant $\Sigma^1_{d,n}$ is exactly the resultant of the space of system

$$\begin{cases} \alpha(x,y) = 0 \\ \beta(x,y) = 0 \end{cases}$$

with α and β as above.

Furthermore, the cohomology of the complement of this resultant has already been computed by Vassiliev in [Vas15, Theorem 2]. More precisely, the space $\mathbf{P}X^1_{d,n}$ is exactly the one denoted by $\mathbf{C}^D \backslash \Sigma_{\mathbf{C}}$, with $D = 2d+2-n$, in [Vas15].

Vassiliev's result establishes that $H^\bullet(\mathbf{P}X^1_{d,n};\mathbf{Q})$ is an exterior algebra generated by two classes in degrees $1,3$. Moreover, the weights of these generators and their product, in the mixed Hodge structure of $H^\bullet(\mathbf{P}X^1_{d,n};\mathbf{Q})$, are $2,4$ and 6, respectively.

Thus, by noticing that the generalized version of Leray-Hirsch theorem applies for the action of GL_2, we have that

$$H^i(X^1_{d,n}/H_n;\mathbf{Q}) = \begin{cases} \mathbf{Q}, & i = 0; \\ 0, & \text{otherwise.} \end{cases}$$

Assume then $n = 0$.

Any $f \in V^1_{d,0}$ is a bihomogeneous polynomial in $\mathbf{C}[x_0,x_1,y_0,y_1]$ of bi-degree $(1,d)$, i.e. it is homogeneous of degree 1 in the variables x_0,x_1 and of degree d in the variables y_0,y_1. If we fix a point $[x_0,x_1] = [X_0,X_1] \in \mathbf{P}^1$, and thus one of the two families of rulings in $\mathbf{P}^1 \times \mathbf{P}^1$, then f is of the form $f(X_0,X_1,y_0,y_1) = X_0 g(y_0,y_1) + X_1 h(y_0,y_1)$, with $g,h \in \mathbf{C}[y_0,y_1]$ homogeneous of degree d.

Recall that the vanishing locus of f is a divisor of class $E_0 + dF_0$. Hence f is singular if and only if its vanishing locus contains a line of the ruling, namely the one containing F_0. This is equivalent to require the polynomials g and f to share a common factor.

Therefore, the discriminant in the space of polynomials of the form $f(X_0,X_1,y_0,y_1)$ is again the resultant of a space of system

$$\begin{cases} g(y_0,y_1) = 0 \\ h(y_0,y_1) = 0. \end{cases}$$

By applying again Vassiliev's result in [Vas15, Theorem 2], the rational cohomology of the complement of the resultant is exactly that of $\mathbf{P}X^1_{d,n}$ with $n > 0$.

Finally, we recover the cohomology of $\mathbf{P}X^1_{d,0}$ by looking at the E_2-page of the Leray spectral sequence associated to the projection $[x_0, x_1] \mapsto [X_0, X_1]$:

Table IV.1

4	$\mathbf{Q}(-3)$		$\mathbf{Q}(-4)$
3	$\mathbf{Q}(-2)$		$\mathbf{Q}(-3)$
2			
1	$\mathbf{Q}(-1)$		$\mathbf{Q}(-2)$
0	$\mathbf{Q}$		$\mathbf{Q}(-1)$
	0	1	2

Notice that also in this case a generalized version of Leray-Hirsch applies for the action of $SL_2 \times SL_2$, therefore the differentials highlighted above have rank 1.

Thus, the rational cohomology of $\mathbf{P}X^1_{d,0}$ is isomorphic to that of $SL_2 \times SL_2$ and we obtain the same result as in the $n > 0$ case:

$$H^i(X^1_{d,0}/H_0; \mathbf{Q}) = \begin{cases} \mathbf{Q}, & i = 0; \\ 0, & \text{otherwise.} \end{cases}$$

3 Smooth curves of higher gonality

Let us now consider the case $k > 3$.

Similarly to what we have already seen in the $k = 3$ case, if for instance $n \geq 1$, a section in $V^k_{d,n}$ is equivalent to a polynomial f of degree d in the weighted ring $\mathbf{C}[x, y, z]$, with $\deg x = \deg y = 1$ and $\deg z = n$, of the form

$$f(x, y, z) = \alpha(x, y)z^k + \beta(x, y)z^{k-1} + \dots$$

Assuming that d is large enough with respect to n, the only differences that we may get from the $k = 3$ case, when studying the Vassiliev spectral sequence, are determined by configuration spaces having at least 3 points on the same line of the ruling.

However these configurations have all trivial twisted Borel-Moore homology by Lemma I.4.1. Hence the first columns of the Vassiliev spectral sequence converging to the Borel-Moore homology of $\Sigma^k_{d,n}$ are exactly the same ones as in the spectral sequence converging to the Borel-Moore homology of $\Sigma^3_{d,n}$, in Table III.1.

The only difference then is given by the stable range, which will now depend on k.

More precisely, the proof of Theorem 1.1 follows by an analogue of Lemma III.2.1.

Lemma 3.1. *Fix $k > 3$, $N \geq 1$. Then for any $n \geq 0$, the restriction of*

$$\{(f, p_1, \ldots, p_N) \in V^k_{d,n} \times B(\mathbb{F}_n, N) | p_1, \ldots, p_N \in \mathrm{Sing}(f)\} \xrightarrow{\pi} B(\mathbb{F}_n, N)$$

to the locus where no more than two points p_i lie on the same line of the ruling is a vector bundle of rank $\dim V^k_{d,n} - 3N$ provided $d \geq 2N + kn - 1$ holds.

Remark 13. By the same reasons explained in Remark III.7, also in this case it is sufficient to prove the above lemma for N points $\{p_1, \ldots, p_N\} \subset \mathbb{F}_n$, all lying on distinct lines of the ruling.

Proof. For $n = 0$ the claim follows from exactly the same argument of Lemma III.2.1, also for $n = 0$.

Assume then $n > 0$. Similarly to the proof of Lemma III.2.1, we fix a set of N distinct points $\{p_1, \ldots, p_N\} \subset \mathbb{F}_n$, all lying on distinct lines of the ruling. Then, let us consider the evaluation map

$$\mathbf{C}\left[x, y, z\right]_d \xrightarrow{ev} M_{3,N}(\mathbf{C})$$

defined exactly in the same way.

Consider first the case where $p_1 \in \mathbb{F}_n \backslash E_n$, with $p_1 = [1, 0, 0]$, after an appropriate change of coordinates. Define then the polynomials φ_i's exactly as in the proof of Lemma III.2.1, with $r \geq kn + 1$. The polynomials φ_i have degree $d \geq 2N + kn - 1$, and it is easy to check that their images through the evaluation map are again linearly independent generators for the subspace in $M_{3,N}(\mathbf{C})$ of matrices with zeros outside the first column.

On the other hand, if p_1 belongs to the exceptional divisor, and it is of the form $p_1 = ([0, 0, 1], [1, 0]) \in Bl_{[0,0,1]}\mathbf{P}(1, 1, n)$, after an appropriate change of coordinates, we define

$$\varphi_0 = x^{r-kn} z^k \ell_2^2 \cdots \ell_N^2,$$

$$\varphi_1 = x^{r-kn-1} y z^k \ell_2^2 \cdots \ell_N^2,$$

$$\varphi_2 = x^{r-(k-1)n} z^{k-1} \ell_2^2 \cdots \ell_N^2.$$

The polynomials φ_i have again degree $d \geq 2N + kn - 1$, and their images through the evaluation map are again linearly independent generators for the subspace in $M_{3,N}(\mathbf{C})$ of matrices with zeros outside the first column.

By generalizing this to all other columns, this proves the surjectivity of the evaluation map, and thus the statement for $n > 0$. □

Finally, let us observe that, also in this case, the generalized version of Leray-Hirsch theorem applies for the action of GL_2 on $X^k_{d,n}$, if $n > 0$, and for the action of all H_0 on $X^k_{d,0}$.

This also follows from the fact that the first columns of the Vassiliev's spectral sequence, converging to the Borel-Moore homology of $\Sigma^k_{d,n}$, are exactly the ones in the spectral sequence converging to the Borel-Moore homology of $\Sigma^3_{d,n}$, provided that the lemma above applies.

Taking then the quotient by the action of $\mathbf{C}^*$, when $n \geq 1$, yields Theorem 1.1. Precisely, we have that, for $k \geq 3$ and $n \geq 1$, the rational cohomology of $X^k_{d,n}/H_n$, in degree $i < \frac{d-kn+1}{2}$, is either

$$H^i(X^k_{d,n}/H_n; \mathbf{Q}) = \begin{cases} \mathbf{Q}, & i = 0, \\ \mathbf{Q}(-1), & i = 2, \\ \mathbf{Q}(-3), & i = 5, \\ \mathbf{Q}(-4), & i = 7, \\ 0 & \text{otherwise}; \end{cases}$$

or

$$H^i(X^k_{d,n}/H_n; \mathbf{Q}) = \begin{cases} \mathbf{Q}, & i = 0, \\ \mathbf{Q}(-1), & i = 2, \\ \mathbf{Q}(-2), & i = 3, \\ \mathbf{Q}(-2), & i = 4, \\ \mathbf{Q}(-3), & i = 5, \\ \mathbf{Q}(-4), & i = 7, \\ 0, & \text{otherwise}; \end{cases}$$

while the rational cohomology of $X^k_{d,0}/H_0$, in degree $i < \frac{d-1}{2}$, is

$$H^i(X^k_{d,0}/H_0; \mathbf{Q}) = \begin{cases} \mathbf{Q}, & i = 0, \\ \mathbf{Q}(-3), & i = 5, \\ 0, & \text{otherwise}. \end{cases}$$

4 Hyperelliptic case

In this final section, we discuss the case $k = 2$.

In the previous one, we proved that for $k > 3$, the stable cohomology ring of $X^k_{d,n}/H_n$ is the same as the one found for $k = 3$ in chapter III, with tighter stable range. This was motivated by the fact that, when applying Gorinov-Vassiliev's method, configuration spaces not having the expected codimension were those with at least 3 points on the same line of the ruling and these have trivial twisted Borel-Moore homology by [Vas99, Lemma 2].

On the contrary, for $k = 2$, configurations not having the expected condimension are also those with two points on the same line of the ruling. These do not have trivial twisted Borel-Moore homology, therefore we do not expect a similar behavior.

In fact we can compute the first columns in the Vassiliev's spectral sequence and show that these are different to the ones found for $k \geq 3$. Moreover, we cannot determine if the cohomology stabilizes, but, if this is the case, we expect the stable ring to be supported on an infinite number of degrees.

Let $f \in V^2_{d,n}$. Configurations with 2 points on the same line of the ruling will have codimension 5 instead of 6. Notice also that $(2E_n + dF_n) \cdot F_n = 2$, hence we cannot have more that 2 singular points on each ruling (if the curve is irreducible): if we do have two distinct singular points on a line of the ruling, then the curve must be reducible with that line of the ruling as a component.

Thus, we can distinguish our singular points between couples of points belonging to the same line of the ruling and single points on distinct lines of the ruling. We will indeed define the configurations of singularities in Gorinov-Vassiliev's method according to this distinction.

Definition 10. A configuration of points in $\mathbb{F}_n$ is of type (h, l) if it is defined by h couples of points on h distinct lines of the ruling, plus l points on l distinct lines of the ruling, different from the previous h ones.

The corresponding configuration space will be denoted by $X_{h,l}$.

Remark 14. When $n = 0$, we are considering the ruling containing F_0.

From this definition, the configuration space $X_{h,l}$ consists of $2h+l$ points. Moreover, the codimension of the vector subspace in $V^2_{d,n}$ of elements being singular at least at $X_{h,l}$ is $5h + 3l$.

These configuration spaces are fiber spaces

$$X_{h,k} \to F(\mathbf{P}^1, h+l), \tag{IV.1}$$

with fiber $B(\mathbf{P}^1, 2)^h \times (\mathbf{P}^1)^l$.

As usual, we want to compute the Borel-Moore homology of $X_{h,l}$ with twisted coefficients. Since the h lines of the ruling contain 2 points, the local system of coefficients that we have to consider on the base space is the one induced by the trivial representation on $\mathfrak{S}_h$ and the by the sign representation on $\mathfrak{S}_l$, which is the representation $\mathbb{S}_{h,1^l} \oplus \mathbb{S}_{h+1,1^{l-1}}$ of $\mathfrak{S}_{h+l}$.

In the following we will compute the Vassiliev's spectral sequence converging to $\bar{H}_\bullet(\Sigma^2_{d,n}; \mathbf{Q})$ for small values of $h+l$, precisely for $h+l \leq 2$.

Here is the list the families $X_{h,l}$ with $h+l \leq 2$ ordered by increasing codimension $[c_{h,l}]$ and increasing number of points.

(0, 1) One point, [3].

(1, 0) Two points on the same line of the ruling, [5].

(0, 2) Two points on distinct lines of the ruling, [6].

(1, 1) Three points, two of which on the same line of the ruling, [8] .

(2, 0) Four points, two on each of two lines of the ruling, [10] .

The computation of the Borel-Moore homology of the associated strata is obtained as usual, by recalling that, if $X_{h,l}$ consists of configurations of $2h+l$ points, then the stratum $F_{h,l}$ is a $\mathbf{C}^{\dim V^2_{d,n} - c_{h,l}} \times \mathring{\Delta}_{2h+l-1}$-bundle over $X_{h,l}$, where $\mathring{\Delta}_{2h+l-1}$ is an $(2h+l-1)$-dimensional open simplex.

Columns $(0,1)$**,** $(1,0)$

The space $F_{0,1}$ is a $\mathbf{C}^{\dim V^2_{d,n}-3}$-bundle over $X_{0,1} \cong \mathbb{F}_n$.
The space $F_{1,0}$ is a $\mathbf{C}^{\dim V^2_{d,n}-5} \times \mathring{\Delta}_1$-bundle over $X_{1,0} \xrightarrow{B(\mathbf{P}^1,2)} \mathbf{P}^1$.

Column $(0,2)$

The space $F_{0,2}$ is a $\mathbf{C}^{\dim V^2_{d,n}-6} \times \mathring{\Delta}_1$-bundle over $X_{0,2} \xrightarrow{\mathbf{P}^1 \times \mathbf{P}^1} F(\mathbf{P}^1, 2)$.

As we noticed above, we need to compute the Borel-Moore homology of $X_{0,2}$ with twisted coefficients. The Borel-Moore homology of the fiber $\mathbf{P}^1 \times \mathbf{P}^1$ is defined by 4 classes. Three of these are invariant with respect to the involution exchanging the two copies of $\mathbf{P}^1$, and only one of them is anti-invariant. Therefore, we need to consider $\bar{H}_\bullet(F(\mathbf{P}^1, 2); \pm\mathbf{Q})$ for the invariant classes and $\bar{H}_\bullet(F(\mathbf{P}^1, 2); \mathbf{Q})$ for the anti-invariant one. Precisely, the twisted Borel-Moore homology of $X_{0,2}$ can be deduced from the following spectral sequence.

TABLE IV.2

4	$\mathbf{Q}(3)$	
3		
2	$\mathbf{Q}(2)$	$\mathbf{Q}(3)$
1		
0	$\mathbf{Q}(1)$	
	2	4

Column $(1, 1)$

The space $F_{1,1}$ is a $\mathbf{C}^{\dim V_{d,n}^2 - 8} \times \mathring{\Delta}_2$-bundle over $X_{1,1} \xrightarrow{B(\mathbf{P}^1,2)\times\mathbf{P}^1} F(\mathbf{P}^1, 2)$.

Since the induced representation of the trivial representation of $\mathfrak{S}_1$ in $\mathfrak{S}_2$ is the regular representation, we need to consider, for each non-trivial class of the Borel-Moore homology of the fiber, that of the base space with the local system of coefficients induced by both trivial and sign representations.

Column $(2, 0)$

The space $F_{2,0}$ is a $\mathbf{C}^{\dim V_{d,n}^2 - 10} \times \mathring{\Delta}_3$-bundle over $X_{2,0} \xrightarrow{B(\mathbf{P}^1,2)^2} F(\mathbf{P}^1, 2)$.

Here, as we already explained, we will consider the Borel-Moore homology of $F(\mathbf{P}^1, 2)$ with coefficients $\mathbf{Q}$.

Putting all together, the first columns of the E^1-page of the spectral sequence will look as the ones represented in Table IV.3 (twisted by $\mathbf{Q}(-\dim V_{d,n}^2)$).

Notice that the resulting cohomology is divisible by that of GL_2 and this agrees with the fact that the generalized version of Leray-Hirsch theorem can be applied for the action of GL_2 on $X_{d,n}^2$.

TABLE IV.3

-3	$\mathbf{Q}(-1)$				
-4					
-5	$2\mathbf{Q}(-2)$				
-6					
-7	$\mathbf{Q}(-3)$	$\mathbf{Q}(-3)$			
-8			$2\mathbf{Q}(-3)$		
-9		$\mathbf{Q}(-4)$			
-10			$\mathbf{Q}(-4)$	$\mathbf{Q}(-4)$	
-11					
-12			$\mathbf{Q}(-5)$	$2\mathbf{Q}(-5)$	
-13					
-14				$\mathbf{Q}(-6)$	$\mathbf{Q}(-6)$
	$(0,1)$	$(1,0)$	$(0,2)$	$(1,1)$	$(2,0)$

Observe also that, since all Hirzebruch surfaces have the same cellular decomposition, for $n = 0$, the corresponding spectral sequence is given by Table IV.3 as well. Moreover, as in the case $k \geq 3$, here the generalized version of Leray-Hirsch theorem can be applied for the whole H_0. This forces the differentials in the E^1-page represented in Table IV.3 to have maximal rank.

Therefore we have, in degree $i < 8$,

$$H^i(X^2_{d,0}/H_0; \mathbf{Q}) = \begin{cases} \mathbf{Q}, & i = 0, \\ 0, & \text{otherwise}; \end{cases} \tag{IV.2}$$

while for $n > 0$, after considering the $\mathbf{C}^*$-bundle $X^2_{d,n}/GL_2 \to X^2_{d,n}/H_n$, we have

$$H^i(X^2_{d,n}/H_n; \mathbf{Q}) = \begin{cases} \mathbf{Q}, & i = 0, \\ \mathbf{Q}(-1), & i = 2, \\ 0 & \text{otherwise}. \end{cases} \tag{IV.3}$$

Here, the bound $i < 8$ will be explained later.

As predicted, the resulting cohomology ring is different from that of $H^\bullet(X^k_{d,n}/H_n; \mathbf{Q})$. Furthermore, we do not expect it to be zero after a certain degree.

Indeed, the Borel-Moore homology of the strata corresponding to $h + l \geq 3$ are not trivial a priori and they can be computed from (IV.1) and the $\mathfrak{S}_{h+l}$-equivariant isomorphism between $F(\mathbf{P}^1, h + l)$ and $\mathcal{M}_{0,h+l} \times PGL_2$.

We will limit ourselves to compute the strata corresponding to $h + l = 3$ in order to verify that, in degree $i < 8$, there are indeed no other classes than the ones in (IV.2) and (IV.3). The corresponding columns in the spectral sequence will look like the following.

TABLE IV.4

	(0,3)	(1,2)	(2,1)	(3,0)
-10		$\mathbf{Q}(-6)$		
-11				
-12			$\mathbf{Q}(-7)$	
-13		$\mathbf{Q}(-8)$		
-14			$\mathbf{Q}(-8)$	
-15			$\mathbf{Q}(-9)$	
-16				$\mathbf{Q}(-9)$
-10			$\mathbf{Q}(-10)$	
-11				
-12				$\mathbf{Q}(-11)$

We see that, in fact, the first non-trivial class is in degree 8 and the other columns with respect to $h + l > 3$ would also give contributions in degree $i \geq 8$ since the corresponding codimension gets higher as $h + l$ grows.

Appendix

A Configurations of singularities for a genus 5 trigonal curve

Here we produce the complete list of families of configurations in Σ_1 satisfying the conditions in List I.1.

In the following, c denotes the codimension in V_1 of the vector subspace of elements that are singular at least at the corresponding configuration and n is the number of points defining the configuration.

LIST A.1: List of configurations of singularities for a genus 5 trigonal curve

	Description	c	n
1.	A point on E;	3	1
2.	A general point;	3	1
3.	Two points on E;	5	2
4.	Two points on a line F of the ruling;	6	2
5.	A point on E + a general point;	6	2
6.	Two general points;	6	2
7.	Three points on E;	6	3
8.	Three points on a line F of the ruling;	7	3
9.	Four points on E	7	4
10.	The exceptional divisor E;	7	
11.	Two points on E + a general point;	8	3
12.	A line of F the ruling;	8	

13.	A point on E + two points on a line F of the ruling;	9	3
14.	A general point + two points on a line F of the ruling;	9	3
15.	A point on E + two general points;	9	3
16.	Three general points;	9	3
17.	Three collinear points;	9	3
18.	A point on E + two points on a line F of the ruling + $\{E \cap F\}$;	9	4
19.	Three points on E + a general point;	9	4
20.	Two points on E + two points on a line F of the ruling;	10	4
21.	A point on E + three points on a line F of the ruling;	10	4
22.	A general point+ three points on a line F of the ruling ;	10	4
23.	Four collinear points;	10	4
24.	Two points on E + two points on a line F of the ruling + $\{E \cap F\}$;	10	5
25.	A point on E + three points on a line F of the ruling + $\{E \cap F\}$;	10	5
26.	Two points on E + two general points;	11	4
27.	Three points on E + two points on a line F of the ruling;	11	5
28.	Two points on E + three points on a line F of the ruling;	11	5
29.	Three points on E + two points on a line F of the ruling + $\{E \cap F\}$;	11	6
30.	Two points on E + three points on a line F of the ruling + $\{E \cap F\}$;	11	6
31.	A line of the ruling + a general point;	11	
32.	A general line;	11	
33.	Two points on each of two lines of the ruling;	12	4
34.	Two points on a line of the ruling + two general points;	12	4
35.	A point on E + three collinear points;	12	4
36.	A general point + three collinear points;	12	4
37.	A point on E + three general points;	12	4
38.	Four general points;	12	4
39.	A point on E + two points on a line F of the ruling + a general point;	12	4
40.	A point on E + two points on a line F of the ruling + a general point + $\{E \cap F\}$;	12	5
41.	Three points on E + two general points;	12	5
42.	A point on E + three points on a line F of the ruling + a general point;	13	5
43.	Three points on a line of the ruling + two general points;	13	5
44.	Three points on a line of the ruling + two points on another ruling;	13	5
45.	Two points on E + two points on a line F of the ruling + a general point;	13	5
46.	A point on E + four collinear points;	13	5
47.	A general point+ four collinear points;	13	5
48.	A point on E + three points on a line F of the ruling + a general point + $\{E \cap F\}$;	13	6
49.	Two points on E + two points on a line F of the ruling + a general point + $\{E \cap F\}$;	13	6

50.	Two points on E + three collinear points;	14	5
51.	Two points on E + three general points;	14	5
52.	Two points on a line of the ruling + three collinear points;	14	5
53.	Five points on a non-degenerate conic;	14	5
54.	Two points on a line of the ruling + three collinear points + the point of intersection;	14	6
55.	Three points on each of two rulings;	14	6
56.	Two points on each of two rulings F_1, F_2 + a point on E + $\{E \cap F_1\}$ $+\{E \cap F_2\}$;	14	7
57.	A line of the ruling + two general points;	14	
58.	A general line + a general point;	14	
59.	A general point + two points on each of two lines of the ruling;	15	5
60.	A point on E + two general points + two points on a line of the ruling;	15	5
61.	Three general points + two points on a line of the ruling;	15	5
62.	A point on E + a general point + three collinear points;	15	5
63.	Two general points + three collinear points;	15	5
64.	Two points on each of two intersecting lines + the point of intersection;	15	5
65.	A point on E + four general points;	15	5
66.	Five general points;	15	5
67.	Three points on E + three collinear points;	15	6
68.	Three points on E + three general points;	15	6
69.	Two points on E + four collinear points;	15	6
70.	Two lines of the ruling;	15	
71.	A line of the ruling + a general line;	15	
72.	A non-degenerate conic;	15	
73.	Three points on a line of the ruling + two points on another ruling + a general point	16	6
74.	Three points on a line of the ruling + three general points;	16	6
75.	A point on E + a general point + four collinear points;	16	6
76.	Two general points + four collinear points;	16	6
77.	Two points on E + two points on a ruling F + two general points + $\{E \cap F\}$;	16	7
78.	Three points on E + four collinear points;	16	7

79.	Six points on a non-degenerate conic;	17	6
80.	Five points on a non-degenerate conic + a general point;	17	6
81.	Three points on E + four points on a non-degenerate conic.	17	7
82.	Two points on a line F of the ruling + three points on a line L + a general point + $\{F \cap L\}$;	17	7
83.	Three points on each of two lines of the ruling + a general point;	17	7
84.	Three points on each of two intersecting lines + the point of intersection;	17	7
85.	Three points of intersection between two non-degenerate conics, one of which is on E + four points of intersection with a line;	17	7
86.	Three points of intersection between two non-degenerate conics, none of which are on E + four points of intersection with a line;	17	7
87.	Four points of intersection of two non-degenerate conics + three points of intersection with a line of the ruling;	17	7
88.	A point on E + two points on each of two rulings F_1, F_2 + a general point + $\{E \cap F_1\}$ +$\{E \cap F_2\}$;	17	8
89.	Two points on E + three points on a line L + a point on a line F of the ruling + $\{E \cap F\}$ + $\{F \cap L\}$;	17	8
90.	Three points of intersection of two conics, each meeting a line F of the ruling and E at one point + $\{E \cap F\}$;	17	8
91.	Two points of intersection between two lines F_1, F_2 of the ruling and a line L + 6 points of intersection with a non-degenerate conic meeting each line at two distinct points;	17	8
92.	Three points of intersection between a line F of the ruling and two lines L_1, L_2 + five points of intersection with a non-degenerate conic meeting each line twice and F only once, outside E;	17	8
93.	Three points of intersection between a line F of the ruling and two lines L_1, L_2 + five points of intersection with a non-degenerate conic meeting each line twice and F at $\{E \cap F\}$;	17	8
94.	Four points of intersection between E, two lines F_1, F_2 of the ruling and a line L + five points of intersection with a non-degenerate conic meeting L twice and E, F_1, F_2 once;	17	9
95.	Three points of intersection between E and three lines F_1, F_2, F_3 of the ruling + 6 points of intersection with a non-degenerate conic meeting each ruling at two distinct points;	17	9
96.	The points of intersection between two lines of the ruling and three general lines;	17	9
97.	The points of intersection between E, three lines of the ruling and two intersecting lines;	17	10
98.	A line of the ruling + 3 general points;	17	
99.	A line + 2 general points;	17	
100.	The whole $Bl_P\mathbf{P}^2$	18	

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